Renormalization in the Winter model

U. G. Aglietti* and P. M. Santini†
Dipartimento di Fisica, Università di Roma "La Sapienza" and INFN, Sezione di Roma, 00185 Rome, Italy
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We show that metastable states in the Winter model can be related to the eigenstates of a particle in a box by means of renormalization and mixing.

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I. INTRODUCTION

The theory of unstable states in quantum mechanics [1–6] has applications virtually in any branch of physics: statistical and condensed-matter physics [7], atomic and molecular physics [8], nuclear physics [9,10], quantum-field theory and particle physics [11], and so on. In this paper we show that the particle states inside the cavity of the Winter model [12–17] can be obtained from the states of a particle in a box by means of renormalization and mixing. The Winter model describes the coupling of a cavity with the outside and is given, after proper rescaling [17] (see the next section), by the Hamiltonian

$$\hat{H} = -\frac{\partial^2}{\partial x^2} + \frac{1}{\pi g}\delta(x-\pi) \qquad (1)$$

in the half line $0 \leqslant x < \infty$ and with vanishing boundary conditions at zero $\psi(x=0,t)=0$. The distribution $\delta(y)$ is the Dirac delta function. Equation (1) describes a model with one parameter $g \in \mathbb{R}$, the inverse of the area of the potential barrier in $x=\pi$ (up to a factor π). The metastable states are nothing but wave packets initially (for example, at $t=0$) concentrated inside the cavity, i.e., in the interval $0 < x < \pi$. The time evolution of metastable states is controlled by wave propagation and imperfect multiple reflections on the right cavity wall, in $x=\pi$, leading to a leakage of the wave amplitude outside it. For $0 < g \ll 1$ (a high barrier), there is weak coupling of the cavity with the outside and resonant long-lived states come into play. The idea is that, by means of them, we can describe the dynamics of the particles initially inside the cavity *as if* the outside did not exist.

Let us briefly discuss the motivations that led us to further investigate the Winter model.

Historically, this model has been used for a semiquantitative analysis of α decay in heavy nuclei (see [9] and references therein), as the superposition of the nuclear and electrostatic potentials can be roughly described by a bump function (a general potential with compact support can be approximated with a δ function for large wavelengths). Of course, there are clearly much more realistic models in this context, to be analyzed with numerical methods.

The Winter model is currently used in quantum chemistry to check metastable properties of more complex models describing some specific phenomenology [14,15], as a source for exact quantum-decay solutions [16], or as a testing ground for various resonant-state formalisms (Green's functions, spectral decompositions, etc. [2]). Its relevance lies in the fact that it is the simplest Hamiltonian system exhibiting metastable states, for which an almost exact analytic treatment is possible. On the contrary, more phenomenologically oriented models, involving, for example, a sequence of δ-like barriers or additional potentials modeling some media or detector interaction [2], have to be analyzed by means of specific numerical techniques. In this framework, the Winter model provides a safe and simple check of results obtained for more complex models, being a limiting case.

There is also a link of the Winter model to a big and active field of current research: quantum chromodynamics on the lattice. The box eigenfunctions play the role in the Winter model of the interpolating fields in quantum-field theory. The latter are operators acting on the vacuum by exciting a multiplicity of different particle states, which one is usually interested in separating, as we explicitly can in the Winter model by means of the inverse of the mixing matrix $U^{-1}(g)$.

Finally, the Winter model also has a pedagogical relevance, as it enables a student in physics or chemistry to fully understand the decay of states in quantum mechanics without the use of perturbation theory (Fermi's golden rule). Indeed, by using perturbative methods, one is always faced with the problem of separating what is intrinsic to the system from what is instead just a consequence of the approximations. On the theoretical side, the application of perturbation theory to decay phenomena is not straightforward (while of course correct) because an arbitrarily small perturbation produces a drastic change in the spectrum of the theory: The discrete spectrum completely disappears. In the same spirit, resonance properties of the Winter model for small $|g|$ were discussed in the book of applications of quantum mechanics by Flügge [13].

Our results involve a second-order computation in g extending the $O(g)$ results in [17], in which we repeated the original Winter computation, finding additional contributions in the time evolution of unstable states, which were absent in [12]. These new terms have a small strength $O(g) \ll 1$ compared to the old ones, but decay generally slower in time, with the smallest decay width. These contributions therefore dominate the evolution of all unstable states except the lowest one at large times and cannot be neglected. In this paper we show that such contributions can in principle be rotated away by means of a linear transformation $U(g)$ in the infinite-dimensional vector space of the resonances.

Since the Winter model is, as discussed above, a kind of attractor of many quantum-decay models, it is clear that the occurrence of nondiagonal terms has to be a general phenomenon in metastable systems. The implications of the mixing terms for the checks discussed above are still to be

*Ugo.Aglietti@roma1.infn.it
†Paolo.Santini@roma1.infn.it

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investigated. Let us stress that it would have been difficult to imagine the existence of such mixing terms in complex models without an exhaustive analytic study of the Winter model.

To show the relevance of renormalizing the Winter model, let us briefly discuss the importance of renormalization in understanding the dynamics of many physical theories. It is common practice in physics to compare a given physical system with a simplified one in which some interactions are omitted. These interactions can be related to control parameters in experiments involving, for example, external electric or magnetic fields or can be treated theoretically as variable quantities. The idea of renormalization is that, as one of the main dynamical effects, switching on an interaction in a physical system modifies the parameters of the starting, noninteracting, system; once renormalization has been made, the residual effects of the interaction are substantially weaker than before renormalization. When the coupling setting the strength of the interaction gets too large, there is usually little connection between the free and the interacting system and renormalization often loses its meaning.

In condensed-matter physics, renormalization is related to the so-called adiabatic continuity principle [7]: By adiabatically (i.e., slowly) turning on an interaction, the free states of the system go 1-1 onto the interacting states by means of a flow of the parameters such as masses, couplings, etc. A typical example is the normal Fermi liquid, i.e., a system with a repulsive interaction among electrons in the Fermi sphere [18]. In quantum-field theory [19–22], the relation between free parameters and interacting ones (masses, couplings, and field normalizations) is often singular because of the lack of intrinsic energy scales cutting off the quantum fluctuations at large energies: That leads to the well-known ultraviolet infinities. One also encounters renormalization in nonrelativistic quantum mechanics with δ-functions potentials [23].

It is remarkable that solving the nonrelativistic Schrödinger equation for nuclei (which are strongly interacting many-body systems), in order to obtain the low-energy excitations and the scattering cross sections, can be greatly simplified by implementing renormalization-group ideas, as recently discovered [10]. One finds the phenomenon of generation of many-body operators by renormalization-group flow, the problem of the stability under change of the ultraviolet cutoff, etc., which are typical of perturbative quantum-field-theory computations, in a completely different framework.

In classical physics, renormalization is usually implemented by the method of multiple scales [24,25]. In the case of a free anharmonic oscillator, for example, renormalization amounts to the absorption of secular terms into a shift of the harmonic frequency. These terms are formally resonances produced by forcing terms occurring in the perturbative expansion, are incompatible with energy conservation, and spoil the convergence of the perturbative expansion at large times. After renormalization, such strong-coupling effects completely disappear; only a small coupling between the harmonics is left and a uniform approximation in time is obtained.

Let us remark that the adiabatic continuity principle is subjected to relevant violations. Let us quote, for example, the cases of the energy gap in the Bardeen-Cooper-Schriffer theory of classical superconductors [26] or the mass gap in massless quantum chromodynamics [27]. These phenomena are typically characterized by functions that have an essential singularity when the interaction coupling g goes to zero, of the form $e^{1/g}$ for $g < 0$, making nonsmooth the connection between the interacting system and the related free one. In these cases, the relation between the free system and the interacting one is highly nontrivial and the residual interaction is of nonperturbative character.

Let us end the Introduction by observing that, even though renormalization is implemented and interpreted in quite different ways in different contexts, it is a ubiquitous phenomenon in physics, like the unstable states cited above, a thing that certainly could not be expected *a priori*.

II. WINTER MODEL

The Hamiltonian operator of Winter model reads

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \lambda\delta(x - L), \tag{2}$$

where m is the particle mass, λ is a coupling constant, δ is the Dirac delta function, and $x = L > 0$ is the support of the potential. Formulas can be simplified by going to a proper adimensional coordinate via

$$x = \frac{L}{\pi}x' \tag{3}$$

and rescaling the Hamiltonian as

$$\hat{H} = \frac{\hbar^2\pi^2}{2mL^2}\hat{H}'. \tag{4}$$

The new (adimensional) Hamiltonian then takes the form in which it appeared in the Introduction,

$$\hat{H}' = -\frac{\partial^2}{\partial x'^2} + \frac{1}{\pi g}\delta(x' - \pi), \tag{5}$$

and contains the single real parameter

$$g = \frac{\hbar^2}{2m\lambda L}. \tag{6}$$

The time-dependent Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{7}$$

now reads

$$i\frac{\partial\psi}{\partial t'} = \hat{H}'\psi, \tag{8}$$

where

$$t' \equiv \frac{\hbar\pi^2}{2mL^2}t. \tag{9}$$

Let us omit primes from now on for the sake of simplicity. It is possible to rescale the Winter Hamiltonian (2) in slightly different ways, as done, for example, by the Winter model itself, but the main point is that we deal in any case with a one-parameter model.

A. Spectrum

For a positive coupling, $g > 0$, i.e., for a repulsive potential, the Hamiltonian of the Winter model in Eq. (1)

has a continuum spectrum only, with eigenfunctions of the form

$$\psi(x;k,g) \propto \left(-\frac{i}{2}\exp(ikx)+\frac{i}{2}\exp(-ikx)\right)\theta(\pi-x) + [a(k,g)\exp(ikx)+b(k,g) \times \exp(-ikx)]\theta(x-\pi) \tag{10}$$

and eigenvalues

$$\epsilon(k)=k^2. \tag{11}$$

The step function $\theta(x)=1$ for $x>0$ and 0 otherwise and the coefficients $a(k,g)$ and $b(k,g)$ have the following expressions:

$$a(k,g)=-\frac{i}{2}+\frac{1}{4\pi gk}[\exp(-2i\pi k)-1], \tag{12}$$

$$b(k,g)=\frac{i}{2}+\frac{1}{4\pi gk}[\exp(+i2\pi k)-1]. \tag{13}$$

These coefficients have the following two symmetries (which will be relevant in the discussion of the spectrum as well as of the time evolution):

$$a(-k,g)=-b(k,g), \tag{14}$$

$$a(k,g)^*=b(k^*,g^*), \tag{15}$$

where the asterisk denotes complex conjugation. In general, k is a real quantum number but, because of Eq. (14), the eigenfunctions are odd functions of k, so one can assume $k>0$, implying that there is no energy degeneracy (trivial S matrix). By normalizing the eigenfunctions as

$$\int_0^\infty \psi^*(x;k',g)\psi(x;k,g)dx=\delta(k-k'), \tag{16}$$

where $\delta(q)$ is the Dirac delta function, the normalization factor reads

$$\mathcal{N}(k,g)=\frac{1}{[2\pi a(k,g)b(k,g)]^{1/2}}. \tag{17}$$

The final expression for the eigenfunctions therefore can be written as

$$\psi(x;k,g)=\frac{1}{\sqrt{2\pi}}\left\{\left(-\frac{i\exp(ikx)}{2\,[a(k,g)b(k,g)]^{1/2}}+\frac{i\exp(-ikx)}{2\,[a(k,g)b(k,g)]^{1/2}}\right)\theta(\pi-x) + \left[\left(\frac{a(k,g)}{b(k,g)}\right)^{1/2}\exp(ikx)+\left(\frac{b(k,g)}{a(k,g)}\right)^{1/2}\exp(-ikx)\right]\theta(x-\pi)\right\}. \tag{18}$$

Note that, because of continuum normalization, the amplitude of the eigenfunctions outside the wall is always $O(1)$, no matter which values are chosen for k and g, while inside the cavity the amplitude has a nontrivial dependence on k and g. For $|g|\ll 1$, the amplitude of $\psi(x;k,g)$ inside the cavity shows marked peaks for $k\approx n-gn$, where $1\leqslant n\ll 1/|g|$ is an integer, because

$$|a(n-gn,g)|=|b(n-gn,g)|=\frac{|g|}{2}\sqrt{\pi^2n^2+1}+O(g^2) \approx \frac{\pi}{2}|g|n\ll 1. \tag{19}$$

As is usually the case, peaks become less marked for increasing n.

III. TEMPORAL EVOLUTION OF UNSTABLE STATES

The eigenfunctions of a particle in a box of length $L=\pi$ with Hamiltonian

$$\hat{H}_0=-\frac{\partial^2}{\partial x^2} \tag{20}$$

are given, as is well known, by

$$\psi_0^{(l)}(x,t)=\sqrt{\frac{2}{\pi}}\sin(lx)e^{-il^2t}, \tag{21}$$

where $l=1,2,3,\ldots$ is a positive integer and $0\leqslant x\leqslant\pi$. We study the time evolution of wave functions $\psi^{(l)}(x,t;g)$ that coincide at $t=0$ with the free eigenfunctions in Eq. (21) in the interval $x\in[0,\pi]$ (the cavity) and vanish outside it:

$$\psi^{(l)}(x,0)=\begin{cases}\sqrt{2/\pi}\sin(lx) & \text{for } 0\leqslant x\leqslant\pi\\ 0 & \text{for } \pi<x<\infty.\end{cases} \tag{22}$$

The initial conditions above make the limit $g\to 0$ easy because the wave functions $\psi^{(l)}(x,t;g)$ become eigenfunctions of Winter Hamiltonian in that limit. For $g\neq 0$, however, we will see in the next section that there are more natural initial conditions to consider.

The spectral representation in eigenfunctions of the wave function of the unstable state at time t has the explicit expression

$$\psi^{(l)}(x,t;g)=\left(\frac{2}{\pi}\right)^{3/2}\int_0^\infty p^{(l)}(k;x,g)e^{-ik^2t}dk, \quad 0\leqslant x\leqslant\pi,\quad g>0, \tag{23}$$

where

$$p^{(l)}(k;x,g)=(-1)^l l\frac{\sin(k\pi)}{k^2-l^2}\frac{\sin(kx)}{4a(k,g)b(k,g)}. \tag{24}$$

The integral on the right-hand side (rhs) of Eq. (23) can be exactly evaluated with numerical methods for t not too large because high-frequency oscillations occur in the factor

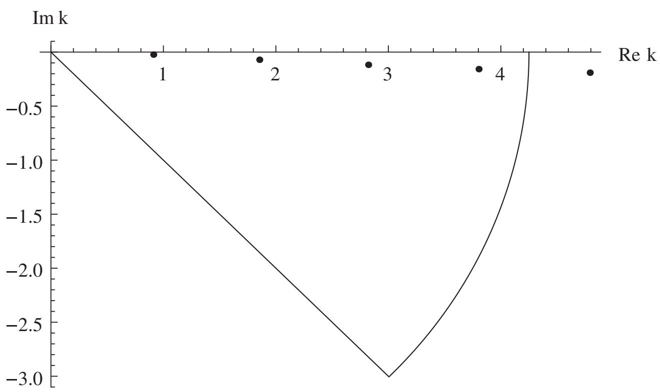

FIG. 1. Rotation of the integration contour in the complex k plane (see the text) and (simple) zeros of the function $b(k,g)$ for $g = 0.1$ lying in the fourth quadrant.

e^{-ik^2t} in the integrand for $t \to +\infty$. In order to study the large-time behavior we therefore have to develop analytic techniques.

A. Small-time expansion

For small times $t \ll 1$, the wave function in Eq. (23) exhibits a power behavior [8,12,17] that is not relevant to our discussion and will not be treated further.

B. Asymptotic expansion for large times

To obtain explicit analytic formulas, we expand the integral for large t. The steepest-descent method suggests to replace the integral on the rhs of Eq. (23) by the integral over the steepest-descent ray $(0,\infty e^{-i\pi/4})$, on which the fast oscillations of the integrand are absent (see Fig. 1). Therefore, the state $\psi^{(l)}(x,t;g)$ is decomposed in a natural way into the sum of two quite different contributions:

$$\psi^{(l)}(x,t;g) = \psi^{(l)}_{ex}(x,t;g) + \psi^{(l)}_{pw}(x,t;g), \tag{25}$$

where

$$\psi^{(l)}_{ex}(x,t;g) \equiv -2\pi i \left(\frac{2}{\pi}\right)^{3/2} \sum_{n=1}^{\infty} \mathrm{Res}\big[p^{(l)}(k;x,g)e^{-ik^2t}, k^{(n)}(g)\big], \tag{26}$$

$$\psi^{(l)}_{pw}(x,t;g) \equiv e^{-i\pi/4} \left(\frac{2}{\pi}\right)^{3/2} \int_0^{\infty} p^{(l)}(ke^{-i\pi/4};x,g)e^{-k^2t}dk. \tag{27}$$

Here $\mathrm{Res}\,[f(k);a]$ denotes the residue of the (analytic) function f at the point $a \in \mathbb{C}$ and $k^{(n)}(g)$ is a simple pole of the integrand lying in the last octant of the complex k plane for $n \in \mathbb{N}_+$(see Fig. 1), to be evaluated in the next section.

In general, the contribution $\psi^{(l)}_{ex}(x,t;g)$ exhibits an exponential decay, while the contribution $\psi^{(l)}_{pw}(x,t;g)$ exhibits a power decay as $t \gg 1$. Let us consider the above contributions in turn.

1. Exponential contributions

The explicit expression of the exponential part of the unstable wave function at time $t \geqslant 0$ reads

$$\psi^{(l)}_{ex}(x,t;g) = -2\pi i \left(\frac{2}{\pi}\right)^{3/2} \sum_{n=1}^{\infty} \mathrm{Res}\bigg[(-1)^l l \frac{\sin(k\pi)}{k^2 - l^2} \times \frac{\sin(kx)\exp(-ik^2t)}{4a(k,g)b(k,g)}; k^{(n)}(g)\bigg]. \tag{28}$$

The Hamiltonian is Hermitian (the physical case) for real g only, which we assume from now on. The integrand [the first argument in the large square brackets in Eq. (28)], as a function of the complex k variable, has removable singularities at the positive integers $k = l$ and pole singularities corresponding to the zeros of the functions $a(k,g)$ and $b(k,g)$ constrained by the conditions

$$\mathrm{Im}k^{(n)}(g) < 0, \quad \mathrm{Re}k^{(n)}(g) > |\mathrm{Im}k^{(n)}(g)|. \tag{29}$$

The transcendental equation

$$b(k,g) = 0 \tag{30}$$

has simple zeros for $|g| \ll 1$ of the form

$$k^{(n)}(g) = n - ng + ng^2 - i\pi n^2 g^2 + O(g^3), \tag{31}$$

where n is a nonzero integer. All these zeros lie in the lower half of the complex k plane, i.e., have $\mathrm{Im}k^{(n)}(g) < 0$, and satisfy also the second condition in Eq. (29) for $n > 0$. In general, the function $k^{(n)}(g)$ is the branch with $k^{(n)}(0) = n$ of the multivalued analytic function $k(g)$ satisfying $b(k(g),g) = 0$. Numerical computation actually shows that conditions (29) remain satisfied up to values of g of order one. The zeros leave the last octant $(-\pi/4 < \theta < 0)$ for very large values of $|g|$, where the unstable-state description becomes irrelevant. Because of Eq. (15), which for real g reads

$$a(k,g) = b(k^*,g)^*, \tag{32}$$

the zeros of the equation $a(k,g) = 0$ are complex conjugates of the ones of Eq. (30), therefore lie in the upper half k plane, and consequently do not enter the residue sum.

The only nontrivial residue to evaluate is therefore

$$\mathrm{Res}\left[\frac{1}{b(k,g)}; k^{(n)}(g)\right] = \lim_{k \to k^{(n)}(g)} \frac{k - k^{(n)}(g)}{b(k,g)} = \frac{1}{(\partial b/\partial k)(k,g)|_{k=k^{(n)}(g)}} = \frac{-2igk^{(n)}(g)}{1 + g[1 - 2\pi i k^{(n)}(g)]}, \tag{33}$$

where, after the evaluation of the derivative, we have simply replaced $k \to k^{(n)}(g)$ and used the relation

$$\exp[2\pi i k^{(n)}(g)] = 1 - 2\pi i g k^{(n)}(g), \tag{34}$$

which is true for any solution of Eq. (30). We then have the following exact expression in terms of the zero set

$\{k^{(n)}(g)\}$:

$$\psi_{ex}^{(l)}(x,t;g) = -2\pi i \left(\frac{2}{\pi}\right)^{3/2} (-1)^l l \sum_{n=1}^{\infty} \frac{1}{4a[k^{(n)}(g),g](\partial b/\partial k)[k^{(n)}(g),g]} \frac{\sin[k^{(n)}(g)\pi]}{k^{(n)}(g)^2 - l^2} \sin[k^{(n)}(g)x] E^{(n)}(t;g)$$
$$= \sqrt{\frac{2}{\pi}}(-1)^l 2lg \sum_{n=1}^{\infty}(-1)^n \frac{k^{(n)}(g)[1-2\pi i g k^{(n)}(g)]^{1/2}}{[l^2 - k^{(n)}(g)^2]\{1+[1-2\pi i k^{(n)}(g)]g\}} \sin[k^{(n)}(g)x] E^{(n)}(t;g), \tag{35}$$

where we have defined the time evolution factors

$$E^{(n)}(t;g) \equiv \exp[-i\varepsilon^{(n)}(g)t] = \exp\left[-i\omega^{(n)}(g)t - \tfrac{1}{2}\Gamma^{(n)}(g)t\right]. \tag{36}$$

Since the energies are complex for $g \neq 0$, on the last member we have split them into real and imaginary parts as

$$\varepsilon^{(n)}(g) = [k^{(n)}(g)]^2 = \omega^{(n)}(g) - \frac{i}{2}\Gamma^{(n)}(g), \tag{37}$$

where $\omega^{(n)}(g)$ is the frequency and $\Gamma^{(n)}(g)$ is the decay width of the pole state n. Note that $E^{(n)}(0;g) = 1$, as it should. In Eq. (35) we have chosen the principal branch of the complex square root $-\pi < \arg z \leqslant \pi$ ($1^{1/2} = 1$). In deriving the last member in Eq. (35) we have also used a relation obtained by taking the square root of Eq. (34):

$$\exp[i\pi k^{(n)}(g)] = (-1)^n[1-2\pi i g k^{(n)}(g)]^{1/2}. \tag{38}$$

The sign in front of the square root is fixed by taking the limit $g \to 0$ on both sides, i.e., by setting $g = 0$ and replacing $k^{(n)}(g) \to n$. The equality between the first and the last member in Eq. (35) can be written in compact form as

$$\psi_{ex}^{(l)}(x,t;g) = \sqrt{\frac{2}{\pi}}(-1)^l 2lg \sum_{n=1}^{\infty}(-1)^n F_l(k^{(n)}(g);g) \times \sin[k^{(n)}(g)x]E^{(n)}(t;g), \tag{39}$$

where

$$F_l(z;w) \equiv \frac{z(1-2\pi i z w)^{1/2}}{(l^2 - z^2)[1+(1-2\pi i z)w]}. \tag{40}$$

Once the poles $\{k^{(n)}(g)\}$ have been exactly evaluated (with numerical methods, for example) for a large set of integers $1 \leqslant n \leqslant N$ with $N \gg 1$, one can insert them in the known function F_l on the rhs of Eq. (39). This way one obtains an (almost) exact evaluation of the exponential part of the wave function. In the following sections, however, we present an expansion for $g \ll 1$ that allows for explicit analytic expressions. Equation (35) is conveniently rewritten as

$$\psi_{ex}^{(l)}(x,t;g) = \sum_{n=1}^{\infty} V(g)_{ln}\theta^{(n)}(x,t;g), \tag{41}$$

where the entries of the mixing matrix $V(g)$ read

$$V(g)_{ln} \equiv \frac{g(-1)^{l+n}2lk^{(n)}(g)[1-2\pi i g k^{(n)}(g)]^{1/2}}{[l^2 - k^{(n)}(g)^2]\{1+[1-2\pi i k^{(n)}(g)]g\}}. \tag{42}$$

We have defined the pole wave functions (which evolve diagonally with time)

$$\theta^{(n)}(x,t;g) \equiv \sqrt{\frac{2}{\pi}} \sin[k^{(n)}(g)x]E^{(n)}(t;g). \tag{43}$$

For $|g| \ll 1$ there is a similarity between the pole wave functions above and the eigenfunctions in Eq. (18) for $k \simeq n - gn \in \mathbb{R}$. Let us stress, however, the differences between the true exact eigenstates, having real energies and lying in the continuum spectrum, and the resonance states, normalizable states with complex energy describing dynamics in a simple but approximate way for a finite amount of time only [7].

2. Matrix notation

To simplify formulas, it is convenient to introduce matrix notation. Let us define an infinite column vector containing all the pole states

$$\Theta(x,t;g) \equiv \begin{pmatrix} \theta^{(1)}(x,t;g) \\ \theta^{(2)}(x,t;g) \\ \vdots \\ \theta^{(n)}(x,t;g) \\ \vdots \end{pmatrix} \tag{44}$$

and an infinite diagonal matrix representing the evolution of the pole states

$$\mathcal{E}(t;g) \equiv \mathrm{diag}[E^{(1)}(t;g), E^{(2)}(t;g), \ldots, E^{(n)}(t;g), \ldots]. \tag{45}$$

In more standard notation

$$\mathcal{E}(t;g) = \begin{pmatrix} E^{(1)}(t;g) & 0 & \cdots & 0 \\ 0 & E^{(2)}(t;g) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & E^{(n)}(t;g) & \cdots \\ 0 & 0 & \cdots & \cdots \end{pmatrix}. \tag{46}$$

The temporal evolution of the pole states can be rewritten in matrix notation as

$$\Theta(x,t;g) = \mathcal{E}(t;g)\Theta(x,0;g). \tag{47}$$

Similarly, let us define an infinite column vector containing the metastable wave functions

$$\Psi(x,t;g) = \begin{pmatrix} \psi^{(1)}(x,t;g) \\ \psi^{(2)}(x,t;g) \\ \vdots \\ \psi^{(n)}(x,t;g) \\ \vdots \end{pmatrix} \tag{48}$$

as well as the vectors $\Psi_{ex}(x,t;g)$ and $\Psi_{pw}(x,t;g)$ containing the exponential and power parts, respectively.[1] Equation (25) now reads

$$\Psi(x,t;g) = \Psi_{ex}(x,t;g) + \Psi_{pw}(x,t;g). \tag{49}$$

Setting $t=0$ in Eq. (48), one obtains a list of the initial conditions for all $l = 1,2,3,\ldots$ [see Eq. (22)]:

$$\Psi(x,0) = \sqrt{\frac{2}{\pi}}\theta(\pi - x)\begin{pmatrix} \sin(x) \\ \sin(2x) \\ \vdots \\ \sin(nx) \\ \vdots \end{pmatrix}. \tag{50}$$

Equation (41) reads in the new notation

$$\Psi_{ex}(x,t;g) = V(g)\Theta(x,t;g) = V(g)\mathcal{E}(t;g)\Theta(x,0;g). \tag{51}$$

Let us remark that since there are no power corrections in time, Eq. (51) does not reproduce the initial value in Eq. (50) for $t=0$.

C. Power contributions

The integral $\psi^{(l)}_{pw}(x,t;g)$, over the ray $(0,\infty e^{-i\pi/4})$ in the complex k plane, can be exactly evaluated with numerical methods without problems for any time $t \geqslant 0$, as it does not involve any oscillation. It is also convergent at the initial time $t=0$; in other words, the decomposition in Eq. (25) does not spoil the convergence at $t=0$. However, for large times $t \gg 1$, the integral takes the dominant contribution from a neighborhood of $k=0$, where the integrand is analytic and can therefore be expanded in powers of k:

$$p^{(l)}(k;x,g) = \frac{g^2}{(1+g)^2}\sum_{j=1}^{\infty} p^{(l)}_j(x,g)k^{2j}. \tag{52}$$

The first two coefficients explicitly read

$$p^{(l)}_1(x,g) = \frac{(-1)^{l+1}}{l}\pi x, \tag{53}$$

$$p^{(l)}_2(x,g) = \frac{(-1)^{l+1}}{l}\pi x\left[\frac{1}{l^2} + \frac{\pi^2}{6} + \frac{2}{3}\frac{\pi^2 g}{1+g} - \frac{\pi^2 g^2}{(1+g)^2} - \frac{x^2}{6}\right]. \tag{54}$$

Substituting the series on the rhs of Eq. (52) into the integral over k on the rhs of Eq. (27), exchanging the integral with the series, and performing the change of variable $\nu = k^2 t$, one obtains the following asymptotic expansion:

$$\psi^{(l)}_{pw}(x,t;g) \approx \frac{\sqrt{2}}{\pi^{3/2}}\frac{e^{-i\pi/4}g^2}{(1+g)^2}\sum_{j=1}^{\infty}\frac{(-i)^j p^{(l)}_j(x,g)}{t^{j+1/2}} \times \int_0^{\infty} d\nu\, \nu^{j-1/2}e^{-\nu} \tag{55}$$

$$= \frac{\sqrt{2}}{\pi}\frac{e^{-i\pi/4}g^2}{(1+g)^2}\sum_{j=1}^{\infty}\frac{(-i)^j(2j-1)!!}{2^j}\frac{p^{(l)}_j(x,g)}{t^{j+1/2}},$$
$$0 \leqslant x \leqslant \pi, \quad t \gg 1, \tag{56}$$

whose first two terms read

$$\psi^{(l)}_{pw}(x,t;g) \approx \frac{e^{i\pi/4}}{\sqrt{2}}\frac{(-1)^l}{l}\frac{g^2}{(1+g)^2}\frac{x}{t^{3/2}} \times \left\{1 - \frac{3i}{2t}\left[\frac{1}{l^2} + \frac{\pi^2}{6} + \frac{2}{3}\frac{\pi^2 g}{1+g} - \frac{\pi^2 g^2}{(1+g)^2} - \frac{x^2}{6}\right] + O\left(\frac{1}{t^2}\right)\right\}. \tag{57}$$

Let us make a few remarks. The physical interpretation of the small-k expansion is that states with very low momenta are produced in the decay at asymptotic times [12]. The above asymptotic expansion is uniformly valid for all $g \geqslant 0$ since the coefficients $p^{(l)}_j(x,g)$ are uniformly bounded in that region [see Eq. (52)]. The exponent 3/2 controlling the power decay $\psi_{pw} \approx 1/t^{3/2}$ does not depend on l and g; power corrections, however, vanish for $g \to 0$ (the impermeable cavity).

We are in complete agreement with [12] as far as the asymptotic power behavior in time is concerned; however, we remark that our results for the power corrections in t are valid for any g, i.e., they do not involve any expansion in g. In particular, one can take the limit $g \to \infty$, in which the potential barrier disappears.

IV. RESONANCES

For $g \ll 1$, i.e., for weak coupling, there is a large time slice between a preexponential small-t region [8,12,17] and a postexponential one related to the powerlike decay just discussed,

$$1 \ll t \lesssim \frac{\ln(1/g)}{g^2}, \tag{58}$$

in which the unstable wave functions $\psi^{(l)}(x,t;g)$ evolve to a good approximation as a superposition of pole states, i.e., of resonances (see Fig. 2). Relation (58) is a consequence of the first-order results in the next section. It is clear that nonexponential contributions do not have a resonance interpretation: They constitute an intrinsic limit of the scheme.

A. First-order computation $O(g)$

By expanding in powers of g the mixing matrix

$$V(g) = \sum_{k=0}^{\infty} g^k V^{(k)}, \tag{59}$$

one obtains up to first order

$$V^{(0)} = Id, \tag{60}$$

$$V^{(1)} = -\tfrac{1}{2}Id + A, \tag{61}$$

where A is the real antisymmetric matrix with entries

$$A_{l,n} \equiv (-1)^{l+n}\frac{2ln}{l^2 - n^2} \quad \text{for} \quad l \neq n \tag{62}$$

[1] In general, we denote the vectors by uppercase greek letters and their components by the corresponding lowercase letters.

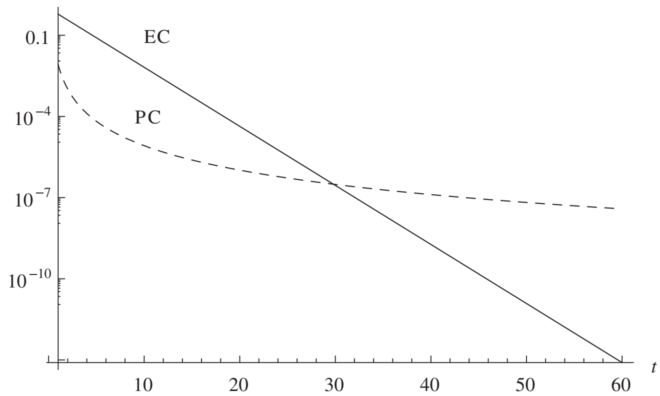

FIG. 2. Time evolution of the modulus square integrated over the cavity of the exponential contribution (EC) (solid line) $\int_0^\pi |\psi_{ex}^{(1)}(x,t;g)|^2 dx$ and power contribution (PC) (dashed line) $\int_0^\pi |\psi_{pw}^{(1)}(x,t;g)|^2 dx$ to the wave function of the fundamental state $\psi^{(1)}(x,t;g)$ for $g = 0.2$ [see Eq. (25)]. Up to $t \approx 30$, the exponential contribution dominates over the power one, which controls the asymptotic behavior. The scale on the vertical axis is logarithmic.

and $A_{l,l} = 0$. The frequencies and widths entering the pole wave functions have the following lowest-order expressions:

$$\omega^{(n)}(g) \equiv [\mathrm{Re}k^{(n)}(g)]^2 - [\mathrm{Im}k^{(n)}(g)]^2 = n^2(1-2g) + O(g^2), \tag{63}$$

$$\Gamma^{(n)}(g) \equiv -4\,\mathrm{Re}k^{(n)}(g)\mathrm{Im}k^{(n)}(g) = 4\pi n^3 g^2 + O(g^3). \tag{64}$$

Since it is convenient to have some freedom in the normalization of the pole states, let us introduce the renormalization constants

$$Z^{(n)}(g) = 1 + \sum_{k=1}^{\infty} g^k z_k^{(n)}, \tag{65}$$

where $z_k^{(n)}$ are (in general complex) coefficients, which depend in general on n: They have to be determined by imposing chosen renormalization conditions. We define the renormalized pole states $\xi^{(n)}(x,t;g)$ by dividing $\theta^{(n)}(x,t;g)$ by $Z^{(n)}(g)$,

$$\xi^{(n)}(x,t;g) \equiv \frac{\theta^{(n)}(x,t;g)}{Z^{(n)}(g)} = \frac{1}{Z^{(n)}(g)}\sqrt{\frac{2}{\pi}}\sin[k^{(n)}(g)x]E^{(n)}(t;g), \tag{66}$$

and define the renormalized mixing matrix entries $U(g)_{l,n}$ by multiplying $V_{l,n}(g)$ by the same factor

$$U(g)_{l,n} = V(g)_{l,n} Z^{(n)}(g) \tag{67}$$

(no sum over n is implied). In order to introduce matrix notation, let us represent the renormalization constants through the diagonal matrix

$$\mathcal{Z}(g) \equiv \mathrm{diag}[Z^{(1)}(g), Z^{(2)}(g), \ldots, Z^{(n)}(g), \ldots]. \tag{68}$$

In components, Eq. (68) reads

$$\mathcal{Z}(g)_{l,n} = \delta_{ln} Z^{(n)}(g), \tag{69}$$

where $\delta_{ln} = 1$ for $l = n$ and zero otherwise is the Kronecker delta. The matrix renormalization constant also possesses a power-series expansion

$$\mathcal{Z}(g) = Id + \sum_{k=1}^{\infty} g^k \mathcal{Z}^{(k)}, \tag{70}$$

where $\mathcal{Z}^{(k)}$ are diagonal matrices. The renormalized mixing matrix then reads

$$U(g) = V(g)\mathcal{Z}(g). \tag{71}$$

In terms of the renormalized pole states, Eq. (51) reads

$$\Psi_{ex}(x,t;g) = U(g)\Xi(x,t;g) = U(g)\mathcal{E}(t;g)\Xi(x,0;g). \tag{72}$$

By multiplying with each other the power series of $V(g)$ and $\mathcal{Z}(g)$, one obtains the power expansion for $U(g)$,

$$U(g) = \sum_{k=0}^{\infty} g^k U^{(k)}, \tag{73}$$

where

$$U^{(n)} = \sum_{k=0}^{n} V^{(k)} \mathcal{Z}^{(n-k)}. \tag{74}$$

Because of the change of wavelength due to the interaction, i.e., to $g \neq 0$, the pole states $\theta^{(n)}(x,t;g)$ are not normalized to one at the initial time $t = 0$, unlike the initial conditions. If we introduce normalized pole states, i.e., satisfying the condition

$$\int_0^\pi |\xi^{(n)}(x,0;g)|^2 dx = 1, \tag{75}$$

we obtain

$$Z^{(n)}(g) = 1 + \frac{g}{2} + O(g^2). \tag{76}$$

Therefore, up to first order,

$$U^{(0)} = Id, \tag{77}$$

$$U^{(1)} = A. \tag{78}$$

It is remarkable that condition (75) has the effect of removing the diagonal contributions from the first-order mixing matrix $U^{(1)}$. Let us also note that the matrix $U(g) = Id + gA + \cdots$ represents an infinitesimal rotation in the infinite-dimensional vector space of the normalized pole states. So the natural question to be treated in the next section is what happens in higher orders in g.

1. Comparison with Winter results

We are in disagreement with [12] regarding the exponential behavior of the excited metastable states, i.e., of $\psi_{ex}^{(l)}(x,t;g)$ with $l > 1$. Let us show that in detail. Equation (72) reads in components

$$\psi_{ex}^{(l)}(x,t;g) = \xi^{(l)}(x,0;g)\exp\left[-i\omega^{(l)}(g)t - \frac{1}{2}\Gamma^{(l)}(g)t\right] + \sum_{n\neq l}^{1,\infty} g\frac{(-1)^{l+n}2ln}{l^2-n^2}\xi^{(n)}(x,0;g) \times \exp\left[-i\omega^{(n)}(g)t - \frac{1}{2}\Gamma^{(n)}(g)t\right]. \quad (79)$$

As usual, l labels the initial state and n the pole state. The diagonal term on the rhs of Eq. (79) (the one with $n = l$) is in agreement with the rhs of Eq. (2a) in [12]. In [12] however, the nondiagonal contributions (the ones with $n \neq l$), entering the sum on the rhs, are not included. These terms have a coefficient suppressed by a power of $g \ll 1$ compared to the diagonal one, but have a slower exponential decay for $n < l$ [$\Gamma^{(n)} \propto n^3$; see Eq. (64)], and therefore dominate in the exponential time region (i.e., before power effects take over). A reasonable approximation is to truncate the sum on the rhs of Eq. (79) to $n < l$, as contributions with $n > l$ are suppressed both by a power of $g \ll 1$ and by the large widths. For example, for $l = 2$ Eq. (79) may be approximated by neglecting higher poles as

$$\psi_{ex}^{(2)}(x,t;g) \simeq \xi^{(2)}(x,0;g)\exp[-i\omega^{(2)}(g)t - 16\pi g^2 t] - \tfrac{4}{3}g\xi^{(1)}(x,0;g)\exp[-i\omega^{(1)}(g)t - 2\pi g^2 t]. \quad (80)$$

As a measure of the size of the above terms, let us take the square of the modulus integrated over the cavity ($0 < x < \pi$)

$$\int_0^{\pi} |\cdots|^2 dx. \quad (81)$$

As shown in Fig. 3, for $g = 0.1$ there is a large temporal region, from $t \simeq 5$ up to $t \simeq 160$, where the nondiagonal contribution from the first pole

$$\frac{16}{9}g^2\int_0^{\pi}|\xi^{(1)}(x,0;g)|^2 dx\,\exp(-4\pi g^2 t) \quad (82)$$

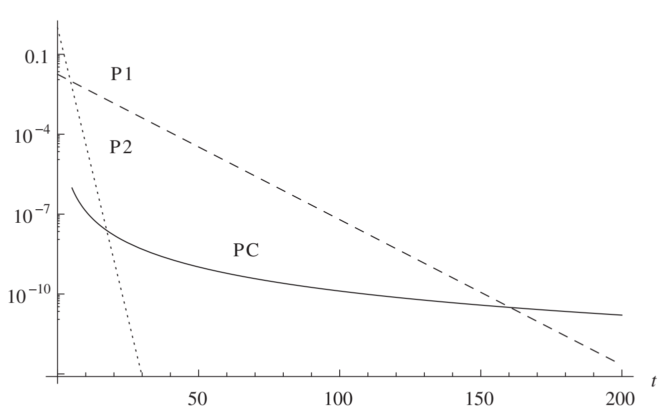

FIG. 3. Time evolution of the contributions to the $l = 2$, i.e., first excited, state for $g = 0.1$. The dotted line denotes the second pole contribution (P2), the dashed line denotes the first pole contribution (P1), and the solid line denotes the power contribution (PC).

dominates over the diagonal one from the second pole

$$\int_0^{\pi}|\xi^{(2)}(x,0;g)|^2 dx\,\exp(-32\pi g^2 t) \quad (83)$$

in the temporal evolution of the first excited state ($l = 2$). For very large times $t \gtrsim 160$, the power contribution dominates over the exponential ones and we enter the (true) asymptotic region. For general g, the nondiagonal $l = 1$ contribution dominates over the diagonal one $l = 2$ for

$$t > t^* \simeq \frac{1}{28\pi g^2}\ln\left(\frac{9}{16g^2}\right). \quad (84)$$

For $t = t^*$ the signal-to-noise ratio (i.e., the modulus squared of the wave function normalized at $t = 0$ integrated over the cavity) is decreased from one down to $\simeq 2(16g^2/9)^{8/7}$ (the factor 2 comes from the two resonances).

To summarize, neglecting the nondiagonal contributions is a reasonable approximation only for the time evolution of the lowest-lying state $\psi^{(1)}(x,t;g)$. The presence of the nondiagonal terms shows that the evolution of general unstable states is far more complicated than implied by the analysis in [12]. The occurrence and the relevance of such off-diagonal terms has been noted in [17]. A physical interpretation of such an effect will be presented in the next section.

2. Physical interpretation of pole state mixing

In order to express the metastable wave functions of the Winter model $\psi^{(l)}(x,t;g)$ in terms of the eigenfunctions of the particle in a box, one has to diagonalize the time evolution. That is achieved by counterrotating the vector containing the initial conditions, i.e., by considering the evolution not of $\Psi(x,0;g)$ but of

$$\Phi(x,0;g) \equiv U^{-1}(g)\Psi(x,0;g). \quad (85)$$

By using the first equality in Eq. (72), it is immediately shown that

$$\Phi(x,t;g) = \Xi(x,t;g) = \mathcal{E}(t;g)\Xi(x,0;g). \quad (86)$$

By looking at the vector equation (85) component by component, the new initial conditions read

$$\phi^{(l)}(x,0;g) = \sqrt{\frac{2}{\pi}}\theta(\pi - x)\sum_{n=1}^{\infty}[U^{-1}(g)]_{ln}\sin(nx), \quad (87)$$

each evolving as a single pole wave function

$$\phi^{(l)}(x,t;g) = \xi^{(l)}(x,t;g). \quad (88)$$

The experimental meaning of Eq. (85) or (87) is clear: In order to observe a diagonal time evolution as in the free case (21), one has to prepare the initial state as the coherent superposition of free eigenfunctions given by Eq. (85) or (87). In the case of excited states $l > 1$, the superposition in Eq. (85) or (87) also has the effect of subtracting the contributions from smaller l, which decay slower in time and therefore tend to dominate the evolution, as discussed in [17]. If the matrix $U(g)^{-1}$ is computed in an approximate way (as a truncated power series in g, for example), there is a small residual contamination in the time evolution of the lth state from the lower ones, which becomes substantial asymptotically in time. In other words, the problem of isolating the lth mode for all times can in principle be solved only with an exact computation of $U(g)$.

Let us now explicitly evaluate the initial wave function that evolves diagonally in t, according to Eq. (87). To order g,

$$U(g)^{-1} = 1 - gA + O(g^2). \tag{89}$$

The sum of the trigonometric series on the rhs of Eq. (87) reads

$$\phi^{(l)}(x,0;g) = \sqrt{\frac{2}{\pi}}\left[\left(1-\frac{g}{2}\right)\sin(lx) - glx\cos(lx)\right] \times \theta(\pi - x) + O(g^2). \tag{90}$$

The rhs of Eq. (90) is the expansion to $O(g)$ of

$$\sqrt{\frac{2}{\pi}}\left(1-\frac{g}{2}\right)\sin[l(1-g)x]\theta(\pi - x). \tag{91}$$

It is tempting to think that higher orders in g actually lead to the result (91): The term $g^2A^2/2$ in the second-order correction $g^2U^{(2)}$ actually confirms this guess. However, let us warn the reader that wave-function renormalization does not account for all the second-order effects (see the next section).

The interpretation of Eq. (91) is straightforward: The counterrotation of the initial wave function in index space amounts to the shift of the wave vector $l \to k^{(l)}(g) = l(1 - g) + \cdots$ in momentum space, with a consequent change of normalization. In other words, in order to have a diagonal evolution in t of the initial wave function, the latter has to be *prepared* with the *corrected* wave vector $k^{(l)}(g)$, which is dynamically generated from $l = k^{(l)}(0)$, the free one. Temporal evolution is then simply given by multiplication by the factor $E^{(l)}(t,g)$ in Eq. (36).

Let us note that the wave function in Eq. (91) has a finite jump $O(g)$ at the right border of the cavity, in $x = \pi$. The Fourier series in Eq. (87) exhibits indeed the Gibbs phenomenon in $x = \pi$, as the coefficients decay asymptotically $\approx 1/n$ for $n \to \infty$.[2] It is remarkable that we obtain a discontinuous initial wave function, while the eigenfunctions only have a discontinuous first derivative [12,13,17]. Let us remark, however, that the results above are obtained by means of power series in g that we have not shown to be convergent and are probably only asymptotic. The Fourier series therefore should be truncated to some finite order in g, regularizing the discontinuity.

B. Second-order computation $O(g^2)$

In this section we push the perturbative expansion for $g \ll 1$ up to $O(g^2)$ included in order to obtain more accurate results and to get some insight into the general structure of the expansion, if any. The exact expression in Eq. (42) for the mixing matrix in terms of the exact solutions $k^{(n)}(g)$ of the equation $b(k,g) = 0$ indeed is not very illuminating. By inserting the small-g expansion for the poles pushed one order further with respect to previous section,

$$k^{(n)}(g) = n - ng + (n - i\pi n^2)g^2 + \left(\tfrac{4}{3}\pi^2 n^3 + 3i\pi n^2 - n\right)g^3 + O(g^4), \tag{92}$$

we obtain for the mixing matrix

$$V^{(0)} = Id, \tag{93}$$

$$V^{(1)} = A - \tfrac{1}{2}Id, \tag{94}$$

$$V^{(2)} = \tfrac{1}{2}A^2 - A + \tfrac{3}{8}Id + i\pi AH - \tfrac{3}{2}i\pi H, \tag{95}$$

where for convenience we have repeated the lowest-order results and H is the real diagonal matrix

$$H \equiv \mathrm{diag}(1,2,3,\ldots,n,\ldots). \tag{96}$$

The coefficients entering the pole wave functions read

$$\omega^{(n)}(g) = n^2(1 - 2g + 3g^2) + O(g^3), \tag{97}$$

$$\Gamma^{(n)}(g) = 4\pi n^3 g^2(1 - 4g) + O(g^4). \tag{98}$$

Equation (95) has been obtained by using the explicit expression

$$V_{ln}^{(2)} = \delta_{ln}\left(\frac{1}{4} - \frac{\pi^2}{6}l^2 - i\pi\frac{3}{2}l\right) + (1-\delta_{ln}) \times \left[\frac{(-1)^{l+n}2ln}{l^2-n^2}(i\pi n - 1) + \frac{(-1)^{l+n+1}2ln(l^2+n^2)}{(l^2-n^2)^2}\right], \tag{99}$$

together with the formula

$$\frac{1}{2}(A^2)_{ln} = (1-\delta_{ln})(-1)^{l+n+1}\frac{2ln(l^2+n^2)}{(l^2-n^2)^2} - \delta_{ln}\left(\frac{\pi^2}{6}l^2 + \frac{1}{8}\right). \tag{100}$$

Note that the squared matrix is symmetric, as it should, being the square of an antisymmetric matrix. Equation (100) has been derived by means of the identities[3]

$$\sum_{k\neq m}^{1,\infty}\frac{1}{k^2-m^2} = \frac{3}{4m^2}, \quad \sum_{k\neq m}^{1,\infty}\frac{k^2}{(k^2-m^2)^2} = \frac{\pi^2}{12} + \frac{1}{16m^2}, \tag{101}$$

which hold for m a positive integer. Let us remark that it is not trivial that the matrix A^2 does exist, as its entries involve the summation of infinite series, which in effect turn out to be (absolutely) convergent. By looking at the asymptotic form of the coefficients of A and A^2 given above, it is straightforward to show that A^3 and A^4 also exist. We expect that all the positive powers of A do exist.

Let us now discuss renormalization at second order. The explicit expansion of $U(g)$ up to $O(g^2)$ reads

$$U^{(0)} = Id, \tag{102}$$

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[2] The related vertical slope in $x = \pi$ can be derived by differentiating Eq. (87) with respect to x and then setting $x = \pi$.

[3] These identities are obtained from $\sum_{k\neq m}^{-\infty,+\infty} 1/(k-m) = 0$ and $\sum_{k\neq m}^{-\infty,+\infty} 1/(k-m)^2 = \pi^2/3$, respectively, by splitting the sums into positive and negative indices and rearranging them in order to have a single sum (the first identity can also be found in [28]).

$$U^{(1)} = V^{(1)} + \mathcal{Z}^{(1)} = A - \tfrac{1}{2} Id + \mathcal{Z}^{(1)}, \tag{103}$$

$$U^{(2)} = V^{(2)} + V^{(1)}\mathcal{Z}^{(1)} + \mathcal{Z}^{(2)} = \tfrac{1}{2}A^2 - A + \tfrac{3}{8} Id + i\pi AH - \tfrac{3}{2} i\pi H + \left(A - \tfrac{1}{2} Id\right)\mathcal{Z}^{(1)} + \mathcal{Z}^{(2)}. \tag{104}$$

As shown in previous section, if we set

$$\mathcal{Z}^{(1)} = \tfrac{1}{2} Id, \tag{105}$$

we obtain at first order

$$U^{(1)} = A, \tag{106}$$

so that at second order we get

$$U^{(2)} = \tfrac{1}{2}A^2 - \tfrac{1}{2}A + \tfrac{1}{8} Id - \tfrac{3}{2} i\pi H + i\pi AH + \mathcal{Z}^{(2)}. \tag{107}$$

We may set, for example,

$$\mathcal{Z}^{(2)} = -\tfrac{1}{8} Id + \tfrac{3}{2} i\pi H, \tag{108}$$

to give

$$U^{(2)} = \tfrac{1}{2}A^2 - \tfrac{1}{2}A + i\pi AH. \tag{109}$$

The renormalized mixing matrix then finally reads

$$U(g) = Id + gA + \tfrac{1}{2}g^2A^2 - \tfrac{1}{2}g^2A + i\pi g^2 AH + O(g^3). \tag{110}$$

Let us notice that, by introducing the renormalized coupling

$$g_r \equiv g - \tfrac{1}{2}g^2 + O(g^3), \tag{111}$$

the formula above can be simplified a bit:

$$U(g_r) = Id + g_r A + \tfrac{1}{2}g_r^2 A^2 + i\pi g_r^2 AH + O\left(g_r^3\right). \tag{112}$$

The inverse matrix is given up to second order in g_r by

$$U^{-1}(g_r) = Id - g_r A + \tfrac{1}{2}g_r^2 A^2 - i\pi g_r^2 AH + O\left(g_r^3\right). \tag{113}$$

As anticipated in previous section, while the term $1/2 g_r^2 A^2$ can be absorbed in the box eigenfunctions by means of wave-vector renormalization, that is not true for the term $-i\pi g_r^2 AH$. The latter has a large size

$$(AH)_{l,n} = \begin{cases} \frac{(-1)^{l+n} 2ln^2}{l^2-n^2} & \text{for} \quad l \neq n \\ 0 & \text{otherwise} \end{cases} \tag{114}$$

and produces also a highly singular behavior in the counter-rotated box eigenfunctions considered in the previous section, $U(g)^{-1}\Psi(x,0;g)$, because

$$(AH)_{l,n} = O(1) \quad \text{for} \quad n \to \infty \; (l \text{ fixed}). \tag{115}$$

The detailed investigation of such effects, for which we are not able to provide at present a physical interpretation, requires the study of the convergence properties of the series in g involved, which is beyond the scope of the present paper.

1. Exponentiation

The first three terms on the rhs of Eq. (112) are actually the expansion of

$$\exp(g_r A) = Id + g_r A + \tfrac{1}{2}g_r^2 A^2 + O\left(g_r^3\right), \tag{116}$$

so it is not difficult to conjecture that higher orders in g_r will lead to the exponential above. The problem is that we are not sure that the conjectured exponentiation is legitimate, i.e., that it includes *all* the *leading* terms order by order in g. To $O(g^2)$ we encountered indeed the big term AH, which will presumably produce iterates of similar size in higher orders, which we are unable to control. Furthermore, since $\mathcal{Z}^{(2)}$ is a diagonal matrix, whatever value is chosen for it, we cannot cancel the term AH in $U^{(2)}$ with an *ad hoc* renormalization condition. A third-order computation in g, which is in principle straightforward while technically cumbersome, could probably reveal further structure of the perturbative expansion.

V. DISCUSSION

Let us now discuss the renormalized wave functions $\phi^{(l)}(x,t;g)$. The main qualitative difference between the free case and the interacting one is that in the latter case there are nonzero widths. The appearance of an imaginary part in the *ab initio* real energy is a second-order effect in g.[4] Once a nonzero width is allowed, the key point is that the $\phi^{(l)}(x,t;g)$ have a form similar to the eigenfunctions of the free system $\psi_0^{(l)}(x,t)$ in Eq. (21). The differences between the free case and the interacting one, as long as $0 < g \ll 1$, can be relegated to small modifications of the parameters entering the free wave functions $\psi_0^{(l)}(x,t)$. In other words, switching on the interaction, i.e., going from $g = 0$ to $0 < g \ll 1$, produces finite renormalizations only. Let us discuss these renormalizations in turn.

(i) The normalization coefficient $Z^{(l)}(g)$ has a modulus greater than one for $0 < g \ll 1$ and reduces to one in the free case $g = 0$; it has a first-order correction in g and is the analog of the field renormalization constant Z in quantum-field theory [17]. Unlike the most common cases (QED, for example), $Z^{(l)}(g)$ is not real because the one-particle states are unstable.

(ii) The wave vector $k^{(n)}(g)$ is renormalized to first order in g by the interaction and reduces to the free case for $g \to 0$: $k^{(l)}(0) = l$. It acquires an imaginary part at second order in g, related to the decay width. That implies the disappearance of the node of the wave function around $x = \pi$ and a (small) exponential growth of $\phi^{(l)}(x,t;g)$ by going from the impermeable wall in $x = 0$ toward the permeable one in $x = \pi$.

(iii) The real part of the energy $\omega^{(l)}(g)$ is also renormalized to first order in g by the interaction and reduces to the free case for $g \to 0$: $\omega^{(l)}(0) = l^2$. Note that the free dispersion relation $\omega = k^2$ is not renormalized at first order.

Let us make a few remarks.

We do not expand in powers of g the wave functions $\phi^{(l)}(x,t;g)$, but only the parameters $k^{(n)}(g)$, $\omega^{(n)}(g)$, etc., entering them through the functions appearing in $\psi_0^{(l)}(x,t)$. That implies that we are resumming classes of higher-order

[4]Nonzero widths are clearly not in contradiction with the unitarity of the fundamental theory because we are looking at a subsystem, i.e., an open system [4,14].

corrections in g in the wave function, in the spirit of renormalization in quantum-field theory [19–22] and statistical mechanics [29] or the method of multiple scales in classical physics [24,25].

The decay widths grow faster with increasing n than the frequencies

$$\Gamma^{(n)}(g) \propto n^3, \quad \omega^{(n)}(g) \propto n^2. \tag{117}$$

Since our renormalized theory has meaning only for

$$\Gamma^{(n)}(g) \ll \omega^{(n)}(g), \tag{118}$$

we cannot take n too large. Therefore, while in principle the state vectors and the evolution or mixing matrices are infinite, in practice, for any fixed g one has to make a truncation in n according to the condition (118) (see [18]). This limitation is also reasonable from a physics viewpoint: High-energy particles pass through the barrier in $x = \pi$ without difficulty and therefore there is no sense in including them in the description of the dynamics inside the cavity. By restricting on n one is also cutting off small wavelengths $\lambda \lesssim 2\pi/n$ and therefore is limiting space resolution.

VI. CONCLUSION

In this work we have shown that the evolution according to the Winter model of the eigenfunction of a particle inside an impermeable cavity (i.e., a box) with *any* quantum number $l = 1,2,3,\ldots$ is not controlled asymptotically by the corresponding lth resonance, as intuitively expected and as stated in [12], but *always* by the *first* resonance $l = 1$. This phenomenon originates from $O(g)$ coupling terms between the eigenfunctions of the particle in a box (box eigenfunctions hereafter) and the resonances, which we have evaluated with next-to-leading-order accuracy, i.e., up to second order in the coupling $g \ll 1$. Because of this mixing, metastable dynamics is far more complicated than implied by the Winter results and common arguments. Roughly speaking, the idea suggested by our results is that the box eigenfunctions are not natural initial states as far as decay is concerned: Natural initial states are dynamically generated from the box eigenfunctions. With the exception of the fundamental state, such box eigenfunctions do not even approximately evolve as single elementary excitations, as claimed by Winter, but as coherent superpositions of many elementary excitations. Furthermore, mixing effects are quite large, as mixing matrix entries decay rather slowly as we move away from the main diagonal [as $1/n$ to $O(g)$, where n is the distance from the diagonal; see Eqs.(52) and (89)]. The physical picture is that time evolution produces, in addition to the expected decay of box eigenfunctions, also a finite rotation of them, represented by the infinite matrix $U(g)$. Therefore, in order to have a natural initial state evolving simply, i.e., diagonally, with time, one has to counterrotate the box eigenfunctions by means of the matrix $U^{-1}(g)$. One of the main dynamical effects of such a counterrotation is to adjust the wavelengths of the box eigenfunctions to those of the elementary excitations of the model. In physical terms, the time evolution of box eigenfunctions produces a rearrangement of their wavelengths to the characteristic wavelengths of the system. Let us stress, however, that wavelength renormalization does not exhaust the effects contained in the matrix $U^{-1}(g)$ (a complete physical interpretation is still missing). Since the Winter model is a limiting case of many different models, it is clear that the occurrence of nondiagonal terms has to be a general phenomenon in metastable systems. The implications of the mixing terms for more complex models are still to be investigated and could modify their current understanding. Let us stress that it would have been quite difficult to imagine the existence of such mixing terms in more phenomenologically relevant models, but also much more complicated, without our exhaustive analytic study of the Winter model. We have also shown that the resonant states of the Winter model can be related to the box eigenfunctions by means of renormalization of the parameters entering the free eigenfunctions, after allowing for nonzero widths.

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