Free-particle wave function and Niederer's transformation

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The solutions to the free Schrödinger equation discussed by P. Strange (arXiv:1309.6753) and A. Aiello (arXiv:1309.7899) are analyzed. It is shown that their properties can be explained with the help of Niederer's transformation.

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I. INTRODUCTION

Recently Strange [1] considered a specific solution to the one-dimensional free-particle Schrödinger equation in which the space and time dependence are not separable. The solution exhibits a few peaks in probability density which accelerate with time. It has been analyzed [1] in quantum and semiclassical regimes along the same lines as the Airy accelerating solution constructed by Berry and Balazs [2]. Strange's solution was further generalized to two dimensions by Aiello [3].

In the present Brief Report we show that the form and properties of such a solution are direct consequences of Niederer's transformation [4] relating, both on the classical and quantum levels, the harmonic oscillator and the free particle of the same mass. We generalize the results presented in Ref. [1] concerning the behavior of the probability density and its relation to the properties of the family of classical trajectories. It is also shown that the solutions under consideration can be easily generalized to any dimension.

II. NIEDERER'S TRANSFORMATION

Niederer [4] (see also Ref. [5]) constructed the mapping which transforms the harmonic oscillator motion into the free one. Given the d-dimensional oscillator described by the Lagrangian

$$L = \frac{m}{2}\dot{\vec{x}}^2 - \frac{m\omega^2}{2}\vec{x}^2, \tag{1}$$

we consider the following point transformation:

$$\tau = \frac{1}{\omega}\tan(\omega t), \quad t = \frac{1}{\omega}\arctan(\omega\tau),$$
$$\vec{y} = \frac{\vec{x}}{\cos(\omega t)}, \quad \vec{x} = \vec{y}(1+\omega^2\tau^2)^{-\frac{1}{2}}. \tag{2}$$

One easily checks the following identity:

$$L dt = \frac{m}{2}\left(\frac{d\vec{y}}{d\tau}\right)^2 d\tau - d\left(\frac{m\omega}{4}\sin(2\omega t)\vec{y}^2\right), \tag{3}$$

which tells us that $\vec{y}(\tau)$ describes free motion provided $\vec{x}(t)$ obeys the harmonic oscillator equation of motion. In particular, the mapping (2) transforms harmonic motion in the interval $(-\frac{\pi}{2\omega},\frac{\pi}{2\omega})$ (half of the period) into the free dynamics for $-\infty < t < \infty$.

Transformation described by Eqs. (2) has its quantum counterpart. Namely, if $\psi(\vec{x},t)$ obeys the Schrödinger equation for harmonic oscillator, then

$$\chi(\vec{y},\tau) = (1+\omega^2\tau^2)^{-\frac{1}{4}} e^{\frac{im\omega^2\tau}{2(1+\omega^2\tau)}\vec{y}^2}\psi \times \left(\vec{y}(1+\omega^2\tau^2)^{-\frac{1}{2}}, \frac{1}{\omega}\arctan(\omega\tau)\right) \tag{4}$$

is a solution to the free Schrödinger equation. The structure of the relation given by Eq. (4) is transparent. First, the arguments of the wave function are replaced by the appropriate functions of the new ones according to the classical formulas (2). Then two factors are added: The first one accounts for proper normalization while the second one is related to the fact that under the transformation (2) the Lagrangian transforms by a total derivative [cf. Eq. (3)].

The inverse transformation reads

$$\psi(\vec{x},t) = [\cos(\omega t)]^{-\frac{1}{2}} e^{-\frac{im\omega}{2}\tan(\omega t)\vec{x}^2}\chi\left(\frac{\vec{x}}{\cos(\omega t)}, \frac{1}{\omega}\tan\omega t\right). \tag{5}$$

Taking $d = 1$ and $\psi(x,t)$ as the eigenfunction of the harmonic oscillator Hamiltonian,

$$\psi(x,t) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}} e^{-i\omega(n+\frac{1}{2})t} e^{-\frac{m\omega x^2}{2}} H_n((m\omega)^{\frac{1}{2}}x), \tag{6}$$

one arrives at formula (2) from the Strange paper [1] provided the identification $\omega = \frac{1}{t_c}$ has been made.

One can also easily construct multidimensional solutions. For example, let us take $d = 2$ and let $\psi(\vec{x},t)$ be a common eigenfunction of energy and angular momentum; up to a normalization factor,

$$\psi_{n,l}(\vec{x},t) = e^{-i\omega\left(n+\frac{|l|}{2}+\frac{1}{2}\right)t} e^{-\frac{m\omega r^2}{2}} r^{|l|} F(-n,|l|+1,m\omega r^2). \tag{7}$$

Applying Eq. (4) to $\psi_{0,l}$, one arrives at the wave function considered by Aiello [3]. In a similar way one can construct three- and higher-dimensional examples.

Let us now come back to the one-dimensional case. We would like to analyze the behavior of the probability density for the solutions under consideration. Let $\psi_n(x,t)$ be the nth stationary solution. Then Eq. (4) implies

$$|\chi(y,\tau)|^2 = (1+\omega^2\tau^2)^{-\frac{1}{2}}\rho_n(y(1+\omega^2\tau^2)^{-\frac{1}{2}}), \tag{8}$$

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$\rho_n(x)$ being a time-dependent probability density for the stationary state of the harmonic oscillator. Now, $\rho_n(x)$ has $n+1$ maxima. Therefore, the probability density $|\chi(y,\tau)|^2$ has $n+1$ peaks $y_n(\tau)$ which travel with time along hyperbolic trajectories,

$$y_n(\tau) = y_n(0)(1+\omega^2\tau^2)^{\frac{1}{2}}, \tag{9}$$

and broaden according to $(1+\omega^2\tau^2)^{\frac{1}{2}}$ law. This generalizes to any n the findings of Ref. [1] for $n = 2$.

Berry and Balazs [2] found the solution to the free Schrödinger equation in the form of a wave packet propagating in space without distortion and with constant acceleration. Although not being square integrable it does not provide a counterexample to the Ehrenfest theorem, the appearance of accelerated motion for the free particle is slightly disturbing. Therefore, Berry and Balazs proposed an explanation of this fact based on considering the family of classical trajectories that are related, via semiclassical analysis, to the wave function under consideration; the acceleration of the packet is ascribed to the curvature of the envelope of this family.

In the case of the solution considered by Strange, which is normalizable, the Ehrenfest theorem cannot be broken. However, the probability density possesses several peaks which travel with acceleration. As shown by Strange in the $n = 2$ case, their acceleration can be also related to the behavior of some family of classical trajectories. The picture based on Niederer's transformation allows us to give a simple interpretation for any n. The nth eigenstate of the harmonic oscillator Hamiltonian corresponds to the family of classical trajectories parametrized by an angle α,

$$x(t,\alpha) = \sqrt{\frac{2E_n}{m\omega^2}}\cos(\omega t+\alpha), \tag{10}$$

which, in view of Eq. (2), yields the family of free trajectories,

$$y(\tau,\alpha) = \sqrt{\frac{2E_n}{m\omega^2}}(\cos\alpha - \omega\tau\sin\alpha). \tag{11}$$

The envelope of this family is obtained by solving

$$\frac{\partial y(\tau,\alpha)}{\partial\alpha} = 0 \tag{12}$$

with respect to α and reinserting into Eq. (11). This yields

$$y(\tau) = \pm\sqrt{\frac{2E_n}{m\omega^2}}(1+\omega^2\tau^2)^{\frac{1}{2}}. \tag{13}$$

Now, $x_\pm = \pm\sqrt{\frac{2E_n}{m\omega^2}}$ are the classical turning points restricting the classically allowed region for the harmonic oscillator motion with the energy E_n. In the semiclassical regime, $n \gg 1$, $n\hbar$ fixed, the extreme maxima of probability density are placed at points differing by an $O(\frac{1}{n})$ distance from the turning points. Therefore, Eq. (4) tells us that Eq. (13) describes, in the semiclassical limit, the motion of extreme maxima of the probability density for the corresponding free Schrödinger equation.

III. CONCLUSION

We have shown that the "exotic" solutions to the free Schrödinger equation considered recently [1,3] can be easily generated by using Niederer's transformation [4,5]. Their slightly peculiar properties are naturally explained in terms of this transformation.

We considered here only solutions to the free Schrödinger equation, which are the images, under the Niederer transformation, of stationary states of harmonic oscillators. However, there are also other interesting states of the harmonic oscillator which could generate some interesting free wave functions. For example, coherent states yield the Gaussian packet of constant width oscillating according to the classical equations of motion. This and other similar cases will be studied elsewhere.

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