Optical simulation of Majorana physics

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We show a procedure to classically simulate the Majorana equation in $1+1$ dimensions via two one-dimensional photonic crystals. We use a decomposition of the Majorana equation into two Dirac equations and propose an approach that uses a bichromatic refractive index distribution and nearest-neighbor couplings of the type found in Glauber-Fock lattices. This allows us to escape the restriction of staying near the Brillouin zone imposed by the classical simulation of Dirac dynamics with bichromatic lattices. Furthermore, it is possible to simulate the evolution of Gaussian wave packets under the Majorana-Dirac equation with light impinging only into the first waveguide of our bichromatic-Glauber-Fock lattice.

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I. INTRODUCTION

Arrays of coupled photonic waveguides have been excellent classical simulators of quantum and relativistic physics [1,2]. In particular, they have allowed one of the first visualizations of Dirac physics phenomena like *Zitterbewegung* [3], Klein tunneling [4], and Dirac solitons [5] in tabletop experiments. The Majorana equation is a Lorentz covariant generalization of the Dirac equation [6,7]. It is the first relativistically invariant theory and the first application of the infinite-dimensional representations of the Lorentz group [8]. If supplemented by the Majorana condition of charge invariance, i.e., Majorana fermions, it is equivalent to the Dirac equation. In modern gauge theories, chiral spinors answer to two-component Majorana equations [9,10]. This has rekindled the interest in Majorana dynamics, and recently, both a procedure to implement nonphysical operations related to Majorana physics [11] and a scheme to simulate the Majorana equation [12] with trapped ions setups have been proposed. These proposals bring Majorana physics into the quantum optics laboratory, following the pioneering work in the quantum simulation of relativistic Dirac physics, e.g., *Zitterbewegung* [13,14] and the Klein paradox [15,16].

Here we go beyond the quantum optics realizations and show that it is possible to simulate Majorana physics by propagation of classical light in one-dimensional photonic crystals. For this reason we will present the quantum simulation of the Majorana equation in $1+1$ dimensions in the following section to provide the basic set of transformations that will move us to and from its classical simulation. Then, we will propose a photonic analog in the form of two one-dimensional photonic crystals that simulate the Majorana dynamics via the propagation of two initial light distributions for each one. In our proposal we use a combination of experimentally demonstrated bichromatic [3] and Glauber-Fock [17] lattices. This bichromatic-Glauber-Fock lattice allows us to evade the restriction of staying near the Brillouin zone that appears when using bichromatic lattices to classically simulate Dirac physics. Our proposal imposes no restriction and manages to classically simulate the evolution of Gaussian wave functions under Dirac dynamics by a single beam of light impinging the first waveguide of the array. Finally, we will show that by exploiting the quantum-classical analog it is straightforward to construct the impulse functions of our lattices.

II. QUANTUM SIMULATION OF MAJORANA PHYSICS

One of us has been part of a proposal to implement a quantum simulation of the Majorana equation in a trapped-ion setup [12]. Here, we give a brief summary of the topic in order to provide the basis to construct a classical analog. We start from the Majorana equation [6,7],

$$i\gamma^{\mu}\partial_{\mu}\psi = m_M\psi_c, \tag{1}$$

where the symbol ψ_c stands for charge conjugation of the spinor ψ, $\psi_c \equiv \gamma^2\psi^*$, and γ_μ are the Dirac matrices, the symbol ∂_μ is shorthand notation for partial derivation with respect to μ, the superscript asterisk represents complex conjugation, m_M is the Majorana mass, and we have set $\hbar = c = 1$. In $1+1$ dimensions it is possible to rewrite it as

$$i\partial_t\psi = \hat{\sigma}_x\hat{p}_q\psi - im\hat{\sigma}_y\psi^*, \tag{2}$$

where $\hat{\sigma}_i$, with $i = x,y,z$, are the Pauli matrices, $\hat{p}_q$ is the dimensionless momentum, m stands for the modified Majorana mass in units of momentum, and the Majorana field is given as a two-dimensional complex vector, $\psi = (\psi^{(1)},\psi^{(2)})$. Then, we can combine the field and its charge conjugate,

$$i\partial_t(\psi+\psi^*) = \hat{\sigma}_x\hat{p}_q(\psi+\psi^*) + im\hat{\sigma}_y(\psi-\psi^*), \tag{3}$$

$$i\partial_t(\psi-\psi^*) = \hat{\sigma}_x p_q(\psi-\psi^*) - im\hat{\sigma}_y(\psi+\psi^*), \tag{4}$$

to create and extended real Hilbert space, where we can write the Majorana equation as a Schrodinger equation,

$$i\partial_t\Psi = [(1_2\otimes\hat{\sigma}_x)\hat{p}_q - m(\hat{\sigma}_x\otimes\hat{\sigma}_y)]\Psi, \tag{5}$$

with the real four-element field given by $\Psi = (\mathrm{Re}(\psi^{(1)}),\mathrm{Re}(\psi^{(2)}),\mathrm{Im}(\psi^{(1)}),\mathrm{Im}(\psi^{(2)}))$, which is related to the Majorana field via the transformation $\psi = \hat{M}\Psi$, with $\hat{M} = (1_2, i1_2)$, where the symbol 1_2 stands for the unit matrix in dimension 2. Note that the unitary operation $\hat{U} = e^{-i\pi\hat{\sigma}_y/4}\otimes e^{-i\pi\hat{\sigma}_x/4}$ providing the basis $\Psi = \hat{U}\Phi$, such that we can write the original Majorana field as $\psi = \hat{M}\hat{U}\Phi$, yields the following Schrödinger equation related to the

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Majorana equation:

$$i\partial_t\Phi = [(1_2 \otimes \hat{\sigma}_x)p_q + m(\hat{\sigma}_z \otimes \hat{\sigma}_z)]\Phi. \tag{6}$$

At this point, we can propose a form for our four-vector and realize that this Schrödinger representation for the Majorana equation leads to two uncoupled Dirac-like equations with positive and negative mass,

$$i\partial_t\phi_\pm = [-i\hat{\sigma}_x\hat{p}_q \pm m\hat{\sigma}_z]\phi_\pm, \tag{7}$$

where $\phi_+ = (\Phi^{(1)},\Phi^{(2)})$ and $\phi_- = (\Phi^{(3)},\Phi^{(4)})$, such that $\Phi = (\phi_+,\phi_-)$.

III. PHOTONIC LATTICE ANALOG

Starting from the separation of the Majorana equation into two Dirac equations in (7), we can follow the idea in [18] and compare the dispersion relation for a bichromatic lattice with the energy-momentum dispersion relation of the Dirac equation and conclude that the Majorana Hamiltonian can be simulated classically by two bichromatic refractive index lattices described by the differential equation set

$$-i\partial_z\mathcal{E}_j = [n \pm (-1)^j m]\mathcal{E}_j + \mathcal{E}_{j+1} + \mathcal{E}_{j-1}, \tag{8}$$

as long as the simulation stays near the boundary of the Brillouin zone. Also, we have introduced a bias refractive index n that only introduces an overall phase factor.

We do not want such a strong restriction between the momentum of the simulated wave packet and the characteristics of the bichromatic lattice. In order to get rid of this restraint, we map the adimensional linear momentum operator to a combination of bosonic creation (annihilation) operators $\hat{a}^\dagger$ ($\hat{a}$),

$$\hat{p}_q = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}). \tag{9}$$

Then, we can rewrite the two uncoupled Dirac equations that quantum simulate the Majorana equation as a Schrödinger equation with effective Hamiltonians:

$$H_\pm = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a})\hat{\sigma}_x \pm m\hat{\sigma}_z, \tag{10}$$

after a $\pi/2$ rotation around the $\hat{a}^\dagger\hat{a}$ axis, $\hat{R} = e^{i\pi\hat{a}^\dagger\hat{a}/2}$, providing a new basis φ such that $\phi = \hat{R}\varphi$. It is straightforward to notice that (10) is equivalent to the Rabi Hamiltonian [19] with null field frequency. Thus, we can set up a photonic lattice analog following previous work on the classical simulation of the Rabi Hamiltonian [20,21]. In summary, if we split the corresponding Hilbert space into two parity subspaces given by

$$\{|+,j\rangle\} = \{|g,0\rangle,|e,1\rangle,|g,2\rangle,\ldots\}, \tag{11}$$

$$\{|-,j\rangle\} = \{|e,0\rangle,|g,1\rangle,|e,2\rangle,\ldots\}, \tag{12}$$

we can write the evolution of any given initial state $|\varphi_{\pm,\pm}\rangle = \sum_j \mathcal{E}_{\pm,\pm,j}|\pm,j\rangle$, where the first subindex is related to the positive (negative) mass Hamiltonian and the second is related to the positive (negative) parity, as the vector differential set

$$i\partial_t E_{\pm,\pm} = H_{\pm,\pm}E_{\pm,\pm}, \tag{13}$$

where the vector of amplitudes is given by $E_{a,b} = (\mathcal{E}_{a,b,0},\mathcal{E}_{a,b,1},\ldots)$ and the elements of the four matrices describing the dynamics reduce to two matrices as

$$(H_{+,+})_{j,k} = (H_{-,-})_{j,k} \tag{14}$$

$$= -m(-1)^j\delta_{j,k} + \sqrt{\frac{k}{2}}\delta_{j+1,k} + \sqrt{\frac{j}{2}}\delta_{j-1,k}, \tag{15}$$

$$(H_{+,-})_{j,k} = (H_{-,+})_{j,k} \tag{16}$$

$$= m(-1)^j\delta_{j,k} + \sqrt{\frac{k}{2}}\delta_{j+1,k} + \sqrt{\frac{j}{2}}\delta_{j-1,k}. \tag{17}$$

By making the change $t \to z$ we obtain, up to a constant phase factor, a differential set describing two photonic lattices where the individual refractive indices are bichromatic [22] and the nearest-neighbor couplings go as those in Glauber-Fock photonic lattices [23,24]. It is straightforward to realize that the magnitude of the effective mass m will provide three dynamics regimes: $m \gg 1/\sqrt{2}$, $m \sim 1/\sqrt{2}$, and $m \ll 1/\sqrt{2}$.

Before advancing further, we want to discuss the meaning of light impinging the jth waveguide in one of our photonic lattices, which, after dropping two of the subindices referring to the parity and the mass sign, is equivalent to writing

$$|\psi_j(0)\rangle = |j\rangle \tag{18}$$

$$= \int dq|q\rangle\Psi_j(q), \tag{19}$$

with

$$\Psi_j(q) \equiv \langle q|j\rangle \tag{20}$$

$$= \frac{1}{\sqrt{2^j j!}}\left(\frac{1}{\pi}\right)^{1/4} e^{-q^2/2}H_j(q), \tag{21}$$

where $H_n(x)$ is the nth Hermite polynomial. In other words, a beam of light impinging just the first waveguide of the array is equivalent to an initial Gaussian wave function in dimensionless canonical space, $\Psi_0(q) = e^{-q^2/2}/\pi^{1/4}$. Light impinging just the jth waveguide will simulate an initial wave function in canonical space with a jth Hermitian-Gaussian distribution. Any given initial wave function in dimensionless canonical space $\psi(q)$ can be constructed from an adequate superposition of classical fields impinging the photonic crystal,

$$\psi(q) = \sum_j \mathcal{E}_j(0)\Psi_j, \tag{22}$$

where the initial field amplitudes at each waveguide $\mathcal{E}_j(0)$ correspond to the decomposition of the initial wave function in the orthonormal basis provided by (21),

$$\mathcal{E}_j(0) = \int dq\,\psi(q)\Psi_j^*(q). \tag{23}$$

Note that this has to be done for each set of initial conditions to be propagated in each photonic crystal. Figure 1 shows the numerical propagation of a Gaussian wave packet in canonical space Ψ_0 simulated by a beam of light impinging the first waveguide when the effective mass belongs to different regimes: $m = 0.1/\sqrt{2}$ [Fig. 1(a)], $m = 1/\sqrt{2}$ [Fig. 1(b)], and $m = 10/\sqrt{2}$ [Fig. 1(c)]. Note that for an initial Gaussian packet, as the effective mass becomes negligible, $m \to 0$,

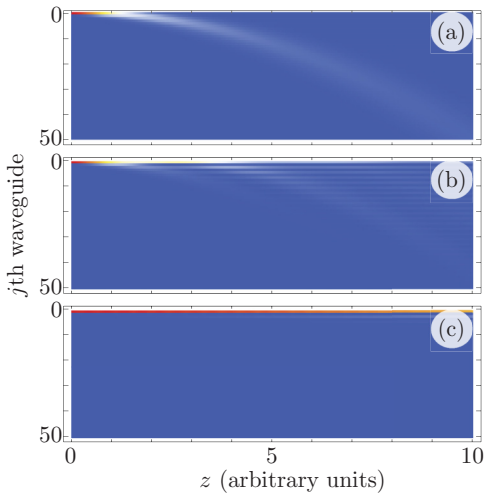

FIG. 1. (Color online) Time evolution of the modulus squared of an initial Gaussian wave packet, $\Psi_0(q) = e^{-q^2/2}/\pi^{1/4}$, simulated by the propagation of light impinging the first waveguide of an array of photonic waveguides described by the matrix $H_{+,+}$ or $H_{-,-}$ with the following effective masses: (a) $m = 0.1/\sqrt{2}$, (b) $m = 1/\sqrt{2}$, and (c) $m = 10/\sqrt{2}$.

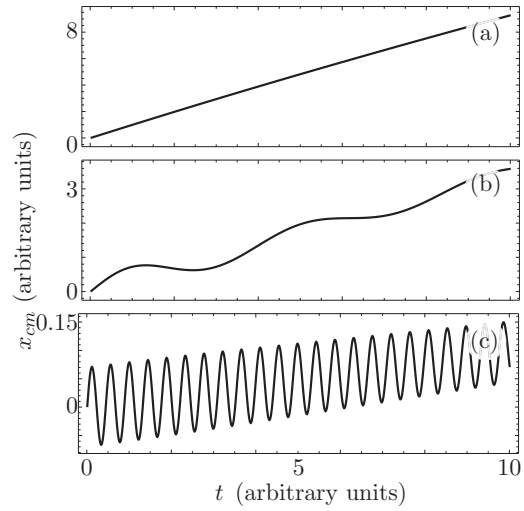

FIG. 2. Evolution of the center of mass for an initial Gaussian wave packet, $\Psi_0(q) = e^{-q^2/2}/\pi^{1/4}$, under Dirac dynamics recovered from the propagation of an initial beam impinging the first waveguide in an array of photonic waveguides described by the matrix $H_{+,+}$ or $H_{-,-}$ related to effective-mass parameters described in Fig. 1: (a) $m = 0.1/\sqrt{2}$, (b) $m = 1/\sqrt{2}$, and (c) $m = 10/\sqrt{2}$.

the amplitude distribution will become closer to a Poisson distribution with mean and variance given by $t^2/2$, $\psi(q,t) = \sum_j e^{-t^2/4}(it)^j/\sqrt{j!}$ [23], while in the complete opposite case, $m \to \infty$, the amplitude distribution will become propagation invariant. With our classical simulation we have direct access to the center of mass of the intensity distribution for variable effective-mass parameters. Please be aware that in our simulation the center of mass for the intensity does not coincide with the center of mass of the wave function in canonical space,

$$q_{cm} = \sum_k \sqrt{\frac{k+1}{2}}[\mathcal{E}_k^*(t)\mathcal{E}_{k+1}(t) + \mathcal{E}_{k+1}^*(t)\mathcal{E}_k(t)], \quad (24)$$

and furthermore, in order to study the center of mass under the original dynamics we have to account for all the rotations to transform back to the original frame, x_{cm}; e.g., for the Dirac dynamics part of our classical simulation,

$$x_{cm} = \sum_k i\sqrt{\frac{k+1}{2}}[\mathcal{E}_k^*(t)\mathcal{E}_{k+1}(t) - \mathcal{E}_{k+1}^*(t)\mathcal{E}_k(t)]. \quad (25)$$

Figure 2 shows the evolution of the center of mass of an initial Gaussian wave packet under Dirac dynamics for different effective masses, $m = 0.1/\sqrt{2}$ [Fig. 2(a)], $m = 1/\sqrt{2}$ [Fig. 2(b)], and $m = 10/\sqrt{2}$ [Fig. 2(c)], recovered from the classical simulation equivalent to the propagation of a beam impinging the first waveguide of the photonic crystal.

We are also interested in providing an impulse function for our photonic crystal. For this reason we start from the matrix differential set (13), where the matrix $H_{a,b}$ is constant and allows us to write the time propagator as

$$U(t) = e^{-iH_{a,b}t} \quad (26)$$

$$= \cos\Omega(\hat{q})t - \frac{i}{\Omega(\hat{q})}\sin\Omega(\hat{q})t\,H_{a,b}, \quad (27)$$

with

$$\Omega(\hat{q}) = \sqrt{m^2 + \tfrac{1}{2}(\hat{a} + \hat{a}^\dagger)^2} \quad (28)$$

$$= \sqrt{m^2 + \hat{q}^2}, \quad (29)$$

where $\hat{q}$ is the dimensionless canonical position operator. This expression is related to the dispersion relation of the photonic lattice and the energy states of the Dirac equation. These results and the action of the matrix $H_{a,b}$ over the components of the parity basis allow us to calculate the impulse function (field amplitude) at the kth waveguide for an input in the jth waveguide of the corresponding photonic lattice; for example, for the waveguide array described by the matrices $H_{+,+}$ and $H_{-,-}$ we can write the impulse function as

$$I_{j,k}^{(+,+)} = I_{j,k}^{(-,-)} \quad (30)$$

$$= \int dq \left\{ \cos\Omega(q)t - \frac{i[-m(-1)^j + q]}{\Omega(q)} \times \sin\Omega(q)t \right\} \Psi_k^*(q)\Psi_j(q). \quad (31)$$

Comparing this expression to numerical propagation in the photonic lattice shows good agreement between the results. It is straightforward to use the impulse function to calculate the propagation of any initial light field simulating any given initial Majorana field.

IV. CONCLUSIONS

We have shown a scheme to classically simulate Majorana physics in $1+1$ dimensions with two one-dimensional photonic crystals. Our approach is based on the quantum simulation of the Majorana equation through two Dirac equations with positive or negative mass. We have proposed a way to classically simulate the Dirac equation combining two one-dimensional photonic crystals that have already been produced experimentally; one is the bichromatic photonic lattice, and the other is the Glauber-Fock photonic lattice. Each Dirac equation has an optical analog in a set of two bichromatic-Glauber-Fock lattices, i.e., a waveguide array where the refractive index of individual waveguides alternates and the coupling goes as the square root of the waveguide number. We have also demonstrated that instead of using four photonic lattices, it is enough to use two waveguide arrays with the propagation of two adequate initial conditions in each one to simulate the Majorana equation. Finally, by using the equivalence between the quantum and classical simulations we were able to give the impulse function of the photonic lattices. These impulse functions allow us to calculate the propagation of any initial light distribution simulating any given Majorana field.

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