Success probabilities for universal unambiguous discriminators between unknown pure states

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A universal programmable discriminator can perform discrimination between two unknown states, and the optimal solution can be approached via discrimination between the two averages over the uniformly distributed unknown input pure states, as has been widely discussed in previous works. In this paper, we consider the success probabilities of the optimal universal programmable unambiguous discriminators when applied to the pure input states. More precisely, the analytic results of the success probabilities are derived with the expressions of the optimal measurement operators for the universal discriminators, and we find that the success probabilities are independent of the dimension d while the amount of copies in the two program registers is equal. The success probability of the programmable unambiguous discriminator can asymptotically approach that of the usual unambiguous discrimination (state comparison) as the number of copies in the program registers (data register) goes to infinity.

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The discrimination between quantum states is a basic tool in quantum information processing, and this is a nontrivial problem since an unknown state cannot be perfectly cloned [1]. Usually, there are two basic strategies to achieve state discrimination: minimum-error discrimination (MD) [2–4], with a minimal probability for the error, and unambiguous discrimination (UD) [5], with a minimum probability of inconclusive results. In those works, a quantum state is chosen from a set of known states, but we do not know which and we want to determine the actual states.

The discrimination problems above are dependent on the set of states to be distinguished, and the device for the discrimination is not universal but specifically designed for the states. As in the spirit of programmable quantum devices [6], it is interesting to design a discrimination device that does not need to change as the input states change. Such a universal device that can unambiguously discriminate between two unknown qubit states was first constructed by Bergou and Hillery [7]. In this programmable quantum device, two possible states enter two program registers as "programs," respectively, and the data register is prepared with a third state (guaranteed to be one of the two possible states) which one wishes to identify. One amazing feature of this discriminator is that the states in the device can be unknown, which means no classical knowledge on the states is provided, and it is capable of distinguishing between any pair of states in this device.

Later, the generalizations and the experimental realizations of programmable discriminators were introduced and widely discussed [8–13]. The problems with multiple copies in program and data registers or with high-dimensional states in the registers were considered. Furthermore, the case in which each copy in the registers is the mixed state was also treated [11]. In most of these works, either the unambiguous discrimination or the minimum-error scheme was used, and quite recently, the two strategies were unified in Ref. [13] by introducing an error margin. With multiple copies and high-dimensional pure states in the registers, the optimal solution and success probability are hard to obtain. However, by taking the average of the unknown states, the task is equivalent to the discrimination of known mixed states in previous works. Because the equivalent mixed states are highly symmetrical objects, the success probabilities for discrimination between them, or the average success probabilities (ASPs), can be derived and then the optimal solution is obtained using the symmetry properties. The ASPs can be used to determine the optimality of the discriminators, i.e., the discriminators with the maximal ASPs are optimal. In fact, as mentioned before, the unknown pure states instead of the average mixed states are distinguished in most cases, and the ASP is not the success probability when the device works. Therefore, it is useful and meaningful to obtain the success probabilities for the programmable discrimination between the pure states, or pure success probabilities (PSPs) for short. Moreover, if we have the PSP, we take its average and we can obtain the ASP. Unfortunately, except for the results in a few special simple cases [7,8], PSPs for the general case with multiple copies and high-dimensional states have still not been obtained.

The main purpose of this paper is to evaluate the PSPs of the optimal universal unambiguous discriminators between two unknown pure states for the general case. Following the optimal measurement operators for the universal unambiguous discriminator in Ref. [9], we show that the PSPs are related to Wigner's D function and the Clebsch-Gordan (CG) coefficients for the angular momentum coupling. The analytic results of PSPs can be obtained with the exact expressions of Wigner D function and the CG coefficients.

I. UNIVERSAL PROGRAMMABLE UNAMBIGUOUS DISCRIMINATORS

The general case of the programmable discriminators has been systematically studied [9], and the optimal solutions are obtained with the representation theory of the U(d) group and the Jordan basis method. The discriminator consists of two program registers A and C, and one data register B. It is assumed that systems A and C are prepared in the states $|\phi_1\rangle$ and $|\phi_2\rangle$, each with n_A and n_C copies, respectively, and system B is prepared in n_B copies of either $|\phi_1\rangle$ or $|\phi_2\rangle$, with *a priori*

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probabilities η_1 and η_2, such that $\eta_1+\eta_2=1$. Such a device can measure and then may distinguish the total input states

$$|\Phi_1\rangle = |\phi_1\rangle_A^{\otimes n_A}|\phi_1\rangle_B^{\otimes n_B}|\phi_2\rangle_C^{\otimes n_C},\\ |\Phi_2\rangle = |\phi_1\rangle_A^{\otimes n_A}|\phi_2\rangle_B^{\otimes n_B}|\phi_2\rangle_C^{\otimes n_C}, \tag{1}$$

with the unknown qudit states $|\phi_1\rangle$ and $|\phi_2\rangle$ in d-dimensional Hilbert space $\mathcal{H}$. Mathematically, this programmable discriminator is defined by the elements of a universal POVM $\{\Pi_1,\Pi_2,\Pi_0\}$, where $\Pi_1(\Pi_2)$ is associated with the input state $|\Phi_1\rangle(|\Phi_2\rangle)$, and Π_0 corresponds to an inconclusive result. Without loss of generality, we assume that $n_A \geq n_C$, and we set $n_1=n_A+n_B, n_2=n_B+n_C, N=n_A+n_B+n_C$.

It is not difficult to see that the state $|\Phi_1\rangle$ lies in the tensor space $\mathbb{H}_1=\mathcal{H}^{[n_1]}\otimes\mathcal{H}^{[n_c]}$, while $|\Phi_2\rangle$ in $\mathbb{H}_2=\mathcal{H}^{[n_A]}\otimes\mathcal{H}^{[n_2]}$. $\mathcal{H}^{[n]}$ is the totally symmetric subspace of the n-partite Hilbert space $\mathcal{H}^{\otimes n}$, and $[n]$ is a Young diagram. The irreducible basis $|{}^{[\nu_1]}_{\omega_1}\rangle_1$ forming the complete orthogonal basis of $\mathbb{H}_1$ can be obtained by the coupling of the irreducible bases of $\mathcal{H}^{[n_1]}$ and $\mathcal{H}^{[n_c]}$, where all the possible Young diagrams for $[\nu_1]$ constitute a set $S_1=\{[N-k,k]:k=0,1,\ldots,n_C\}$ and $\omega_1=1,2,\ldots,d^{[\nu_1]}$, with $d^{[\nu_1]}$ the dimension of the irreducible space labeled by $[\nu_1]$. Similarly, one has the complete orthogonal basis $|{}^{[\nu_2]}_{\omega_2}\rangle_2$ for $\mathbb{H}_2$, and besides the Young diagrams in S_1, $[\nu_2]$ can take ones in the set $S_2=\{[N,N-k]:k=n_C+1,n_C+2,\ldots,\min(n_A,n_2)\}$ when $n_A>n_B$. Furthermore, $\mathbb{H}_2=H_2\oplus H_2^{\perp}$, where H_2 is spanned by all the bases $|{}^{[\nu_2]}_{\omega_2}\rangle_2$ with $[\nu_2]\in S_1$ while $H_2^{\perp}$ is spanned by those with $[\nu_2]\in S_2$. $H_2^{\perp}$ is orthogonal to both $\mathbb{H}_1$ and H_2, and the basis $\{|{}^{[\nu_1]}_{\omega_1}\rangle_1:[\nu_1]\in S_1,\omega_1=1,2,\ldots,d^{[\nu_1]}\}$ for $\mathbb{H}_1$ and the basis $\{|{}^{[\nu_2]}_{\omega_2}\rangle_2:[\nu_2]\in S_1,\omega_2=1,2,\ldots,d^{[\nu_2]}\}$ for H_2 form the Jordan basis [9,14].

Now, we can introduce the optimal measurement operators for the universal programmable unambiguous discriminators,

$$\Pi_1 = \sum_{k=1}^{n_C}\sum_{\omega=1}^{d_k}\frac{1-q_{k,1}^{\text{opt}}}{1-O_k^2}|\psi_{k,\omega}^{\perp}\rangle_2\langle\psi_{k,\omega}^{\perp}|,\\ \Pi_2 = \sum_{k=1}^{n_C}\sum_{\omega=1}^{d_k}\frac{1-q_{k,2}^{\text{opt}}}{1-O_k^2}|\psi_{k,\omega}^{\perp}\rangle_1\langle\psi_{k,\omega}^{\perp}|+\mathbb{1}^{\perp},\\ \Pi_0 = \mathbb{1}-\Pi_1-\Pi_2, \tag{2}$$

where $d_k=d^{[N-k,k]}$, and $\mathbb{1}^{\perp}$ and $\mathbb{1}$ are the identity operators on $H_2^{\perp}$ and $\mathbb{H}=\mathbb{H}_1\cup\mathbb{H}_2$, respectively. $|\psi_{k,\omega}^{\perp}\rangle_1$ and $|\psi_{k,\omega}^{\perp}\rangle_2$ are normalized orthogonal vectors to $|{}^{[\nu]}_{\omega}\rangle_1$ and $|{}^{[\nu]}_{\omega}\rangle_2$, respectively, in the subspace spanned by them, and O_k are the inner products of the Jordan basis dependent only on the Young diagram $[\lambda]=[N,N-k]$. The parameter $q_{k,1}^{\text{opt}}$ is taken as

$$q_{k,1}^{\text{opt}} = \begin{cases} 1 & \text{for} \quad \eta_1 < c_k, \\ \sqrt{\frac{\eta_2 d_1}{\eta_1 d_2}}O_k & \text{for} \quad c_k \leq \eta_1 \leq d_k, \\ O_k^2 & \text{for} \quad \eta_1 > d_k, \end{cases} \tag{3}$$

$q_{k,2}^{\text{opt}}=O_k^2/q_{k,1}^{\text{opt}}$, and the boundaries $c_k=d_1O_k^2/(d_2+d_1O_k^2)$ and $d_k=d_1/(d_1+d_2O_k^2)$. Details for this section are discussed in Ref. [9]. We will address the PSPs in the following section.

II. PSPs FOR UNIVERSAL UNAMBIGUOUS DISCRIMINATORS

Since the unambiguous discriminators are universal, the optimal operators Π_0, Π_1, and Π_2 are applicable for the PSPs to the discrimination between the pure states $|\Phi_1\rangle$ and $|\Phi_2\rangle$. Armed with the expressions of the optimal operators in Eq. (2), the optimal success probability reads

$$P = \eta_1\langle\Phi_1|\Pi_1|\Phi_1\rangle + \eta_2\langle\Phi_2|\Pi_2|\Phi_2\rangle. \tag{4}$$

To give the exact expression of PSP, let us consider $\langle\Phi_1|\Pi_1|\Phi_1\rangle$ first. With the expression of Π_1 and the relationship

$$\langle\Phi_1|\psi_{k,\omega}^{\perp}\rangle_2\langle\psi_{k,\omega}^{\perp}|\Phi_1\rangle = \left(1-O_k^2\right)\left\langle\Phi_1\middle|{}^{[\lambda]}_{\omega}\right\rangle_1\left\langle{}^{[\lambda]}_{\omega}\middle|\Phi_1\right\rangle,$$

we have $\langle\Phi_1|\Pi_1|\Phi_1\rangle=\sum_{[\lambda]\in S_1}(1-q_{k,1}^{\text{opt}})\langle\Phi_1|\mathbb{1}^{[\lambda]}|\Phi_1\rangle$, where $\mathbb{1}^{[\lambda]}=\sum_{\omega=1}^{d^{[\lambda]}}|{}^{[\lambda]}_{\omega}\rangle_1\langle{}^{[\lambda]}_{\omega}|$ is the identity operator on the irreducible representation space labeled by $[\lambda]=[N,N-k]$. The following lemma is very useful to calculate the PSPs.

Lemma. The expectation of the operator $\mathbb{1}^{[\lambda]}$ with respect to $|\Phi_1\rangle$ can be expressed as

$$\langle\Phi_1|\mathbb{1}^{[\lambda]}|\Phi_1\rangle = \sum_M\left|\sum_l D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})}C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l}^{\frac{N}{2}-k,M}\right|^2, \tag{5}$$

where $D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})}$ are the Wigner D function [15], and $C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l}^{\frac{N}{2}-k,M}$ are the CG coefficients [16].

For readability, we postpone the detailed proof of this lemma to the technical appendix. Plugging the exact expressions of CG coefficients and Wigner D function into Eq. (5), and by some algebra, we have

$$\langle\Phi_1|\mathbb{1}^{[\lambda]}|\Phi_1\rangle = \frac{(N-2k+1)n_1!n_C!}{k!(N-k+1)!}\left(\sin^2\frac{\beta}{2}\right)^{n_C}\\ \times {}_2F_1\left(n_1-k+1,k-n_C;1;-\cot^2\frac{\beta}{2}\right),$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function [17]. Finally, we can obtain

$$\eta_1\langle\Phi_1|\Pi_1|\Phi_1\rangle = a(1-|\langle\phi_1|\phi_2\rangle|^2)^{n_C},$$

where the relationship $\sin^2\frac{\beta}{2}=1-|\langle\phi_1|\phi_2\rangle|^2$ has been used and

$$a = \eta_1\sum_{k=1}^{n_C}\left(1-q_{k,1}^{\text{opt}}\right)\frac{(N-2k+1)n_1!n_C!}{k!(N-k+1)!}\\ \times {}_2F_1\left(n_1-k+1,k-n_C;1;-\cot^2\frac{\beta}{2}\right).$$

Similar discussions can be carried out for $\langle\Phi_2|\Pi_2|\Phi_2\rangle$, and one can also obtain

$$\eta_2\langle\Phi_2|\Pi_2|\Phi_2\rangle = (b+c)(1-|\langle\phi_1|\phi_2\rangle|^2)^{n_2},$$

with

$$b = \eta_2\sum_{k=1}^{n_C}\left(1-q_{k,2}^{\text{opt}}\right)\frac{(N-2k+1)n_2!n_A!}{k!(N-k+1)!}\\ \times {}_2F_1\left(n_A-k+1,k-n_2;1;-\cot^2\frac{\beta}{2}\right),$$

$$c = \eta_2 \sum_{k=n_C+1}^{\min(n_A,n_2)} \frac{(N-2k+1)n_2!n_A!}{k!(N-k+1)!} \times {}_2F_1\left(n_A-k+1,k-n_2;1;-\cot^2\frac{\beta}{2}\right).$$

Now, the exact values of PSPs can be expressed as

$$P = a(1-|\langle\phi_1|\phi_2\rangle|^2)^{n_C} + (b+c)(1-|\langle\phi_1|\phi_2\rangle|^2)^{n_2}. \quad (6)$$

This is the main result in this paper, and in the following we will use it to discuss some specific cases.

III. REMARKS AND DISCUSSION

(i) From Eq. (3), it is easy to notice that the parameters $q_{k,1}^{\text{opt}}$ and $q_{k,2}^{\text{opt}}$ may be dependent on the dimension d of the Hilbert space $\mathcal{H}$ unless $n_A = n_C$, and thus the values of PSPs are generally dependent on d. As the reference states $|\phi_1\rangle$ and $|\phi_2\rangle$ are completely unknown, there is no priority for one data register to own more copies than the other one, so we set $n_A = n_C = m, n_B = n$ in this section. The PSPs are now independent of the dimension d, which is very different from the cases for the averages of the input states, where the ASPs are always dependent on d. This is reasonable since the average of the overlap $|\langle\phi_1|\phi_2\rangle|$ over the unknown states $|\phi_1\rangle$ and $|\phi_2\rangle$ is always related to d.

(ii) When n goes to infinity while m is finite, from Eq. (6), we have the asymptotic limit of PSP for each finite m,

$$P_m \equiv \lim_{n\to\infty} P = \sum_{k=1}^{m} a_{mk}\left(\cos^2\frac{\beta}{2}\right)^{m-k}\left(\sin^2\frac{\beta}{2}\right)^{k}, \quad (7)$$

where the coefficients a_{mk} in each term can be calculated, and if we further define $a_{00}=1$ and $a_{m0}=1$, the values of $a_{mk} (m \geq k \geq 0)$ can be listed in the following:

$$\begin{array}{lccccccccc}
m=0: & & & & & 1 & & & & \\
m=1: & & & & 1 & & 1 & & & \\
m=2: & & & 1 & & 2 & & 1 & & \\
m=3: & & 1 & & 3 & & 3 & & 1 & \\
m=4: & 1 & & 4 & & 6 & & 4 & & 1 \\
\vdots & \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots
\end{array}$$

In the triangular array above, each entry happens to be the sum of the two upper entries, and this is the celebrated *Pascal's triangle*, a geometric representation of the binomial coefficients. Hence the elements in the array above are the binomial coefficients, say $a_{mk} = \binom{m}{k}$. As a result,

$$P_m = 1 - \left(\cos^2\frac{\beta}{2}\right)^m = 1 - |\langle\phi_1|\phi_2\rangle|^{2m}. \quad (8)$$

The asymptotic limit above for $n\to\infty$ can be achieved in a different approach. As n is infinite, the unknown state in the data register can be exactly reconstructed via quantum state tomography [18,19]. Denote the reconstructed state by $|\phi_0\rangle$, and then $|\phi_0\rangle$ is the state $|\phi_i\rangle$ ($i=1$ or 2) with probability η_i. While the state in the data system is known, the discrimination between the total states comes down to determining whether the state $|\phi_0\rangle$ is equal to the state in program system A or C. With m copies of states in both program systems, we have m pairs of states $|\phi_0\rangle|\phi_1\rangle$ and m pairs of $|\phi_0\rangle|\phi_2\rangle$, and the task can now be completed by the state comparisons [20] of the two states in the $2m$ pairs. More precisely, for each pair $|\phi_0\rangle|\phi_i\rangle (i=1$ or 2), we project the state $|\phi_i\rangle$ onto $|\phi_0^\perp\rangle$, the state orthogonal to $|\phi_0\rangle$ in the space spanned by $|\phi_1\rangle$ and $|\phi_2\rangle$, and the measurement result associated with the projection operator $E_0^\perp = |\phi_0^\perp\rangle\langle\phi_0^\perp|$ gives the right answer of comparison. Thus, the failure probability of the comparison for each pair is $1-\langle\phi_i|E_0^\perp|\phi_i\rangle = |\langle\phi_0|\phi_i\rangle|^2$, and the success probability for at least one pair is $1-\prod_i |\langle\phi_0|\phi_i\rangle|^{2m} = 1-|\langle\phi_1|\phi_2\rangle|^{2m}$, holding for both $|\phi_0\rangle = |\phi_1\rangle$ and $|\phi_0\rangle = |\phi_2\rangle$. Since the discrimination task succeeds unless the state comparisons of the $2m$ pairs of states all fail, the success probability via state comparisons is $P_m' = 1-|\langle\phi_1|\phi_2\rangle|^{2m}$, which is explicitly the same as that in Eq. (8).

(iii) For the averaged input states, the programmable unambiguous discriminator reduces to the usual unambiguous discrimination between known states as m goes to infinity [10,11]. The situation for the pure input states has not been involved, and now we can address this problem here. Since $n_A = n_C$, we have

$$P = \sum_{k=1}^{m}\left(1-\eta_1 q_{k,1}^{\text{opt}} - \eta_2 q_{k,2}^{\text{opt}}\right)\sum_{l=-\frac{n_C}{2}}^{\frac{n_C}{2}-k}\left|D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})} C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l}^{\frac{N}{2}-k,\frac{n_1}{2}+l}\right|^2, \quad (9)$$

where the relationship Eq. (5) has been used. With large m, using the normal approximation for the binomial distribution, one can obtain $|D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})}|^2 \approx \frac{1}{\sqrt{2\pi}\sigma}\exp[-\frac{(m/2+l-\mu)^2}{2\sigma^2}]$, which is a normal distribution with the expectation $\mu = m\cos^2(\beta/2)$ and the variance $\sigma^2 = m\cos^2(\beta/2)\sin^2(\beta/2)$. Therefore, as $m\to\infty$, $|D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})}|^2 \approx \frac{1}{m}\delta(\frac{l}{m}+\frac{1}{2}-\cos^2\frac{\beta}{2})$, and then $l \approx l_0 \equiv (\cos^2\frac{\beta}{2} - 1/2)m$ is required for the nonzero terms in Eq. (9), yielding $P \approx \sum_{k=1}^m (1-\eta_1 q_{k,1}^{\text{opt}} - \eta_2 q_{k,2}^{\text{opt}})|C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0}^{\frac{N}{2}-k,\frac{n_1}{2}+l_0}|^2$. To obtain the asymptotic limit of the CG coefficient $C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0}^{\frac{N}{2}-k,\frac{n_1}{2}+l_0}$ for large m, we first investigate the two expressions $\hat{\boldsymbol{J}}^2/m^2|(\frac{n_1}{2}\frac{n_C}{2})\frac{N}{2}-k,\frac{n_1}{2}+l_0\rangle$ and $\hat{\boldsymbol{J}}^2/m^2|\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0\rangle$, where $\hat{\boldsymbol{J}} = \hat{\boldsymbol{J}}_{AB} + \hat{\boldsymbol{J}}_C$ is the total angular momentum for the whole system. Notice that $|(\frac{n_1}{2}\frac{n_C}{2})\frac{N}{2}-k,\frac{n_1}{2}+l_0\rangle$ is the eigenvector of $\hat{\boldsymbol{J}}^2$ with the eigenvalue $(\frac{N}{2}-k)(\frac{N}{2}-k+1)$, and we easily have (set $\hbar=1$) $\hat{\boldsymbol{J}}^2/m^2|(\frac{n_1}{2}\frac{n_C}{2})\frac{N}{2}-k,\frac{n_1}{2}+l_0\rangle \approx (1-\frac{k}{m})^2|(\frac{n_1}{2}\frac{n_C}{2})\frac{N}{2}-k,\frac{n_1}{2}+l_0\rangle$. Define $\hat{j}^\pm = \hat{j}^x \pm i\hat{j}^y$ for angular momentum operators, and then $\hat{\boldsymbol{J}}^2 = \hat{\boldsymbol{J}}_{AB}^2 + \hat{\boldsymbol{J}}_C^2 + 2\hat{J}_{AB}^z\hat{J}_C^z + \hat{J}_{AB}^+\hat{J}_C^- + \hat{J}_{AB}^-\hat{J}_C^+$. So, with some algebra, we can have $\hat{\boldsymbol{J}}^2/m^2|\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0\rangle \approx \cos^2\frac{\beta}{2}|\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0\rangle$. In the derivation above, the angular momentum relationships $\hat{j}^\pm|JM\rangle = \sqrt{J(J+1)-M(M\pm1)}|JM\pm1\rangle$ have been applied. Now, we know that the angular momentum vectors $|(\frac{n_1}{2}\frac{n_C}{2})\frac{N}{2}-k,\frac{n_1}{2}+l_0\rangle$ and $|\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0\rangle$ are both the eigenvectors of the same Hermitian operator $\hat{\boldsymbol{J}}^2/m^2$, and therefore the CG coefficient $C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0}^{\frac{N}{2}-k,\frac{n_1}{2}+l_0}$ vanishes unless the two corresponding eigenvalues are equal, say $(1-k/m)^2 \approx \cos^2(\beta/2)$ for large

m. Further notice that $\sum_{k=0}^{n_C} |C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0}^{\frac{N}{2}-k,\frac{n_1}{2}+l_0}|^2 = 1$, and we can conclude that $|C_{\frac{n_1}{2}\frac{n_1}{2},\frac{n_C}{2}l_0}^{\frac{N}{2}-k,\frac{n_1}{2}+l_0}|^2 \approx \frac{1}{m}\delta(\frac{k}{m} - 2\sin^2\frac{\beta}{4})$ as $m \to \infty$. Using the results above, Eq. (9) reduces to $P \approx (1 - \eta_1 q_{k,1}^{\rm opt} - \eta_2 q_{k,2}^{\rm opt})|_{k/m=2\sin^2(\beta/4)}$. Further from Eq. (3), and together with $O_k \approx (1 - k/m)^n$ for large m [9], finally one has

$$P_n \equiv \lim_{m\to\infty} P = \begin{cases} \eta_2\left(1 - \cos^{2n}\frac{\beta}{2}\right) & \text{for } \eta_1 < e, \\ 1 - 2\sqrt{\eta_1\eta_2}\cos^n\frac{\beta}{2} & \text{for } e \le \eta_1 \le f, \\ \eta_1\left(1 - \cos^{2n}\frac{\beta}{2}\right) & \text{for } \eta_1 > f, \end{cases} \tag{10}$$

with the boundaries $e = \cos^{2n}\frac{\beta}{2}/(1 + \cos^{2n}\frac{\beta}{2})$ and $f = 1/(1 + \cos^{2n}\frac{\beta}{2})$.

The asymptotic limit for $m \to \infty$ in Eq. (10) is exactly the same as that for the usual unambiguous discrimination between two known sates $|\phi_1\rangle^{\otimes n}$ and $|\phi_2\rangle^{\otimes n}$ with inner product $|^{\otimes n}\langle\phi_1|\phi_2\rangle^{\otimes n}| = \cos^n(\beta/2)$. Actually, this asymptotic limit can indeed be approached by the usual unambiguous discrimination strategy here. As m is infinite, we can also reconstruct the unknown states in program registers via quantum state tomography, and the problem comes down to determining whether the states in the data system are $|\phi_1\rangle^{\otimes n}$ or $|\phi_2\rangle^{\otimes n}$.

(iv) Although they received little attention before, optimal measurement operators are important in the universal unambiguous discrimination. The expressions of these operators in Eq. (2) reveal the symmetry properties in the universal unambiguous discrimination, which are rather useful in the derivation of our results in this paper. Before our work, the PSPs were achieved only for a few simple examples since the general case is much more complicated without the explicit expressions of the optimal measurement operators. More significantly, the expressions of these operators theoretically represent the measurements in the experiments, and with them one can design the optimal universal unambiguous discriminators in the laboratory.

IV. CONCLUSIONS AND SUMMARIES

We derive the analytic expressions of PSPs for the universal programmable unambiguous discriminators using the explicit expressions of the optimal measurement operators given in Ref. [9]. Since pure states are actually input, PSPs are the success probabilities when the device works, and we show that the optimal programmable unambiguous discriminator is equivalent to the usual unambiguous discrimination between the pure input states in the data register as the number of copies in the program registers goes to infinity, and equivalent to a series of state comparisons as the number of copies in the data register goes to infinity. Similar conclusions hold for the averaged input states, and moreover, our result is more general, allowing for the arbitrary *a priori* probabilities η_1 and η_2, arbitrary dimension d, and arbitrary number of copies in the registers. It is the PSPs, instead of the ASPs, that are directly observed in the laboratory, and hence the results in this paper will be useful and helpful for the experimental realization of the universal unambiguous discriminators in the future. We expect that our results could lead to further theoretical or experimental consequences.

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APPENDIX: PROOF OF THE LEMMA

First, notice that $|\phi_1\rangle$ and $|\phi_2\rangle$ are two states in $\mathcal{H}$, so there always exists a certain d-dimensional unitary transformation $u \in \mathrm{U}(d)$ such that

$$\begin{aligned} |\phi_1'\rangle &= u|\phi_1\rangle = |0\rangle, \\ |\phi_2'\rangle &= u|\phi_2\rangle = e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}|0\rangle + e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}|1\rangle, \end{aligned} \tag{A1}$$

where $|0\rangle$ and $|1\rangle$ are two orthogonal bases of $\mathcal{H}$, and the unknown parameters α, β, and γ are Euler angles of the rotation group with $\cos\frac{\beta}{2} = |\langle\phi_1|\phi_2\rangle|$, $\beta \in [0,\pi]$, and $\alpha,\gamma \in [0,2\pi)$. Without any knowledge about $|\phi_1\rangle$ and $|\phi_2\rangle$, it is impossible to determine the exact expression of u, but it does exist, and this is enough. Now, we have a new state $|\Phi_1'\rangle = u^{\otimes N}|\Phi_1\rangle$.

Secondly, recall that $|^{[\lambda]}_{\omega}\rangle_1 (\omega = 1,2,\ldots,d^{[\lambda]})$ are the irreducible bases for the representation labeled by $[\lambda]$, and then one can have $u^{\otimes N}\mathbb{1}^{[\lambda]}(u^{\otimes})^\dagger = \sum_{\omega=1}^{d^{[\lambda]}} |^{[\lambda]}_{\omega}\rangle\langle^{[\lambda]}_{\omega}|$, where the bases $|^{[\lambda]}_{\omega}\rangle (\omega = 1,2,\ldots,d^{[\lambda]})$ are the irreducible basis of both the permutation group S_N and the unitary group $\mathrm{U}(d)$, and can be transformed to $|^{[\lambda]}_{\omega}\rangle_1$ by the subduction coefficients (SDCs) of the permutation group S_N [21]. Then,

$$\langle\Phi_1|\mathbb{1}^{[\lambda]}|\Phi_1\rangle = \sum_{\omega=1}^{d^{[\lambda]}} \left\langle\Phi_1'\middle|\begin{matrix}[\lambda]\\ \omega\end{matrix}\right\rangle\left\langle\begin{matrix}[\lambda]\\ \omega\end{matrix}\middle|\Phi_1'\right\rangle. \tag{A2}$$

Thirdly, $|\phi_1'\rangle$ and $|\phi_2'\rangle$ can be regarded as states of the spin-1/2 system with $|0\rangle$ and $|1\rangle$ the spin-up and spin-down bases. Thus, for the state $|\Phi_1'\rangle$, the system consisting of A and B has the angular momentum $j_{AB} = n_1/2$, while the system C has $j_C = n_C/2$. Based on these, $|\Phi_1'\rangle$ can be expressed as a linear combination of angular momentum basis

$$|\Phi_1'\rangle = \sum_{l=-\frac{n_C}{2}}^{\frac{n_C}{2}} D_{l\frac{n_C}{2}}^{(\frac{n_C}{2})}(\alpha,\beta,\gamma)\left|\frac{n_1}{2}\frac{n_1}{2}\right\rangle\left|\frac{n_C}{2}l\right\rangle, \tag{A3}$$

with α, β, and γ the unknown parameters in Eq. (A1). The bases $|^{[\lambda]}_{\omega}\rangle$ which contribute to Eq. (A2) should be the eigenstates such as $|\frac{n_1}{2}\frac{n_C}{2}(\frac{N}{2}-k)M\rangle$ with the total angular momentum $J = N/2 - k$ and its z component M, where $k = 0,1,\ldots,n_C$ and $M = k - \frac{N}{2}, k - \frac{N}{2} + 1, \ldots, \frac{N}{2} - k$ for each k, and substituting this and Eq. (A3) into Eq. (A2), one can have Eq. (5). ■

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