Relativistic recoil effects for energy levels in a muonic atom within a Grotch-type approach. II. An application to the one-loop electronic vacuum polarization

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We continue our account of relativistic recoil effects in muonic atoms and present explicitly analytic results at first order in electron-vacuum-polarization effects. The results are obtained within a Grotch-type approach based on an effective Dirac equation. Some expressions are cumbersome and we investigate their asymptotic behavior. Previously, relativistic two-body effects due to the one-loop electron vacuum polarization were studied by several groups. Our results found here are consistent with the previous result derived within a Breit-type approach (including ours) and disagree with a recent attempt to apply a Grotch-type approach.

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I. INTRODUCTION

An analytic calculation of relativistic recoil effects in a hydrogenlike atom to order $(Z\alpha)^4 m^2/M$ is possible through the equation [1]

$$E = m + m_R[f_C(Z\alpha) - 1] - \frac{m_R^2}{2M}[f_C(Z\alpha) - 1]^2, \qquad (1)$$

where $f_C(Z\alpha)$ is the dimensionless energy of a Dirac-Coulomb equation, which is indeed well known (see, e.g., [2]). The corrections are of order $O((Z\alpha)^4(m/M)^2 m)$ and $O((Z\alpha)^5(m/M)m)$. The terms in $(Z\alpha)^5$ are due to effects of multiphoton exchange. It is even possible to provide a complete calculation of the m/M recoil effects for pure Coulomb two-body systems by taking into account multiphoton exchange contributions exactly in $(Z\alpha)$ [3,4].

As was shown in our previous paper [5], one can consider a more general problem and the result for the energy takes the form

$$\begin{aligned} E = m + m_R[f_{CN}(Z\alpha,\kappa) - 1] - \frac{m_R^2}{2M}[f_{CN}(Z\alpha,\kappa) - 1]^2 \\ - \frac{m_R^2}{2M}\frac{\partial}{\partial \ln\kappa}[f_{CN}(Z\alpha,\kappa) - 1]^2 \\ - \langle\psi|\left(\frac{V^2}{2M} + \frac{1}{4M}[V,[\mathbf{p}^2,W]]\right)|\psi\rangle, \end{aligned} \qquad (2)$$

where $\kappa = Z\alpha m_R/\mu$, the potential is

$$V = V_C + V_N,$$

where in a certain sense $V_N \sim \varepsilon V_C$, $\varepsilon \ll 1$. Here, W is a specific auxiliary potential, ψ is the wave function of the Dirac problem with the reduced mass, and $f_{CN}(Z\alpha,\kappa)$ is the dimensionless energy for the potential $V_C + V_N$. The momentum scale (i.e., the characteristic inverse radius) of V_N is μ^{-1}. For the case of the Uehling potential, the scale parameter is defined as $\mu = m_e$.

In this paper, we study a correction to the energy in the first order of V_N, so we can write

$$f_{CN}(Z\alpha,\kappa) = f_C(Z\alpha) + f_N(Z\alpha,\kappa),$$

where $f_N(Z\alpha,\kappa)$ is the corresponding dimensionless correction. Since we are interested only in terms of order $\varepsilon(Z\alpha)^4 m^2/M$, we can further simplify this expression as

$$\begin{aligned} E &= m + m_R[f_{CN}(Z\alpha,\kappa) - 1] + \Delta E, \\ \Delta E &= -\frac{m_R^2}{2M}[f_{CN}(Z\alpha,\kappa) - 1]^2 - \frac{m_R^2}{2M}\frac{\partial}{\partial \ln\kappa}\frac{[E^{(\mathrm{NR})}(\kappa)]^2}{m_R^2} \\ &\quad - \langle\psi_{\mathrm{NR}}|\left(\frac{V^2}{2M} + \frac{1}{4M}[V,[\mathbf{p}^2,W]]\right)|\psi_{\mathrm{NR}}\rangle, \end{aligned} \qquad (3)$$

where it is sufficient to apply the nonrelativistic approximation to the energy in the term with derivative

$$[f_{CN}(Z\alpha,\kappa) - 1]|_{\mathrm{nonrel}} = \frac{E^{(\mathrm{NR})}(\kappa)}{m_R}, \qquad (4)$$

as well as to the wave function in the last term.

It is remarkable that in certain respects the relativistic recoil correction beyond the Dirac equation with the reduced mass ΔE is simpler than the solution of the Dirac equation. To order $\varepsilon(Z\alpha)^4 m^2/M$, it requires only nonrelativistic evaluation. In particular, the leading recoil correction, being expressed in pure nonrelativistic terms, does not depend on the total angular momentum j, but only on the angular momentum l. That means that this correction may contribute to the Lamb splitting (a difference between states with the same j, but different l, such as the $2p_{1/2} - 2s_{1/2}$ difference), but not to the fine-structure interval (a difference between states with the same l, but different j, such as the $2p_{3/2} - 2p_{1/2}$ difference).

To validate applicability of this expression for the electron-vacuum-polarization (eVP) effects, we should prove that the relativistic recoil effects can be reduced to the evaluation of

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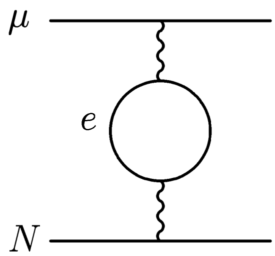

FIG. 1. One-photon-exchange diagram for the eVP contributions. It is responsible for the the Uehling-potential corrections to orders $\alpha(Z\alpha)^2 m$ and $\alpha(Z\alpha)^4 m$.

the one-photon exchange (see Fig. 1) and present explicit expressions for related contributions to V and W.

Apparently, the correction to the potential is the Uehling potential, which can be presented, e.g., in the form [6]

$$V_U(r) = -\frac{\alpha(Z\alpha)}{\pi}\int_0^1 dv\, \rho_e(v)\frac{e^{-\lambda r}}{r}, \tag{5}$$

where

$$\lambda = \frac{2m_e}{\sqrt{1-v^2}},\qquad \rho_e(v) = \frac{v^2(1-v^2/3)}{1-v^2}. \tag{6}$$

The factor α/π plays a role of the parameter ε in our general consideration [5] and $\mu = m_e$. Meanwhile, a construction of W, which is to be preceded by a choice of an appropriate gauge is not trivial (see the discussion in Ref. [7]).

In principle, the correction ΔE can be treated relativistically without any nonrelativistic reduction of the energy and the wave function. However, the higher-order effects which are incorporated in this case are smaller than possible effects of two-photon corrections. In particular, for the eVP contributions, the higher-order relativistic recoil contributions to ΔE are of order $\alpha(Z\alpha)^6 m^2/M$, while the two-photon-exchange diagrams contribute to order $\alpha(Z\alpha)^5 m^2/M$.

In the following sections, we briefly reproduce this discussion and present appropriate results for a relativistic and nonrelativistic correction to the energy due to a Dirac equation with a potential which accounts for the eVP effects. Using them, we present an analytic expression for eVP relativistic recoil corrections in the general case as well as for most interesting particular cases, such as circular and low-lying states. For both kinds of states, we also derive their large-kappa asymptotics. In conclusion, we discuss a comparison with an alternative technique for relativistic recoil corrections based on the Breit-type equation.

II. GROTCH-TYPE EXPRESSION FOR THE EVP CORRECTIONS IN FIRST ORDER IN α

A choice of gauge for the photon propagator is crucial for the explicit presentation of the one-photon contribution and for the value of the two-photon contribution. We have already discussed that in part in Ref. [5] and in detail in Ref. [7].

Indeed, due to the gauge invariance of quantum electrodynamics, any final complete result for any physical calculation does not depend on the choice of the gauge. However, the technical origin of different contributions to such a final result may be different in different gauges. In particular, the physical result for a relativistic recoil correction to order $\alpha(Z\alpha)^4 m^2/M$ does not necessarily come only from the static part of the one-photon-exchange term.

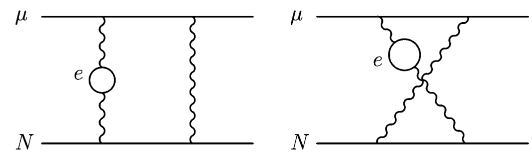

FIG. 2. Two-photon-exchange diagrams for the eVP contribution. Subtraction terms and reducible contributions are omitted. In certain gauges, the two-photon-exchange effects contribute to order $\alpha(Z\alpha)^4 m^2/M$.

Following [7], we use the Coulomb gauge for the free-photon propagator, while the eVP correction to the propagator takes the form

$$\begin{aligned} D^e_{00} &= -\frac{\alpha}{\pi}\int_0^1 dv\,\rho_e \frac{1}{(\mathbf{k}^2+\lambda^2)},\\ D^e_{i0} &= 0,\\ D^e_{ij} &= -\frac{\alpha}{\pi}\int_0^1 dv\,\rho_e \frac{1}{(k^2-\lambda^2)}\left(\delta_{ij} - \frac{k_i k_j}{(\mathbf{k}^2+\lambda^2)}\right). \end{aligned} \tag{7}$$

We note that similarly to the Coulomb gauge, the D_{00} component of the photon propagator does not depend on the energy transfer and $D_{i0} = 0$. This choice is sufficient for vanishing $\alpha(Z\alpha)^4 m^2/M$ contributions from two-photon exchanges (see Fig. 2) and thus the problem of calculations of relativistic recoil effects at this order is reduced to consideration of the one-photon-exchange diagrams (see Fig. 1).

The Grotch-type calculations of the eVP contribution of order $\alpha(Z\alpha)^4 m^3/M^2$ were considered some time ago Refs. [8–10] (for earlier evaluations, see [11,12]). However, the gauge was not appropriate and two-photon-exchange corrections should be added. Those corrections were missed in Refs. [8–10], which produces a discrepancy between the Breit-type calculation [13] and the Grotch-type ones. The situation was clarified in Ref. [7].

Once an appropriate gauge is chosen, we can restrict our consideration to the static part of the one-photon-exchange contribution, including the vacuum polarization. Treating the nucleus nonrelativistically, we arrive at the same equation as for the free one-photon exchange

$$\left(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m + \frac{\mathbf{p}^2}{2M} + V + \frac{1}{2M}\{\boldsymbol{\alpha}\cdot\mathbf{p}, V\} + \frac{1}{4M}[\boldsymbol{\alpha}\cdot\mathbf{p},[\mathbf{p}^2,W]]\right)\psi(r) = E\psi(r), \tag{8}$$

where, however, the effective potentials W and V include eVP effects and, in particular,

$$\begin{aligned} V(r) &= V_C(r) + V_U(r),\\ V_U(\mathbf{k}) &= -4\alpha(Z\alpha)\int_0^1 dv\,\frac{\rho_e(v)}{\mathbf{k}^2+\lambda^2}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} W &= W_C + W_U, \\ W_U(\mathbf{k}) &= -2\frac{V_U(\mathbf{k})}{\mathbf{k}^2+\lambda^2} \\ &= 8\alpha(Z\alpha)\int_0^1 dv\,\frac{\rho_e(v)}{(\mathbf{k}^2+\lambda^2)^2}, \\ W_U(r) &= \frac{\alpha(Z\alpha)}{\pi}\int_0^1 dv\,\rho_e(v)\frac{e^{-\lambda r}}{\lambda}, \end{aligned} \tag{10}$$

and, as given in Refs. [1,5],

$$\begin{aligned} W_C(r) &= -Z\alpha r, \\ W_C(\mathbf{k}) &= \frac{8\pi Z\alpha}{\mathbf{k}^4}. \end{aligned} \tag{11}$$

With an expression for W in hand, we can rewrite the addition to the Hamiltonian, which in the leading order of α, takes the form

$$\begin{aligned} \delta H = &-\underbrace{\left(\frac{V_C^2}{2M}+\frac{1}{4M}[V_C,[\mathbf{p}^2,W_C]]\right)}_{=0} \\ &-\frac{V_U V_C}{M}-\frac{1}{4M}[V_C,[\mathbf{p}^2,W_U]] \\ &-\frac{1}{4M}[V_U,[\mathbf{p}^2,W_C]]. \end{aligned} \tag{12}$$

The expression turns out to be equal to zero for the pure Coulomb case.

For the case of $V_N = V_U$ and $f_N = f_U$, Eq. (3) leads to the expression for the following correction of the first order in α:

$$\begin{aligned} E_U = &\, m_R f_U(Z\alpha,\kappa) - \frac{m_R^2}{M}[f_C(Z\alpha)-1]f_U(Z\alpha,\kappa) \\ &- \frac{m_R^2}{M}[f_C(Z\alpha)-1]\frac{\partial}{\partial\ln\kappa}f_U(Z\alpha,\kappa) \\ &- \langle\psi|\bigg[\frac{V_U V_C}{M}+\frac{1}{4M}[V_C,[\mathbf{p}^2,W_U]] \\ &+ \frac{1}{4M}[V_U,[\mathbf{p}^2,W_C]]\bigg]|\psi\rangle. \end{aligned} \tag{13}$$

The expression includes a nonrelativistic term of order $\alpha(Z\alpha)^2m$ (exact in m/M), a pure relativistic one $\alpha(Z\alpha)^4m$ and a relativistic recoil correction to order $\alpha(Z\alpha)^4m^2/M$. It also contains a higher-order $\alpha(Z\alpha)^6m$ nonrecoil term which may be numerically comparable to $\alpha(Z\alpha)^4m^2/M$ in a certain range of Z.

This expression may be approached analytically or by numerical means. Following, we express it in terms of certain base integrals which we evaluate in closed analytic form following [14,16–21]. The required expressions for the dimensionless $f_C(Z\alpha)$ energy and Dirac-Coulomb wave functions ψ are summarized in the Appendix of our previous paper [5].

To obtain final results, we have to find some derivatives and it is important to have an expression for $f_U(Z\alpha,\kappa)$ (the dimensionless Uehling corrections to the energy levels of the Dirac-Coulomb equation with the reduced mass) in a form suitable for differentiation and it is also available. Not only analytic expressions for relativistic [14,16] (see Sec. III) and nonrelativistic [16–19] (see Sec. III) corrections are known, but also their various asymptotics [14,16,18–21].

For the analytical differentiation, one can take into account that

$$\frac{\partial}{\partial\ln\kappa}f(\kappa/n) = \frac{\partial}{\partial\ln\kappa_n}f(\kappa_n),$$

where $\kappa_n = \kappa/n$ is a combination which naturally appears in various analytic expressions (see Sec. III). For numerical evaluations of the derivative, a more useful relation is

$$\frac{\partial}{\partial\ln\kappa}f(\kappa) = -\frac{\partial}{\partial\ln m_e}f(\kappa).$$

III. UEHLING-POTENTIAL CORRECTION TO THE ENERGY LEVELS OF THE DIRAC-COULOMB EQUATION

The Uehling correction to the energy of the Dirac-Coulomb+Uehling problem was addressed in Ref. [14] for circular states and later was generalized for an arbitrary state [16]. The result reads as

$$\begin{aligned} f_U(Z\alpha,\kappa) &= -\frac{\alpha(Z\alpha)}{\pi m_R}\int_0^1 dv\,\rho_e(v)\langle\psi|\frac{e^{-\lambda r}}{r}|\psi\rangle \\ &= \frac{\alpha(Z\alpha)^2}{\pi n^2}F_{nl_j}(\kappa_n'), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \kappa_n' &= \frac{\eta m_R}{m_e}, \\ \eta &= \frac{Z\alpha}{\sqrt{(n_r+\zeta)^2+(Z\alpha)^2}}, \\ \zeta &= \sqrt{\nu^2-(Z\alpha)^2}, \\ \nu &= (-1)^{j+l+1/2}(j+1/2), \\ n_r &= n-|\nu|. \end{aligned} \tag{15}$$

The parameter κ_n' is different from $\kappa_n = \kappa/n = Z\alpha m/nm_e$. However, in a nonrelativistic approximation

$$\begin{aligned} \kappa_n' &= \frac{Z\alpha m_R}{m_e\sqrt{(n_r+\zeta)^2+(Z\alpha)^2}} \\ &\simeq \frac{Z\alpha m_R}{m_e\sqrt{(n_r+|\nu|)^2+(Z\alpha)^2}} \\ &= \frac{Z\alpha m_R}{m_e n} = \kappa_n. \end{aligned} \tag{16}$$

The function $F_{nl_j}(\kappa)$ can be expressed either in terms of a one-dimensional integral over elementary functions or in terms of a hypergeometric function ${}_3F_2$ [14,16]. Indeed, the correction f_U can be computed numerically for any desired state. However, because of a required expansion and various further considerations, such as examination of the asymptotic behavior, we prefer here analytic or semianalytic results.

In particular, as it was found in Ref. [16]

$$F_{nl_j}(\kappa) = -\sum_{i,k=0}^{n_r} B_{ik}\, K_{2,2\zeta+i+k}(\kappa), \tag{17}$$

where

$$B_{ik} = \left(\frac{\eta n}{Z\alpha}\right)^2 \frac{(-1)^{i+k}(n_r)!}{i!(n_r-i)!k!(n_r-k)!} \frac{\Gamma(2\zeta+n_r+1)\Gamma(2\zeta+i+k)}{\Gamma(2\zeta+i+1)\Gamma(2\zeta+k+1)} \frac{1}{\frac{Z\alpha}{\eta}-\nu} \times\left\{\left[\left(\frac{Z\alpha}{\eta}-\nu\right)^2+(n_r-i)(n_r-k)\right]-\frac{E_C(nl_j)}{m}\left(\frac{Z\alpha}{\eta}-\nu\right)(2n_r-i-k)\right\}. \tag{18}$$

The base integrals, defined as [14,16]

$$K_{abc}(\kappa) = \int_0^1 dv\, \frac{v^{2a}}{(1-v^2)^{b/2}} \left(\frac{\kappa\sqrt{1-v^2}}{1+\kappa\sqrt{1-v^2}}\right)^c = \int_0^1 y^{1-b}(1-y^2)^{a-1/2}\left(\frac{\kappa y}{1+\kappa y}\right)^c,$$
$$K_{bc}(\kappa) = K_{1bc}(\kappa) - \frac{1}{3}K_{2bc}(\kappa). \tag{19}$$

It is easy to obtain for the first derivative of K

$$\frac{\partial K_{bc}}{\partial \kappa} = \frac{c}{\kappa^2}\, K_{b+1,c+1}(\kappa). \tag{20}$$

The integrals K can be also expressed in a closed form [14,16]

$$K_{abc}(\kappa) = \frac{\kappa^c}{2}\, B\left(a+\frac{1}{2}, 1-\frac{b}{2}+\frac{c}{2}\right) {}_3F_2\left(\frac{c}{2}, \frac{c}{2}+\frac{1}{2}, 1-\frac{b}{2}+\frac{c}{2};\ \frac{1}{2}, a+\frac{3}{2}-\frac{b}{2}+\frac{c}{2};\ \kappa^2\right) - \frac{c\,\kappa^{c+1}}{2}\, B\left(a+\frac{1}{2}, \frac{3}{2}-\frac{b}{2}+\frac{c}{2}\right) {}_3F_2\left(\frac{c}{2}+1, \frac{c}{2}+\frac{1}{2}, \frac{3}{2}-\frac{b}{2}+\frac{c}{2};\ \frac{3}{2}, a+2-\frac{b}{2}+\frac{c}{2};\ \kappa^2\right), \tag{21}$$

where $B(\alpha,\beta)$ is the beta function and ${}_3F_2(\alpha,\beta,\gamma;\ \delta,\epsilon;\ z)$ stands for the generalized hypergeometric function (see, e.g., [22]).

The solution above is a solution of the Dirac equation for a particle with mass m. However, as we see from Eq. (3), the two-body energy is the easiest to express in terms of a Dirac equation with the reduced mass, introducing corrections.

In muonic hydrogen for $n=2$, the argument of the hypergeometric function κ_2' is less than unity ($\simeq$0.7), and the hypergeometric series converges well. For $n=1$ in muonic hydrogen or $n=2$ in muonic helium, one has to use analytic continuation of the hypergeometric series or integral representation of the hypergeometric function.

To calculate the term with derivative and the term with δH we need efficient nonrelativistic expressions. The Uehling correction in the nonrelativistic limit is

$$f_U^{(\mathrm{NR})}(Z\alpha,\kappa) = \frac{\alpha(Z\alpha)^2}{\pi n^2} F_{nl}^{(\mathrm{NR})}(\kappa_n), \tag{22}$$

where

$$F_{nl_j}^{(\mathrm{NR})}(\kappa_n) = -\sum_{i,k=0}^{n-l-1} B_{ik}^{(\mathrm{NR})}\, K_{2,2l+i+k+2}(\kappa_n), \tag{23}$$

$$B_{ik}^{(\mathrm{NR})} = \frac{(-1)^{i+k}(n-l-1)!}{i!(n-l-i-1)!k!(n-l-k-1)!} \frac{(n+l)!(2l+i+k+1)!}{(2l+i+1)!(2l+k+1)!} \tag{24}$$

can be expressed in terms of elementary functions. Alternative expressions for the nonrelativistic correction can be found in Refs. [16,19].

In a particular case of the ground state, the result has a simple form [17]

$$F_{10}^{(\mathrm{NR})}(\kappa) = -\frac{1}{3}\left\{-\frac{4+\kappa^2-2\kappa^4}{\kappa^3}A(\kappa) + \frac{4+3\kappa^2}{\kappa^3}\frac{\pi}{2} - \frac{12+11\kappa^2}{3\kappa^2}\right\}, \tag{25}$$

where

$$A(\kappa) = \frac{\arccos(\kappa)}{\sqrt{1-\kappa^2}} = \frac{\ln(\kappa+\sqrt{\kappa^2-1})}{\sqrt{\kappa^2-1}}.$$

The nonrelativistic kernels K_{bc} have only integer subscripts and that allows useful recurrence relations (see [14,18]). Applying them, we arrive at [18]

$$F_{nl}^{(\mathrm{NR})}(\kappa_n) = \frac{(n+l)!}{(n-l-1)!(2n-1)!} \sum_{i=0}^{n-l-1} \frac{1}{(2l+i+1)!} \frac{1}{i!} \left(\frac{(n-l-1)!}{(n-l-i-1)!}\right)^2 \left(\frac{1}{\kappa_n}\right)^{2(n-l-1-i)} \left(\kappa_n^2 \frac{\partial}{\partial \kappa_n}\right)^{2(n-l-i-1)} \times \kappa_n^{2(l+i+1)} \left(\frac{\partial}{\partial \kappa_n}\right)^{2(l+i)} \frac{F_{10}^{(\mathrm{NR})}(\kappa_n)}{\kappa_n^2}. \tag{26}$$

IV. ANALYTIC RESULT FOR THE RELATIVISTIC RECOIL UEHLING CORRECTION

Using the expression for the relativistic Uehling energy equation (14) in terms of base integrals K_{bc} and their various properties [14,16,18–21], for the third term of Eq. (13) we arrive at

$$\frac{\alpha(Z\alpha)^2}{\pi n^2} \frac{m_R^2}{M} (f_C - 1) \times \sum_{i,k=0}^{n_r} B_{ik} \frac{2\zeta + i + k}{\kappa_n'} K_{2,2\zeta+i+k+1}(\kappa_n'). \tag{27}$$

The last term of Eq. (13) corresponds to a matrix element of the additional Hamiltonian (12). For the relativistic wave functions, it can be rewritten in the form

$$-\langle\psi|\left(\frac{V_U V_C}{M} + \frac{1}{4M}[V_C,[\mathbf{p}^2,W_U]] + \frac{1}{4M}[V_U,[\mathbf{p}^2,W_C]]\right)|\psi\rangle = \frac{\alpha(Z\alpha)^2}{2M\pi} \int_0^1 dv\, \rho_e(v) \langle\psi|\frac{\lambda e^{-\lambda r}}{r}|\psi\rangle, \tag{28}$$

which has the same structure of integration over r as f_U (cf. Sec. III) and one can readily obtain for it an expression which differs from one for the Uehling correction (17) only by a factor and the second indices of K_{bc}, arriving at

$$\frac{\alpha(Z\alpha)^4}{\pi} \frac{m_R^2}{M} \frac{\eta}{Z\alpha n^2} \sum_{i,k=0}^{n_r} B_{ik} \frac{K_{3,2\zeta+i+k}(\kappa_n')}{\kappa_n'} \tag{29}$$

or, for the nonrelativistic case,

$$\frac{\alpha(Z\alpha)^4}{\pi} \frac{m_R^2}{M} \frac{1}{n^3} \sum_{i,k=0}^{n_r} B_{ik}^{(\mathrm{NR})} \frac{K_{3,2l+i+k+2}(\kappa_n)}{\kappa_n}. \tag{30}$$

Combining the different parts of the expression (13), we obtain for the correction to the first order of α

$$E_U = m_R f_U(Z\alpha,\kappa) + \Delta E_U,$$

where its relativistic recoil part is

$$\Delta E_U = \frac{\alpha(Z\alpha)^2}{\pi n^2} \frac{m_R^2}{M} \sum_{i,k=0}^{n_r} B_{ik} \left[(f_C - 1) \left(K_{2,2\zeta+i+k}(\kappa_n') + \frac{2\zeta+i+k}{\kappa_n'} K_{2,2\zeta+i+k+1}(\kappa_n') \right) + \frac{Z\alpha\eta}{\kappa_n'} K_{3,2\zeta+i+k}(\kappa_n') \right]. \tag{31}$$

Neglecting higher-order relativistic corrections, we find in order $\alpha(Z\alpha)^4 m^2/M$

$$\Delta E_U^{(\mathrm{NR})} = \frac{\alpha(Z\alpha)^4}{\pi n^3} \frac{m_R^2}{M} \sum_{i,k=0}^{n-l-1} B_{ik}^{(\mathrm{NR})} \left[-\frac{1}{2n} K_{2,2l+i+k+2}(\kappa_n) - \frac{2l+i+k+2}{2n\kappa_n} K_{3,2l+i+k+3}(\kappa_n) + \frac{1}{\kappa_n} K_{3,2l+i+k+2}(\kappa_n) \right]. \tag{32}$$

V. RESULTS FOR PARTICULAR STATES

There are several classes of states of interest. In this section, we consider two of them, namely, circular states ($l = n-1$) and low-lying states ($n = 1,2$). The former are quite insensitive to the nuclear structure, allowing accurate *ab initio* calculations, and may be of a "metrological" interest [23,24], while the latter are the most sensitive to the nuclear structure and may be applied to measure the nuclear charge radius [25,26]. Following, we present results in closed form in terms of the generalized hypergeometric function ${}_3F_2$ for arbitrary Z and κ_n, and additionally the asymptotic behavior for large κ_n is investigated.

A. Circular states

For states with maximal orbital and angular momenta, i.e., $l = n-1$ and $j = l + 1/2$, there is no difference between κ_n and its relativistic analog κ_n' and $n_r = 0$. The expression (31) in this case can be transformed to

$$\Delta E_U = \frac{\alpha(Z\alpha)^2}{n\zeta\pi} \frac{m_R^2}{M} \left[(f_C - 1) \left(K_{2,2\zeta}(\kappa_n) + \frac{2\zeta}{\kappa_n} K_{3,1+2\zeta}(\kappa_n) \right) + \frac{(Z\alpha)^2}{n\kappa_n} K_{3,2\zeta}(\kappa_n) \right] \tag{33}$$

or, neglecting higher-order terms in $Z\alpha$,

$$\Delta E_U^{(\mathrm{NR})} = \frac{\alpha(Z\alpha)^4}{\pi n^4} \frac{m_R^2}{M} \frac{1}{\kappa_n} \left[-\frac{\kappa_n}{2} K_{2,2n}(\kappa_n) + n K_{3,2n}(\kappa_n) - n K_{3,1+2n}(\kappa_n) \right]. \tag{34}$$

The asymptotics of the last expression for large κ_n is

$$\Delta E_U^{(\mathrm{NR})} = \frac{\alpha(Z\alpha)^4}{\pi n^3}\frac{m_R^2}{M}\left[-\frac{1}{3n}\left(\ln(2\kappa_n) - \psi(2n) + \psi(1) + \frac{1}{6}\right) + \frac{2}{3(2n-1)} - \frac{\pi}{4\kappa_n} + \frac{2n-1}{4\kappa_n^2} - \pi\frac{(n-2)(2n+1)}{18\kappa_n^3} + O\left(\frac{1}{\kappa^4}\right)\right], \quad (35)$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

B. Low-lying states

The above results can be applied to the case of the $1s_{1/2}$ and $2p_{3/2}$ states. In particular, for the nonrelativistic case

$$\Delta E_U^{(\mathrm{NR})}(1s) = \frac{\alpha(Z\alpha)^4}{\pi}\frac{m_R^2}{M}\frac{1}{36\kappa^3(\kappa^2-1)}[-6(2\kappa^6 - 3\kappa^4 - 12\kappa^2 + 10)A(\kappa) + 2\kappa(5\kappa^4 + 16\kappa^2 - 30) - 3\pi(3\kappa^4 + 7\kappa^2 - 10)] \quad (36)$$

or, for large κ,

$$\Delta E_U^{(\mathrm{NR})}(1s) = \frac{\alpha(Z\alpha)^4}{\pi}\frac{m_R^2}{M}\left[-\frac{1}{3}\ln(2\kappa) + \frac{17}{18} - \frac{\pi}{4\kappa} + \frac{1}{4\kappa^2} + \frac{\pi}{6\kappa^3} + O\left(\frac{1}{\kappa^4}\right)\right]. \quad (37)$$

Other particular cases of interest are $2s_{1/2}$ and $2p_{1/2}$ states. For the nonrelativistic case, relations for these states can be written in a unified form[1]

$$\Delta E_U^{(\mathrm{NR})}(2l) = \frac{\alpha(Z\alpha)^4}{\pi}\frac{m_R^2}{32M}\left\{-K_{24}(\kappa_2) + \frac{4}{\kappa_2}[K_{34}(\kappa_2) - K_{35}(\kappa_2)] + \frac{2(1-l)}{\kappa_2^3}[\kappa_2 K_{44}(\kappa_2) + 4K_{54}(\kappa_2) - 4K_{55}(\kappa_2)]\right\} \quad (38)$$

or, for large κ,

$$\Delta E_U^{(\mathrm{NR})}(2l) = \frac{\alpha(Z\alpha)^4}{\pi}\frac{m_R^2}{16M}\left\{\frac{1}{3}\left(-\ln(2\kappa_2) + \frac{16-7l}{3}\right) - \frac{\pi}{2\kappa_2} + \frac{l+2}{2}\frac{1}{\kappa_2^2} + \frac{2\pi(1-l)}{3}\frac{1}{\kappa_2^3} + \mathcal{O}\left(\frac{1}{\kappa_2^4}\right)\right\}. \quad (39)$$

The numerical results for the low-lying states in light muonic atoms, obtained from (36) and (38), are summarized in Table I.

[1]To come to this form from Eq. (32), one can use the relation

$$K_{b,c} = \frac{1}{\kappa}K_{b+1,c+1} + K_{b,c+1}.$$

TABLE I. Relativistic recoil eVP corrections for the low-lying levels in muonic hydrogen. The units are $(\alpha/\pi)(Z\alpha)^4 m_R^2/M$.

Atom	$1s$	$2s$	$2p_{1/2}$	$2p_{3/2}$
μH	0.182	0.0381	0.000901	0.000901
μD	0.180	0.0388	0.000968	0.000968
μ^3He	0.122	0.0459	0.00184	0.00184
μ^4He	0.121	0.0459	0.00184	0.00184

VI. COMPARISON WITH THE BREIT-TYPE CALCULATIONS

Evaluations of relativistic recoil effects within the Grotch-type approach developed in this paper and the standard Breit-type technique (see, e.g., [7,13,27]) are complementary. Both produce for the eVP effects not only the leading term of order $\alpha(Z\alpha)^4m^2/M$, but also certain higher-order corrections. While the Grotch-type calculation provides us with partial account of $\alpha(Z\alpha)^6m^2/M$ contributions, the Breit-type calculation leads to an exact (in m/M) result for the $\alpha(Z\alpha)^4m$ contribution.

The additional $\alpha(Z\alpha)^6m^2/M$ terms in the Grotch-type approach are not so important as a simplification of pure recoil contributions. The technique allows us to easily separate the leading $\alpha(Z\alpha)^4m^2/M$ term from the higher-order effects and calculate it much more easily than by means of the Breit-type evaluation. Meanwhile, such an evaluation is completely separated from the nonrecoil relativistic term. On the contrary, the Breit-type evaluation produces both recoil and nonrecoil relativistic contributions within the same calculations, which allows additional crosschecks.

In this section, we describe a rearrangement of the Breit-type evaluation which would allow a direct comparison between the Grotch-type and Breit-type results. In paper [7], we have calculated the relativistic-recoil correction in question for low-lying states in light muonic atoms by both mentioned methods. In both cases, we can expand the correction by powers of m/M:

$$E_U = \frac{\alpha}{\pi}(Z\alpha)^4 m_R\left[C_0 + C_1\frac{m_R}{M} + C_2\left(\frac{m_R}{M}\right)^2 + \cdots\right], \quad (40)$$

where C_0 corresponds to the known nonrecoil Uehling correction to the energy, C_1 is found by the Grotch method, and the Breit method provides both C_1 and C_2. The coefficients C_1 calculated by the two methods agree. That is a unique expansion and the C_0 and C_1 coefficients obtained by both methods should be the same. The Grotch-type approach produces C_0 and C_1, but not C_2.

The conventional Breit-type calculation does not produce the result in such a form which makes the direct comparison difficult. The Breit approach is based on an unperturbed Hamiltonian

$$H^{(0)} = \frac{\mathbf{p}^2}{2m_R} + V(r)$$

and thus the wave function does not include the muon and nuclear mass separately, but only in a combination in the form of the reduced mass $\phi^{(0)} = \phi(r; m_R)$.

In the meantime, the standard Breit equation (see, e.g., [2])

$$V_{\rm Br} = -\left(\frac{1}{m^3}+\frac{1}{M^3}\right)\frac{\mathbf{p}^4}{8} + \frac{Z\alpha}{8}\left(\frac{1}{m^2}+\frac{1}{M^2}\right)4\pi\delta^3(\mathbf{r}) + Z\alpha\left(\frac{1}{4m^2}+\frac{1}{2mM}\right)\frac{\mathbf{L}\cdot\boldsymbol{\sigma}}{r^3} + \frac{Z\alpha}{2mM}4\pi\delta^3(\mathbf{r}) + \frac{Z\alpha}{2mM}\left[\frac{1}{r^3}\mathbf{L}^2 - \mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2\right] \quad (41)$$

explicitly depends on the muon mass m and the nuclear mass M. The eVP addition to the Breit Hamiltonian is of the form [27]

$$V_{\rm Br}^{\rm VP} = \left(\frac{1}{8m^2}+\frac{1}{8M^2}\right)\nabla^2 V_U + \left(\frac{1}{4m^2}+\frac{1}{2mM}\right)\frac{V_U'}{r}\mathbf{L}\cdot\boldsymbol{\sigma} + \frac{1}{2mM}\nabla^2\left[V_U - \frac{1}{4}(rV_U)'\right] + \frac{1}{2mM} \times\left[\frac{V_U'}{r}\mathbf{L}^2 + \frac{\mathbf{p}^2}{2}(V_U - rV_U') + (V_U - rV_U')\frac{\mathbf{p}^2}{2}\right]. \quad (42)$$

As a result, the matrix element of $V_{\rm Br}$ and $V_{\rm Br}^{\rm VP}$ over $\phi^{(0)}(r;m_R)$ depends on m, M, m_R and is not suited for presentation in the form of (40).

To adjust the Breit Hamiltonian to this form, one has to rewrite it as a function of m_R and M, but not m. The related corrections to the Hamiltonian take the form

$$V_{\rm Br} = -\left(\frac{1}{m_R^3}-\frac{3}{m_R^2M}\right)\frac{\mathbf{p}^4}{8} + \frac{Z\alpha}{8}\left(\frac{1}{m_R^2}-\frac{2}{m_RM}\right)4\pi\delta^3(\mathbf{r}) + \frac{Z\alpha}{4m_R^2}\frac{\mathbf{L}\cdot\boldsymbol{\sigma}}{r^3} + \frac{Z\alpha}{2m_RM}4\pi\delta^3(\mathbf{r}) + \frac{Z\alpha}{2m_RM}\left[\frac{1}{r^3}\mathbf{L}^2 - \mathbf{p}^2\frac{1}{r} - \frac{1}{r}\mathbf{p}^2\right] \quad (43)$$

and

$$V_{\rm Br}^{\rm VP}(\mathbf{r}) = \left(\frac{1}{8m_R^2}-\frac{1}{4m_RM}\right)\nabla^2 V_U + \frac{1}{4m_R^2}\frac{V_U'}{r}\mathbf{L}\cdot\boldsymbol{\sigma} + \frac{1}{2m_RM}\nabla^2\left[V_U - \frac{1}{4}(rV_U)'\right] + \frac{1}{2m_RM}\left[\frac{V_U'}{r}\mathbf{L}^2 + \frac{\mathbf{p}^2}{2}(V_U - rV_U') + (V_U - rV_U')\frac{\mathbf{p}^2}{2}\right], \quad (44)$$

and here all terms which contribute to order $(Z\alpha)^4m^3/M^2$ and $\alpha(Z\alpha)^4m^3/M^2$ are neglected.

The perturbations (43) and (44) directly lead to eVP results in Eq. (40). We realized such a rearrangement in Ref. [7] and obtained results by the Breit-type approach, which completely agrees with our Grotch-type approach within an uncertainty of numerical integration (cf. Table I). Such a rearrangement is applicable not only in the first order in α. In particular, we applied it in Ref. [28] to second order in α and obtained in that work relativistic recoil corrections (of the first order in m/M) consistent with our calculation of the relativistic recoil by the Grotch-type method [29].

VII. CONCLUSIONS

In this paper, a method for a calculation of relativistic recoil effects developed previously [5] was applied perturbatively for the one-loop electronic-vacuum-polarization corrections. With the results of the Dirac problem with Coulomb+Uehling potential already known, the evaluation of the additional recoil correction has dealt only with nonrelativistic wave functions. It was performed in closed analytic form in terms of the same base integrals as required for the calculation of the Uehling correction itself (cf. [14,16–19]).

We also found asymptotics for high $Z\alpha m/(m_e n)$ for the circular and low-lying states. The low-lying states $1s, 2s, 2p$ are of particular interest in light muonic atoms because they provide us with perhaps the best opportunity to determine the nuclear charge radius. Some time ago, certain results were obtained within the Breit-type [13,27] and Grotch-type approaches [10]. They have been discussed in part by us in Ref. [7].

Both calculations contain not only the leading eVP relativistic recoil term of order $\alpha(Z\alpha)^4m^2/M$, but also certain higher-order corrections. Here, we explain in details how to compare those calculations. A modification of the effective Breit Hamiltonian is described, which allows us to avoid any higher-order effects in the Breit-type approach. As a result [7], we agree with [13] and disagree with [9,10] and [27]. A discrepancy with the former is due to an inappropriate gauge used in that work, while the discrepancy with the latter is most probably due to a numerical error there.

Here, we applied the technique based on the presentation of the results in terms of base integrals. However, this is not necessary. If it is desired, one can solve the related Dirac equation numerically. As we mentioned, for the relativistic recoil correction by itself even the Dirac equation is not necessary. A nonrelativistic Schrödinger equation with an appropriate potential and subsequent nonrelativistic perturbation theory is sufficient. The approach can be extended further and it has been extended. In a subsequent paper [29], it is successfully applied to the second-order eVP relativistic recoil corrections to order $\alpha^2(Z\alpha)^4m^2/M$.

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