Scaling laws, pressure anisotropy, and thermodynamic effects for blackbody radiation in a finite cavity

Anatoly A. Sokolsky and Maxim A. Gorlach*
Physics Department of the Belarusian State University, 4 Nezalejnosty av., Minsk, 220030, Belarus
(Received 6 November 2013; published 29 January 2014)

Finite-size effects for blackbody radiation in a cavity are revealed. We find out simple scaling laws for thermodynamic functions of blackbody radiation in a finite cavity of arbitrary shape and predict the anisotropy of blackbody radiation pressure in asymmetric cavities. Special thermodynamic effects accompanying cavities merge are discussed.

DOI: 10.1103/PhysRevA.89.013847 PACS number(s): 42.50.Pq, 42.50.Ct, 44.40.+a

I. INTRODUCTION

In 1900 Planck introduced the quantum hypothesis and derived the formula for energy spectral density of blackbody radiation (BBR) in a cavity [1]. This formula was verified experimentally in a wide range of temperatures and frequencies. In the derivation of Planck's formula it is implied that the volume of the cavity containing BBR is sufficiently large (see, e.g. [2], p. 273). Therefore, Planck's formula is inapplicable to BBR in a small cavity at a sufficiently low temperature. A concrete criterion of the Planck's formula validity was formulated by Bijl [3]:

$$T V^{1/3} \gg B \equiv \frac{\hbar c}{k_B} \approx 0.2290 \text{ cm K}, \quad (1)$$

where T is the radiation temperature and V is the cavity volume. The criterion (1) refers also to the Planck's formula consequences (e.g., the Stefan-Boltzmann law, Wien's displacement law, etc.). The following step was done in [4] where the first correction terms to the Stefan-Boltzmann law were calculated for the case of a cubic cavity with ideally conducting walls. Since that time, numerous efforts were made in order to take into account the finite-size effects in thermal radiation theory. The list of main research activities in this field, far from being complete, is given below.

(i) The Weyl's problem [5], which consists in the calculation of the eigenvalues distribution of the vector wave equation in cavities of various shapes and derivation of averaged expressions for the electromagnetic mode density [6–8].

(ii) Refinements of the Stefan-Boltzmann formula and study of cavities having various shapes with ideally conducting walls [6,8,9].

(iii) Corrections to the Stefan-Boltzmann law due to a finite conductivity of cavity walls [10].

(iv) Consideration of radiation from small particles and exploration of the particle size influence on the thermal radiation spectrum [11,12].

The interest in the finite-size effects is growing at the present time due to the increased experimental capabilities. Calculation of thermodynamic functions of a hot quark-gluon plasma was performed in [13] with account of the finite-size effects. In [14] it was shown that experimental detection of the deviations from Planck's formula is within the reach of current experimental capabilities. Finally, in [12] the experimental observation of such deviations was reported: it was detected that in narrow spectral range thermal radiation may exceed the value predicted by Planck's formula. A consistent analysis of the role of the geometry-dependent corrections to Planck's formula is also carried out in [12].

In the present paper, we reveal three finite-size effects for BBR.

(i) Simple scaling laws determining the behavior of BBR thermodynamic functions (energy, pressure, entropy, etc.) when the temperature and/or the cavity volume (at a fixed shape) is changed.

(ii) The existence of essential anisotropy of blackbody radiation pressure due to the cavity asymmetry.

(iii) Special thermodynamic effects taking place when finite cavities with BBR at the same temperature and pressure are merged.

Dealing with these problems, we specified some details of the thermodynamic approach to blackbody radiation in a finite cavity and performed corresponding numerical calculations. The results of numerical calculations illustrate the discussed effects.

II. THERMODYNAMIC RELATIONS FOR BLACKBODY RADIATION IN A FINITE CAVITY AND SCALING LAWS

Let us consider blackbody radiation in a finite cavity of arbitrary shape. In the present paper, we discuss the case of a closed cavity with ideally conducting walls. According to [10], the results obtained under this assumption also give a good approximation of BBR thermodynamic parameters for the case of a cavity with metallic walls of finite conductivity. For the range of conductivities (10^6–10^8) Ohm^{-1} m^{-1} the relative error is less than 0.5%.

It is well known that electromagnetic field in the cavity may be represented as an ensemble of harmonic oscillators corresponding to the eigenmodes of the cavity. The number of these oscillators is infinite for the cavity of arbitrary volume and at any temperature. The total field energy is a sum of oscillator energies: $E = \sum_m E_m$, where m enumerates oscillators of the ensemble [2,15]. As oscillators are noninteracting, statistically independent, and distinguishable, the Gibbs distribution is applicable to the single oscillator:

$$w_m(r) = \exp\left[\frac{F_m - E_m(r)}{k_B T}\right], \quad (2)$$

*Maxim.Gorlach.blr@gmail.com

 ©2014 American Physical Society

where r enumerates states of the mth oscillator and the normalization factor is taken in the form $\exp(\frac{F_m}{k_B T})$; F_m is a free energy of the mth eigenmode. Relation (2) is valid for radiation in small cavities at low temperatures, as well as for "standard" [i.e., corresponding to satisfaction of the Bijl's criterion (1)] case.

Applying the canonical distribution method and using the quantum-mechanical expression for the harmonic-oscillator energy levels $E_m(r) = \hbar\omega_m(r + \frac{1}{2})$, where $r = 0,1,2,\ldots$, ω_m is the frequency of the mth mode, one obtains the expression for the BBR free energy $F \equiv \sum_m F_m$:

$$F = k_B T \sum_m \ln\left[1 - \exp\left(-\frac{\hbar\omega_m}{k_B T}\right)\right]. \tag{3}$$

We omit the vacuum term $F_{\mathrm{vac}} = \frac{1}{2}\sum_m \hbar\omega_m$ in (3), concerning ourselves only with the thermal radiation energy (as a result, the Casimir forces and other vacuum effects will not be taken into account).

The eigenfrequencies of the cavity ω_m depend not only on its volume but also on its geometrical shape. That is why it is reasonable to represent the BBR free energy as a function of the temperature T, cavity volume V, and some dimensionless parameters λ_i defining the cavity shape: $F = F(T,V,\lambda_i)$. The number of such parameters depends on the problem under consideration (for example, only two parameters are needed to describe a cuboid shape). As a consequence, we obtain

$$dF = -S\,dT - p\,dV + \sum_i \Lambda_i\,d\lambda_i, \tag{4}$$

where the BBR entropy S, the BBR pressure p, and the coefficients Λ_i are defined as

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,\lambda_i}, \quad p = -\left(\frac{\partial F}{\partial V}\right)_{T,\lambda_i}, \quad \Lambda_i = \left(\frac{\partial F}{\partial \lambda_i}\right)_{T,V,\lambda_k\neq\lambda_i}. \tag{5}$$

The variable p should be understood as the pressure on the cavity faces averaged in a certain way. The relation between the parameter p and forces acting on cavity walls may be not so simple as in the standard case (see Sec. IV). The expression for the differential of BBR internal energy $E = F + TS$ follows from (4):

$$dE = T\,dS - p\,dV + \sum_i \Lambda_i\,d\lambda_i. \tag{6}$$

We emphasize that the used definition of the internal energy is equivalent to the following one: $E = \sum_{m,r} w_m(r)E_m(r)$. The basic equation (6) is valid for a cavity of arbitrary shape.

Now we introduce dimensionless eigenfrequencies $\tilde{\omega}_m = \omega_m V^{1/3}/c$ (c is the speed of light). As the result, Eq. (3) reads

$$F = k_B T \sum_m \ln\left[1 - \exp\left(-\frac{\hbar c\tilde{\omega}_m}{k_B T V^{1/3}}\right)\right]. \tag{7}$$

From similarity considerations it follows that if the cavity volume is changed at a fixed cavity shape, parameters $\tilde{\omega}_m = \omega_m V^{1/3}/c$ remain constant. Thus we obtain simple scaling law for the BBR free energy in the cavity of arbitrary shape:

$$F = T f(T V^{1/3}), \tag{8}$$

where the function f is fully determined by the cavity geometry. If this function of single variable is calculated (or found from experiment) for the given cavity shape, any of the BBR thermodynamic parameters at any temperature and cavity volume may be easily evaluated. Basing on the Eq. (8), it is easy to obtain corresponding scaling laws in different forms, for example,

$$F = -\frac{4\sigma}{3c}T^4 V\Phi_1(\tau), \quad p = \frac{4\sigma}{3c}T^4\Phi_2(\tau),$$
$$E = \frac{4\sigma}{c}T^4 V\Phi_2(\tau), \quad S = \frac{16\sigma}{3c}T^3 V\Phi_3(\tau), \tag{9}$$

where $\Phi_2(\tau) = \Phi_1(\tau) + \frac{\tau}{3}\frac{d\Phi_1}{d\tau}$, $\Phi_3(\tau) = \Phi_1(\tau) + \frac{\tau}{4}\frac{d\Phi_1}{d\tau}$, $\tau = TV^{1/3}$, and σ is the Stefan-Boltzmann constant. One passes to the standard case in the limit $\tau \to \infty$. In this limit $\Phi_j(\tau) \to 1$ $j = 1,2,3$.

Equations (9) show that the well-known standard relation between p and E is also correct for BBR in a finite cavity [with definition of p from Eq. (5)]:

$$E = 3pV. \tag{10}$$

Asymptotic expressions describing BBR internal energy were deduced for particular cases (e.g., cubic cavity) in [4,8]. These expressions are valid only in limiting cases under the following conditions: either $TV^{1/3} \gg B$ (high-temperature expansion) or $TV^{1/3} \ll B$ (low-temperature expansion). Description of the intermediate temperature region requires numerical computation [see Fig. 1(a)].

III. COMPUTATION TECHNIQUE

To reduce the time needed for numerical calculation of BBR thermodynamic functions, we used the following method. Calculating sums like (7), we used direct summation for eigenfrequencies $\tilde{\omega}_m \leqslant \tilde{\omega}_e$. For eigenfrequencies $\tilde{\omega}_m > \tilde{\omega}_e$ we replaced the rest of the sum by the corresponding integral. For example,

$$\frac{E}{k_B T} = \sum_{\substack{n,\\ \tilde{\omega}_n \leq \tilde{\omega}_e}} g_n \frac{B\tilde{\omega}_n}{Ta}\left[\exp\left(\frac{B\tilde{\omega}_n}{Ta}\right) - 1\right]^{-1} + \int_{\tilde{\omega}_e}^{\infty} \frac{B\tilde{\omega}^3}{\pi^2 Ta}\left[\exp\left(\frac{B\tilde{\omega}}{Ta}\right) - 1\right]^{-1} d\tilde{\omega}, \tag{11}$$

where $B = \frac{\hbar c}{k_B} \approx 0.2290$ cm K, n enumerates different eigenfrequencies, g_n is a degeneracy of the eigenfrequency ω_n, and $a = V^{1/3}$. The other thermodynamic functions were calculated similarly. The net result depends to some extent on the choice of $\tilde{\omega}_e$. This unwanted circumstance may be removed by variation of the $\tilde{\omega}_e$ until the net result becomes insensible to the increase of $\tilde{\omega}_e$. Typical values of the parameter $\tilde{\omega}_e$ used in the calculations were 50 and 100. The computational error estimated by the comparison of the two results was less than 0.05%.

Further we consider the particular case of cuboid cavity with edges X, Y, Z. The shape of such cavity is described by

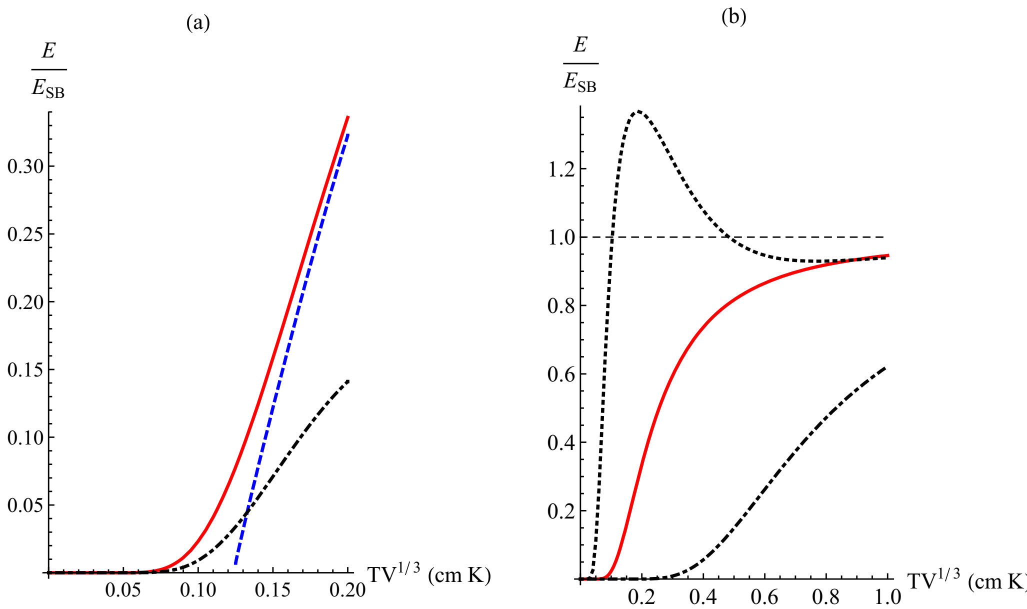

FIG. 1. (Color online) Radiation energy in a cavity normalized to the value predicted by the Stefan-Boltzmann law. (a) Cubic cavity: solid, dashed, and dash-dotted curves correspond to calculated results, high-temperature expansion [9], and low-temperature expansion [9], respectively. (b) Cuboid cavities of different shapes: solid, dotted, and dash-dotted curves correspond to $\alpha = \beta = 1$ (cube), $\alpha = \beta = 10$ (film), and $\alpha = \beta = 10^{-2}$ (rod), respectively.

two dimensionless parameters $\alpha = X/Z$ and $\beta = Y/Z$. Then the normalized eigenfrequencies of the cavity are

$$\tilde{\omega}_n \equiv \frac{\omega_n a}{c} = \frac{\pi}{(\alpha\beta)^{2/3}}\sqrt{n_x^2\beta^2 + n_y^2\alpha^2 + n_z^2\alpha^2\beta^2}, \quad (12)$$

where $n = \{n_x, n_y, n_z\}$. The mode exists if two or three numbers from the set n are nonzero; in the first case degeneracy is $g_n = 1$ and in the second—$g_n = 2$ (additional degeneracy may appear due to the cavity symmetry) [16]. The obtained results for the BBR internal energy are presented in Figs. 1(a) and 1(b). Figure 1(a) illustrates the domain of validity of the asymptotic formulas [9] for the radiation energy. Comprehensive discussion of these asymptotic formulas for a cubic cavity is given in [9]. The influence of the cavity shape on thermodynamic functions of BBR is illustrated in Fig. 1(b).

IV. ANISOTROPY OF BLACKBODY RADIATION PRESSURE IN A FINITE CAVITY

It turns out that the effect of pressure anisotropy takes place for blackbody radiation in a finite cuboid cavity with unequal edges. That is to say, the pressures on the cavity faces defined as $p_x = -\frac{1}{YZ}(\frac{\partial F}{\partial X})_{T,Y,Z}$, $p_y = -\frac{1}{XZ}(\frac{\partial F}{\partial Y})_{T,X,Z}$, and $p_z = -\frac{1}{XY}(\frac{\partial F}{\partial Z})_{T,X,Y}$ may be unequal in the region of sufficiently low temperatures. According to (7) and (12),

$$p_x = \frac{\pi\hbar c}{V}\sum_n \frac{g_n\left(\frac{n_x}{X}\right)^2}{\left[\exp\left(\frac{\hbar\omega_n}{k_B T}\right) - 1\right]\sqrt{\left(\frac{n_x}{X}\right)^2 + \left(\frac{n_y}{Y}\right)^2 + \left(\frac{n_z}{Z}\right)^2}}, \quad (13)$$

where g_n is a degeneracy of the eigenfrequency ω_n. The similar expressions are valid for p_y and p_z. From these expressions it follows that $p_x + p_y + p_z = 3p$ [p is defined by Eq. (5)]; this relation is also obvious from Eq. (10) and from the fact that energy-momentum tensor trace equals zero for the case of free electromagnetic field. Numerical calculation of the pressures p_x, p_y, p_z for the case of asymmetric cavity shows that the pressures are unequal in the region of sufficiently low temperatures [Figs. 2(a) and 2(b)]. One can see that these quantities coincide in the region $TV^{1/3} \gg B$ in accordance with the fact that in the high-temperature region the radiation pressure is always isotropic regardless of the cavity shape. The limits of expressions p_x/p, p_y/p, and p_z/p at $T \to 0$ may be also found analytically. If we consider the case $X < Y < Z$, the smallest eigenfrequency ω_1 corresponding to the set $n_{\text{low}} = \{0,1,1\}$ is $\omega_1 = \pi c\sqrt{Y^{-2} + Z^{-2}}$. It is this frequency that gives a main contribution to the sum (13) at $T \to 0$. Omitting the other terms of the sum, one may find that $p_x \approx 0$, $p_y \approx \frac{A}{Y^2}$, and $p_z \approx \frac{A}{Z^2}$, where the parameter A is defined as $A = \frac{\pi\hbar c}{V}\{[\exp(\frac{\hbar\omega_1}{k_B T}) - 1]\sqrt{Y^{-2} + Z^{-2}}\}^{-1}$. Consequently, $p = \frac{1}{3}(p_x + p_y + p_z) \approx A\frac{Z^2 + Y^2}{3Y^2Z^2}$ and

$$p_x/p \to 0, \quad p_y/p \to \frac{3Z^2}{Y^2 + Z^2},$$
$$p_z/p \to \frac{3Y^2}{Y^2 + Z^2}. \quad (14)$$

For the case $X = 1$ mm, $Y = 2$ mm, and $Z = 3$ mm relations (14) give $p_x/p \to 0$, $p_y/p \to 2.077$, and $p_z/p \to 0.923$. These results are in agreement with ones obtained from the numerical calculations. Consequently, if $X < Y < Z$, we have $p_x < p_z < p_y$ at $T \to 0$. We emphasize that the pressure anisotropy of BBR should also take place for all asymmetric cavities under the condition $TV^{1/3} \lesssim B$. The illustration of the effect is given in Fig. 3.

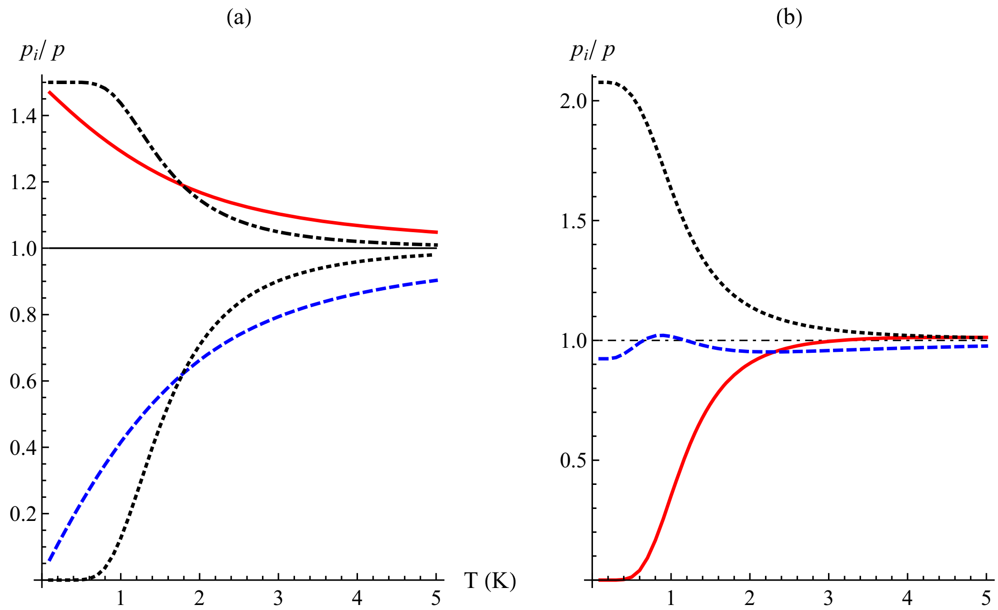

FIG. 2. (Color online) Pressures on the cavity faces normalized to the "average" pressure $p = \frac{E}{3V}$ for cavities of different shapes. (a) Solid and dashed curves correspond to $p_x/p = p_y/p$ and p_z/p for the rod with edges $X = Y = 1$ mm, and $Z = 10$ mm; dash-dotted and dotted curves correspond to $p_x/p = p_y/p$ and p_z/p for the film with edges $X = Y = 10$ mm, and $Z = 1$ mm. (b) Solid, dotted, and dashed curves correspond to p_x/p, p_y/p, and p_z/p for the cuboid with edges $X = 1$ mm, $Y = 2$ mm, and $Z = 3$ mm.

V. FINITE-SIZE EFFECTS ACCOMPANYING CAVITIES MERGE

Thus some of standard BBR thermodynamic properties are invalid in the temperature region $TV^{1/3} \lesssim B$, e.g., BBR free energy appears to be a nonlinear, shape-dependent function of the cavity volume. This affects BBR "behavior" in concrete thermodynamic processes. Below we will analyze the situation where the process takes place due to the finite-size effects, though nothing may occur in the system from the standard point of view.

Let us consider two identical cubic cavities separated by a partition. Then one can imagine that the partition is removed and one cavity of doubled length and volume appears. If temperatures and pressures of BBR in both cavities are equal, nothing will occur in the system from the standard point of view. It turns out that the behavior of the system is another in the intermediate temperature region $TV^{1/3} \lesssim B$.

It is of interest to consider several variants of the cavities merge. These variants are sketched in Fig. 4. If the partition is simply destroyed [Fig. 4(a)], the total energy of the system would conserve ($E =$ const) and the merge of cavities would be a nonequilibrium process. The temperature of the system decreases in this case [Fig. 5(a)] and the entropy increases [Fig. 5(b)]. The qualitative explanation of the effect is as follows. When the partitions between cubic cavities are destroyed, new eigenfrequencies appear for the electromagnetic field inside a composite cavity. That is why the statistical weight of a given system state must increase (when new modes appear, the number of ways to distribute the energy between them increases). Hence the entropy increases. The radiation energy will be redistributed between all modes of the obtained composite cavity. In fact, the energy passes from higher eigenfrequencies to the lower ones (which appear in the composite cavity) and such a redistribution means

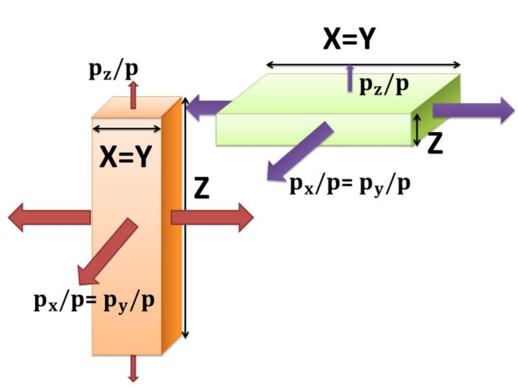

FIG. 3. (Color online) Illustration of the radiation pressure anisotropy in a finite cavity.

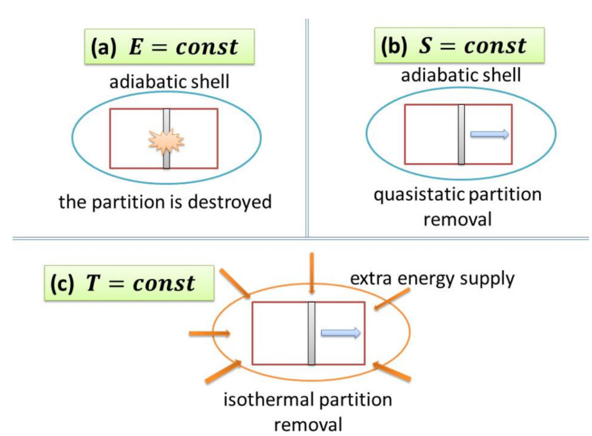

FIG. 4. (Color online) Different variants of cavities merge.

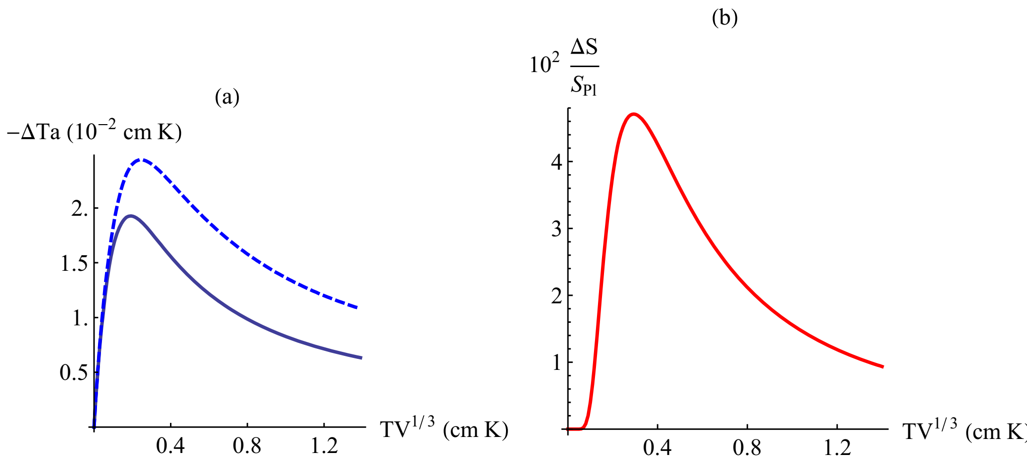

FIG. 5. (Color online) (a) Temperature decrease for BBR in 50 cubes when the partitions between them are removed. Solid and dashed curves correspond to the situations $E = \text{const}$ and $S = \text{const}$, respectively. (b) Entropy increase normalized to the total entropy calculated by the "standard" way for BBR in 50 cubes when the partitions between them are destroyed. Situation $E = \text{const}$. V is a volume of a single cube.

temperature decrease. Another variant is a quasistatic partition removal when the system is enclosed into the adiabatic shell [Fig. 4(b)]. In this case the total entropy S remains constant, the temperature decreases [Fig. 5(a)], and the total energy of the system decreases too [Fig. 6(a)]. The third case shown in Fig. 4(c) corresponds to a quasistatic isothermal partition removal, i.e., there are external energy sources keeping the temperature of the system constant. In this situation, energy supply is needed [Fig. 6(b)]. Calculations are performed for the case when 50 identical cubic cavities are merged.

As the discussed effects are inessential in the region of rather high temperatures and big cavity volumes, they are finite-size effects.

VI. DISCUSSION AND CONCLUSIONS

In the present work, we have explored some specific properties of blackbody radiation in a finite cavity. All the discussed regularities are essential in the case of low temperatures and small cavities. As BBR thermodynamic functions depend essentially on the cavity geometry, the standard thermodynamic approach requires specification. We have derived simple scaling laws for the BBR thermodynamic functions. In the low-temperature region the anisitropy of blackbody radiation pressure in asymmetric cavities is predicted. Special thermodynamic effects accompanying cavities merge are revealed: if the partition between adjacent identical cubic cavities with BBR at the same temperature and pressure is removed, the change of some BBR thermodynamic functions takes place. Qualitative explanation of the above effects is proposed.

ACKNOWLEDGMENT

The authors are grateful to Ilya Feranchuk for his interest in this work.

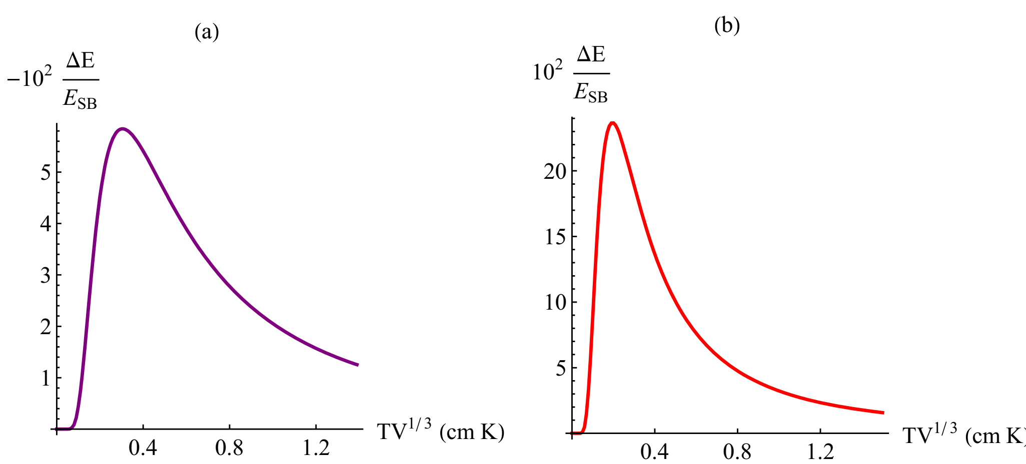

FIG. 6. (Color online) (a) Energy decrease normalized to the total energy calculated by the Stefan-Boltzmann law for BBR in 50 cubes when the partitions between them are removed adiabatically: $S = \text{const}$. (b) Energy increase normalized to the total energy calculated by the Stefan-Boltzmann law for BBR in 50 cubes when the partitions between them are removed isothermally: $T = \text{const}$.

[1] M. Planck, Verh. Dtsch. Phys. Ges. **2**, 237 (1900).
[2] M. Planck, *Introduction to Theoretical Physics. Vol. 5. Theory of Heat* (Macmillan, London, 1949).
[3] D. Bijl, Philos. Mag. **43**, 1342 (1952).
[4] K. Case and S. Chiu, Phys. Rev. A **1**, 1170 (1970).
[5] H. Weyl, Math. Ann. **71**, 441 (1912).
[6] H. Baltes, Helv. Phys. Acta **45**, 481 (1972).
[7] H. Baltes, Infrared Phys. **16**, 1 (1976).
[8] H. Baltes and E. Hilf, *Spectra of Finite Systems* (Bibliographisches Institut, Manheim, 1976).
[9] H. Baltes, Appl. Phys. **1**, 39 (1973).
[10] W. Eckhardt, Opt. Commun. **14**, 95 (1975).
[11] W. McGregor, J. Quant. Spectrosc. Radiat. Transfer **19**, 659 (1978).
[12] A. Reiser and L. Schächter, Phys. Rev. A **87**, 033801 (2013).
[13] F. Gliozzi, J. Phys. A: Math. Theor. **40**, 375 (2007).
[14] A. M. García-García, Phys. Rev. A **78**, 023806 (2008).
[15] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, 4th ed. (Butterworth-Heinemann, London, 1975).
[16] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon Press, New York, 1984).