Relationship between the degree of polarization, indistinguishability, and entanglement

F. De Zela
Departamento de Ciencias, Sección Física, Pontificia Universidad Católica del Perú, Apartado 1761, Lima, Peru
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We address a recently established inequality $\mathcal{P} \geqslant \mathcal{I}$ that constrains the degrees of polarization $\mathcal{P}$ and indistinguishability $\mathcal{I}$. We derive said inequality within three different frameworks, discussing its respective physical meaning. We show that in its original formulation the inequality involved a single degree of freedom, and that only by entangling polarization and path (momentum) in laboratory space, can such an inequality represent a constraint between these degrees of freedom. We show how this could be done with the help of a Mach-Zehnder-like array. We discuss this multipurpose device, which can be employed to address several issues of current interest, such as tests of the complementarity principle, partial coherence stemming from unobserved degrees of freedom, geometric phases of entangled states evolving on the Schmidt sphere, etc. Besides its experimental feasibility, the proposed device serves as a tool for studying common features of quantum and classical entangled states. In particular, it serves for testing a newly proposed measure of coherence, called Bell's measure, using experimental techniques that are independent of those already employed.

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I. INTRODUCTORY REMARKS

It has been recently argued [1] that the degree of polarization ($\mathcal{P}$) and Mandel's degree of indistinguishability [2] ($\mathcal{I}$) are mutually constrained by $\mathcal{P} \geqslant \mathcal{I}$. This remarkable result falls in line with other constraints that address quantum complementarity. This key concept of quantum theory states that full knowledge of some property precludes full knowledge of a conjugate one. The most representative example of this feature is wave-particle duality. When dealing, for instance, with interferometric arrays, "which-path information" (WPI) and visibility $\mathcal{V}$ of the interference pattern are known to be complementary quantities. Now, while wavelike behavior has been quantified in terms of visibility since the early years of classical optics, it is only recently that several measures have been proposed to quantify particlelike behavior, or WPI. One of these measures is Mandel's $\mathcal{I}$, which satisfies $\mathcal{I} \geqslant \mathcal{V}$ [2]. Another one is the so-called "predictability" $\mathcal{W} := |w_2 - w_1|$, where w_i represents the probability that a particle entering a two-arm interferometer goes along path $i = 1,2$. In this case, it holds $\mathcal{W}^2 + \mathcal{V}^2 \leqslant 1$ [3]. Alternatively, one can use $\mathcal{D}$, a parameter introduced by Englert [4], dubbed "distinguishability," which satisfies $\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$.

Whereas quantities such as $\mathcal{I}$ and $\mathcal{W}$ quantify our *a priori* which-way knowledge, $\mathcal{D}$ refers to the available WPI being stored in a which-way marker or detector [4]. Thus, $\mathcal{D}$ requires coupling the path degree of freedom to an auxiliary physical system that serves as a marker through the measurement of some adequate observable. This distinguishing feature that puts $\mathcal{I}$ and $\mathcal{W}$ on one side and $\mathcal{D}$ on the other, is also reflected by the mathematical derivation of the corresponding inequalities. The inequalities satisfied by $\mathcal{I}$ and $\mathcal{W}$ are derived within a single Hilbert space, while the inequality satisfied by $\mathcal{D}$ involves two Hilbert spaces. As for experimental tests, the constraint $\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$ has been confirmed under various configurations that include all-or-nothing cases ($\mathcal{D},\mathcal{V} \in \{0,1\}$) [5], intermediate situations ($\mathcal{D},\mathcal{V} \in]0,1[$) [6], and delayed-choice conditions [7]. The constraint $\mathcal{P} \geqslant \mathcal{I}$ appears to be of the kind of Englert's inequality, i.e., one which involves two degrees of freedom (DOF). If this would be so, then a series of experimental tests could be performed in order to exhibit a complementarity between $\mathcal{P}$ and $\mathcal{I}$ that would mirror the one holding between $\mathcal{V}$ and $\mathcal{D}$. Thus, besides fringe visibility, also polarization would depend on whether WPI is available or not. Now, it is quite astonishing that polarization and wavelike behavior happen to be complementary properties, because producing fully polarized electromagnetic waves is just as possible as producing fully polarized photons. However, when $\mathcal{P} = \mathcal{I}$, a photon behaving like a particle ($\mathcal{I} = 0$) should be completely unpolarized, while with no available WPI ($\mathcal{I} = 1$), light should behave as a completely polarized wave. Furthermore, experimental facts seem to be at odds with polarization being constrained by WPI. Consider, for example, a polarizing beam splitter. It can fully determine a photon's polarization just in connection with path information. As a second example, consider spin-1/2 particles. A constraint similar to $\mathcal{P} \geqslant \mathcal{I}$ would apply in this case as well, due to the kinematical identity between all binary DOF. However, the standard way to fix the spin of a particle is to let it pass through a Stern-Gerlach apparatus and then select one of the two possible paths that the particle can take, thereby "polarizing" it precisely in connection with WPI. These seemingly paradoxical situations prompt us to take a closer look at the derivation of the aforementioned inequality that was presented in [1]. As we will show below, in order to give physical sense to a constraint like $\mathcal{P} \geqslant \mathcal{I}$, two DOF must be involved. That is, by entangling polarization with a second degree of freedom, e.g., the path of the light's beam, $\mathcal{P}$ becomes constrained. This constraint comes from the unresolved degree of freedom with which polarization has been entangled.

The above state of affairs has recently attracted much attention, sparking a series of contributions within the general framework of entanglement and coherence theory [8–13]. It has been realized that partial coherence is not exclusively attributable to random fluctuations, but also to the act of ignoring a degree of freedom with which the observed one is entangled. Kagalwala *et al.* [13], for example, recently explored this issue by dealing with one-dimensional scalar beams such as $E(x) = E_e\psi_e(x) + E_o\psi_o(x)$. This can be seen as an entangled state, as $E(x)$ exists simultaneously in two independent vector spaces. The components E_e and E_o belong to one space, and $\psi_e(x),\psi_o(x)$ to another one, namely, the space of even and odd functions [13]. It should be stressed that entanglement is not necessarily related to nonlocality, which is often referred to as the most prominent feature of quantum mechanics. Indeed, entanglement becomes possible whenever we deal with two vector spaces. There can be different types of entanglement, depending on the nature of the degrees of freedom being involved. There can be, for instance, entanglement between two different modes, such as polarization and path, that are carried along by a single entity, be it a classical light beam or a photon. Alternatively, we can consider a single mode, e.g., spin, being attached to two different, spatially separated particles. This is the case usually addressed in quantum mechanics when dealing with nonlocality.

It is thus possible to have entangled modes in the classical, as well as in the quantum domain. Once the kinematic nature of entanglement has been recognized [9,14–17], fruitful results can be derived. For example, Qian and Eberly have shown that the degree of polarization can be thought of as a measure of entanglement between two degrees of freedom [17]. The two DOF being considered in this case are represented, respectively, by vectors in "laboratory space," Span$\{\mathbf{x},\mathbf{y}\}$, and by vectors in "function space," whose elements are statistical functions E_x,E_y. These are the components of a transverse field $\mathbf{E}=\mathbf{x}E_x+\mathbf{y}E_y$ in the case of a beam. Now, the degree of polarization is usually introduced as a measure of statistical correlations between E_x and E_y. By bringing entanglement to the fore, Qian and Eberly [17] have shown how to generalize polarization beyond the planar-transverse case, as dimensionality of the two entangled spaces becomes irrelevant within their framework. This is so because the tool used to define polarization is Schmidt decomposition, which applies irrespective of dimensionality. Thus, once we have realized that the light field is an intrinsically entangled field, polarization appears as a concept that captures both the vector nature and the statistical nature of that field [17]. Analogous considerations apply for any two entangled degrees of freedom, e.g., those addressed in [13].

Mandel's approach [2] to interference of light beams can be seen under the same perspective. It highlights the connection between statistical correlations and indistinguishability of two interfering beams. The states being considered in this case are of the form $|\psi\rangle=\alpha_1|\psi_1\rangle+\alpha_2|\psi_2\rangle$, with $|\psi_{i=1,2}\rangle$ representing the two-way alternative of an interferometric array. There is a close relationship between the approaches followed by Mandel and by Qian and Eberly. Indeed, in both cases one deals with a space of statistical functions: analytic signals in one case, and field components in the other case. As for the second space, it is laboratory space in Qian-Eberly's treatment, i.e., a two-dimensional one for a beam-type field. In Mandel's case, the second space is path space, which is also two dimensional. As we show below, there is a formal identity between Mandel's degree of indistinguishability $\mathcal{I}$ and a corresponding quantity in Qian-Eberly's approach to polarization. However, such a formal identity should not be promoted to a physical identity, as a consequence of which wave-particle duality could be addressed within the two frameworks. In order to connect polarization and indistinguishability we must go beyond the intrinsic entanglement that Qian-Eberly and Mandel have considered. Thus, we must proceed similarly to Kagalwala *et al.* [13], who managed to entangle two otherwise independent degrees of freedom: polarization and spatial parity. This kind of entanglement should be distinguished from the intrinsic entanglement that was recognized by Qian and Eberly when dealing with polarized states of the form $\mathbf{E}=\mathbf{x}E_x+\mathbf{y}E_y$, or by Mandel when dealing with interferometric states of the form $|\psi\rangle=\alpha_1|\psi_1\rangle+\alpha_2|\psi_2\rangle$.

The present work addresses the above issues with the help of an interferometric array whose primary light source could be either classical or quantal. The array, which is based on a scheme discussed in [18], entangles polarization and path DOF. It allows us to manipulate the coherence of both polarization and path DOF by ignoring one of them. Moreover, we can also manipulate both DOF at the same time, thereby exploring the so-called Schmidt sphere, a parameter space that can be used to describe two-party entanglement, and which is topologically equivalent to the Poincaré sphere. In this way, one can generate Berry phases for entangled states and test results like those derived by Sjöqvist [19].

II. ENTANGLEMENT AND POLARIZATION: THE QIAN-EBERLY APPROACH

Let us briefly discuss Qian and Eberly's approach to polarization [17]. As already said, it generalizes the notion of polarization that applies to light beams with transverse electric vector $\mathbf{E}=\mathbf{x}E_x+\mathbf{y}E_y$, to the case $\mathbf{E}=\mathbf{x}E_x+\mathbf{y}E_y+\mathbf{z}E_z$. Qian and Eberly treated $\mathbf{E}$ as a tensor product of spatial unit vectors, $\mathbf{x},\mathbf{y},\mathbf{z}$, and complex-valued functions E_x,E_y,E_z that belong to a statistical function space. In the case of a beam, i.e., when $E_z=0$, the degree of polarization can be defined in terms of quantities such as $\langle E_xE_y^*\rangle$, where brackets denote ensemble average. These quantities measure statistical correlations between different components of $\mathbf{E}$. In order to generalize the notion of polarization beyond the beamlike case, one can invoke entanglement [17]. Indeed, the degree of polarization turns out to be given by the degree of separability of the two spaces involved. By Schmidt decomposition, it is always possible to express $\mathbf{E}$ in the form $|\mathbf{E}\rangle=\lambda_1|\mathbf{u}_1\rangle|f_1\rangle+\lambda_2|\mathbf{u}_2\rangle|f_2\rangle+\lambda_3|\mathbf{u}_3\rangle|f_3\rangle$, with angle brackets being used only for notational convenience. Complete polarization is tantamount to complete separability, which occurs when two of the λ_i vanish and $\mathbf{E}$ is of the form $\mathbf{E}=\mathbf{u}f$. The other extreme case is that of a fully unpolarized state. It corresponds to a maximally entangled state, for which $\lambda_1=\lambda_2=\lambda_3$. Intermediate cases have a degree of polarization that can be quantified in terms of a parameter K, which is defined by $I^2K=1/[\lambda_1^4+\lambda_2^4+\lambda_3^4]$, where I denotes field intensity [17]. For the beam case, i.e., $|\mathbf{E}\rangle=|\mathbf{x}\rangle|E_x\rangle+|\mathbf{y}\rangle|E_y\rangle$, Qian and Eberly use the parametrization

$$|\mathbf{E}\rangle=\sqrt{I}(\cos\vartheta|\mathbf{x}\rangle|e_x\rangle+\sin\vartheta|\mathbf{y}\rangle|e_y\rangle). \quad (1)$$

Here, $I=\langle E_x|E_x\rangle+\langle E_y|E_y\rangle$ and vectors $|e_i\rangle$ span the subspace of statistical functions. Such a parametrization allows us to take $|e_i\rangle$ unit normalized, $\langle e_i|e_i\rangle=1$, while magnitude and phase of cross correlations can be given by $\alpha=|\alpha|e^{i\delta}:=\langle e_x|e_y\rangle$. By defining $\mathcal{W}=|\mathbf{E}\rangle\langle\mathbf{E}|/I$ and tracing over the subspace of statistical functions one obtains $I\mathcal{W}_{\text{lab}} = I\ \text{Tr}_{fcn}\,\mathcal{W} = \sum_m \langle\phi_m|\mathbf{E}\rangle\langle\mathbf{E}|\phi_m\rangle$, with $\{|\phi_m\rangle\}$ an orthonormal basis for the subspace of statistical functions. One gets $I\mathcal{W}_{\text{lab}} = \sum_{p,q} |\mathbf{p}\rangle\langle\mathbf{q}|\langle E_p|E_q\rangle$, with $p,q = x$ or y. Thus, $\mathcal{W}_{\text{lab}}$ is a 2×2 tensor in the laboratory subspace Span$\{|\mathbf{x}\rangle,|\mathbf{y}\rangle\}$. Its matrix representation in the basis $\{|\mathbf{x}\rangle,|\mathbf{y}\rangle\}$ reads

$$\mathcal{W}_{\text{lab}} = I^{-1}\begin{pmatrix}\langle E_x|E_x\rangle & \langle E_x|E_y\rangle \\ \langle E_y|E_x\rangle & \langle E_y|E_y\rangle\end{pmatrix} = \begin{pmatrix}\cos^2\vartheta & \alpha\cos\vartheta\sin\vartheta \\ \alpha^*\sin\vartheta\cos\vartheta & \sin^2\vartheta\end{pmatrix}. \tag{2}$$

In Qian-Eberly's parametrization the degree of polarization, $\mathcal{P}$, which is given by

$$\mathcal{P} := [1 - 4\ \text{Det}\ \mathcal{W}_{\text{lab}}]^{1/2}, \tag{3}$$

reads

$$\mathcal{P}^2 = \cos^2 2\vartheta + |\alpha|^2 \sin^2 2\vartheta. \tag{4}$$

We note that $\mathcal{P}$ incorporates both the relative magnitude of the field components—through its dependence on ϑ—and their statistical correlation—through its dependence on $|\alpha|$. Now, $\mathcal{W}_{\text{lab}}$ can be seen as the matrix representation of a density operator $\rho_{\mathcal{W}}$ in Span$\{\mathbf{x},\mathbf{y}\}$. Following Mandel [2], we can decompose $\rho_{\mathcal{W}}$ in terms of two operators, ρ_{ID} and ρ_D, which are defined in terms of a fiducial state $|\psi\rangle = \cos\vartheta|\mathbf{x}\rangle + e^{i\delta}\sin\vartheta|\mathbf{y}\rangle$, as $\rho_{ID} = |\psi\rangle\langle\psi|$ and $\rho_D = \cos^2\vartheta|\mathbf{x}\rangle\langle\mathbf{x}| + \sin^2\vartheta|\mathbf{y}\rangle\langle\mathbf{y}|$. Setting $\rho_{\mathcal{W}} = \mathcal{I}_p\rho_{ID} + (1-\mathcal{I}_p)\rho_D$, with $\mathcal{I}_p \in [0,1]$, one finds

$$\mathcal{I}_p = \frac{|(\mathcal{W}_{\text{lab}})_{xy}|}{\sqrt{(\mathcal{W}_{\text{lab}})_{xx}(\mathcal{W}_{\text{lab}})_{yy}}} = |\alpha|, \tag{5}$$

from which it follows, on account of Eq. (4), that

$$\mathcal{P} \geqslant \mathcal{I}_p. \tag{6}$$

In Mandel's framework [2], a decomposition in terms of ρ_{ID} and ρ_D had the purpose of connecting coherence with indistinguishability, the latter referring to the two beams of a Young interferometer. In the present context, "indistinguishability" would refer to the two states $(|\mathbf{x}\rangle,|\mathbf{y}\rangle)$ that coherently contribute to make up the polarization state $|\psi\rangle$, in terms of which ρ_{ID} and ρ_D were constructed. Even though we can talk about "paths" in our two-dimensional polarization space, this has no bearing on wave-particle duality. Thus, the only relevant physical information that $\mathcal{I}_p$ entails is that related to statistical correlations.

It is also useful to see how the above formulation relates to the standard one. In the latter, the polarization matrix is obtained from the column vector $\mathbf{E} = (E_x,E_y)^T$ as the ensemble-averaged direct product $\langle\mathbf{E}\mathbf{E}^\dagger\rangle$. The so obtained matrix—whose components are $\langle E_p E_q^*\rangle$—can be written as

$$\rho_p(\mathcal{P}) = \tfrac{1}{2}(\mathbb{1} + \mathcal{P}\mathbf{n}\cdot\boldsymbol{\sigma}). \tag{7}$$

Here, $\mathbf{n}$ is a unit vector proportional to the Stokes vector, $\mathbb{1}$ is the identity matrix, and $\boldsymbol{\sigma}$ stands for the triple of Pauli matrices. The eigenvectors and eigenvalues of ρ_p are defined through

$$\rho_p|\mathbf{n}_\pm\rangle = \lambda_\pm|\mathbf{n}_\pm\rangle. \tag{8}$$

Because $\mathbf{n}\cdot\boldsymbol{\sigma}|\mathbf{n}_\pm\rangle = \pm|\mathbf{n}_\pm\rangle$, we have $\lambda_\pm = (1 \pm \mathcal{P})/2$ [see Eq. (7)]. Taking $\mathbf{n} = (\sin 2\theta\cos\varphi, \sin 2\theta\sin\varphi, \cos 2\theta)$, we have $|\mathbf{n}_+\rangle = \cos\theta|+\rangle + e^{i\varphi}\sin\theta|-\rangle$ and $|\mathbf{n}_-\rangle = -\sin\theta|+\rangle + e^{i\varphi}\cos\theta|-\rangle$, with $\sigma_3|\pm\rangle = \pm|\pm\rangle$. Thus, besides Eq. (7), we can write ρ_p in two other ways:

$$\rho_p(\mathcal{P}) = \lambda_+|\mathbf{n}_+\rangle\langle\mathbf{n}_+| + \lambda_-|\mathbf{n}_-\rangle\langle\mathbf{n}_-| = \frac{1}{2}\begin{pmatrix}1 + \mathcal{P}\cos 2\theta & \mathcal{P}e^{-i\varphi}\sin 2\theta \\ \mathcal{P}e^{i\varphi}\sin 2\theta & 1 - \mathcal{P}\cos 2\theta\end{pmatrix}. \tag{9}$$

As for the connection between $\mathcal{W}_{\text{lab}}$ and ρ_p, we first note that we can identify the basis $\{|\mathbf{x}\rangle,|\mathbf{y}\rangle\}$ in "laboratory space" $\mathcal{H}_{\text{lab}}$ with the standard basis $\{|+\rangle,|-\rangle\}$. In such a case, scalar products in the space $\mathcal{H}_{fcn}$ of statistical functions must fulfill $\langle E_p|E_q\rangle = \langle E_p E_q^*\rangle$. Qian and Eberly chose a parametrization $(|\alpha|,\vartheta,\delta)$ that is defined by Eq. (1) and $\langle e_x|e_y\rangle = \alpha = |\alpha|e^{i\delta}$. A standard parametrization is given in terms of $(\mathcal{P},\theta,\varphi)$. In order to show the connection between them, let us consider the orthonormal basis vectors $|\phi_\pm\rangle \in \mathcal{H}_{fcn}$ and introduce the operator $\mathcal{T} : \mathcal{H}_{\text{lab}} \mapsto \mathcal{H}_{fcn}$, given by

$$\mathcal{T} = \sqrt{\lambda_+}|\phi_+\rangle\langle\mathbf{n}_+| + \sqrt{\lambda_-}|\phi_-\rangle\langle\mathbf{n}_-|. \tag{10}$$

It satisfies $\mathcal{T}^\dagger\mathcal{T} = \rho_p(\mathcal{P})$. Hence, by defining

$$\begin{aligned}|E_x\rangle &= \mathcal{T}|+\rangle = \sqrt{\lambda_+}\langle\mathbf{n}_+|+\rangle|\phi_+\rangle + \sqrt{\lambda_-}\langle\mathbf{n}_-|+\rangle|\phi_-\rangle, \\ |E_y\rangle &= \mathcal{T}|-\rangle = \sqrt{\lambda_+}\langle\mathbf{n}_+|-\rangle|\phi_+\rangle + \sqrt{\lambda_-}\langle\mathbf{n}_-|-\rangle|\phi_-\rangle,\end{aligned} \tag{11}$$

we see that $\langle E_p|E_q\rangle = \langle p|\mathcal{T}^\dagger\mathcal{T}|q\rangle = \langle p|\rho_p|q\rangle$, with $p,q \in \{+,-\} \equiv \{x,y\}$, so that Eq. (11) gives Eq. (9). Whence, the two parametrizations involved in the definitions of $\mathcal{W}_{\text{lab}}$ and $\rho_p(\mathcal{P})$, respectively, can be traced back to two different ways of defining statistical correlations. The standard way corresponds to Eq. (11), and that of Qian and Eberly to Eq. (1), i.e., $|E_x\rangle = \sqrt{I}\cos\vartheta|e_x\rangle$, and $|E_y\rangle = \sqrt{I}\sin\vartheta|e_y\rangle$.

III. INDISTINGUISHABILITY AND POLARIZATION: THE MANDEL-LAHIRI APPROACH

Let us now discuss in some more detail Mandel's derivation [2] of the constraint $\mathcal{I} \geqslant \mathcal{V}$. We will show how it formally relates to Lahiri's inequality $\mathcal{P} \geqslant \mathcal{I}$. As we saw above, given a density operator $\rho = \sum_{i,j=1,2}\rho_{ij}|\psi_i\rangle\langle\psi_j|$, we can decompose it so as to exhibit its distinguishable and indistinguishable components. Of course, the physical meaning of ρ will depend on the two-state system being considered. Mandel addressed a Young interferometer, for which $|\psi_i\rangle$ refer to the two possible paths that a photon can take: $|\psi_1\rangle = |1\rangle_1|0\rangle_2$ and $|\psi_2\rangle = |0\rangle_1|1\rangle_2$. With respect to a fiducial state $|\psi\rangle = \alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle$, we may define the pure, indistinguishable state $\rho_{ID} := |\psi\rangle\langle\psi|$ and the mixed, distinguishable state $\rho_D := |\alpha_1|^2|\psi_1\rangle\langle\psi_1| + |\alpha_2|^2|\psi_2\rangle\langle\psi_2|$. According to the usual interpretation of a coherent superposition of states, a photon prepared in state ρ_{ID} goes along both paths at the same time, with *a priori* probabilities being determined by the amplitudes α_i. If the photon is instead prepared in state ρ_D, then it goes along one or the other path, with probabilities $|\alpha_i|^2$. Thus, while a system prepared in ρ_{ID} is capable of showing interference, a system prepared in ρ_D is not. By writing $\rho = \mathcal{I}\rho_{ID} + (1-\mathcal{I})\rho_D$, we define $\mathcal{I} \in [0,1]$ in connection with ρ_{ID} and ρ_D.

From $\rho = \sum_{i,j=1,2} \rho_{ij} |\psi_i\rangle\langle\psi_j| = \mathcal{I}\rho_{ID} + (1-\mathcal{I})\rho_D$ we get, as in Eq. (5),

$$\mathcal{I} = \frac{|\rho_{12}|}{\sqrt{\rho_{11}\rho_{22}}}. \quad (12)$$

It turns out that $\mathcal{I}$ can be related to the mutual coherence function of a quantized field, $\gamma_{12} := \Gamma_{12}/[\Gamma_{11}\Gamma_{22}]^{1/2}$, with $\Gamma_{ij} \sim \mathrm{Tr}\{\rho \hat{E}^{(-)}(r_i)\hat{E}^{(+)}(r_j)\}, i,j \in \{1,2\}$. One readily obtains $\gamma_{12} = \rho_{12}/(\rho_{11}\rho_{22})^{1/2}$, hence $|\gamma_{12}| = \mathcal{I}$, and this leads to Mandel's conclusion that the degree of coherence is the same as the degree of indistinguishability. Consider next the field on the screen in Young's array. Phase differences due to propagation cause the appearance of fringes. The visibility of these fringes can be shown [2] to be given by $\mathcal{V} = 2|\gamma_{12}|(\rho_{11}\rho_{22})^{1/2} \leqslant \mathcal{I}$, which is Mandel's constraint.

As Mandel remarked [2], the above procedure is reminiscent of a similar one that Wolf used to define the degree of polarization [20]. It is thus tempting to address polarization in a similar fashion as Mandel did with spatial coherence. This is what Lahiri seems to have attempted in [1]. Now, if we deal with polarization alone and just mirror Mandel's procedure, then we should refer to "paths" only figuratively, i.e., as paths in polarization space. This is so because our setting occurs within a single Hilbert space. If we insist on connecting polarization with spatial degrees of freedom, then we must extend our treatment so as to deal with at least two Hilbert spaces. This is what Qian and Eberly did [17], as explained in the previous section. Here we address the case of a single Hilbert space, as Lahiri did. This corresponds to having traced over the space of statistical functions in Qian-Eberly's approach. We will thus recover some of the results obtained in the previous section.

Lahiri considered the states $|x\rangle := |1\rangle_x|0\rangle_y$ and $|y\rangle := |0\rangle_x|1\rangle_y$, referring to an x-polarized and to a y-polarized photon, respectively. They played the role of Mandel's $|1\rangle := |1\rangle_1|0\rangle_2$, $|2\rangle := |0\rangle_1|1\rangle_2$. The two Hilbert spaces $\mathcal{H}_P = \mathrm{Span}\{|x\rangle,|y\rangle\}$ and $\mathcal{H}_S = \mathrm{Span}\{|1\rangle,|2\rangle\}$ are mathematically the same, i.e., just two-dimensional Hilbert spaces. However, they markedly differ from one another with respect to their physical content. The constraint $\mathcal{P} \geqslant \mathcal{I}$ was derived in [1] by applying Mandel's procedure. That is, given a general polarization state ρ_p, one introduces a normalized fiducial state $|\psi\rangle = \alpha_1|x\rangle + \alpha_2|y\rangle$, in terms of which ρ_p can be decomposed as

$$\rho_p = \mathcal{I}\rho_{ID} + (1-\mathcal{I})\rho_D. \quad (13)$$

In [1], inequality $\mathcal{P} \geqslant \mathcal{I}$ was derived as a direct consequence of the following relationship:

$$\mathcal{P} = [(|\alpha_1|^2 - |\alpha_2|^2)^2 + 4|\alpha_1|^2|\alpha_2|^2\mathcal{I}^2]^{1/2}, \quad (14)$$

which was in turn related to the quantum polarization matrix $\mathrm{Tr}\{\rho \hat{E}^{(-)}(r_i)\hat{E}^{(+)}(r_j)\}$. However, we can derive Eq. (14) without regard to the quantum or classical nature of the field. Indeed, the relationship between $\mathcal{I}$ and $\mathcal{P}$ comes just from expressing ρ_p in two alternative ways: Eq. (13) and Eq. (7). We can thus repeat what we did before within Qian-Eberly's framework. To this end, let us first write Eq. (13) in matrix form with respect to the basis $\{|x\rangle,|y\rangle\}$:

$$\rho_p = \begin{pmatrix} |\alpha_1|^2 & \mathcal{I}\alpha_1^*\alpha_2 \\ \mathcal{I}\alpha_2^*\alpha_1 & |\alpha_2|^2 \end{pmatrix}. \quad (15)$$

Equation (14) follows from equating the above matrix to Eq. (7). Equivalently, we can use the above matrix for calculating

$$\mathcal{P} = \left[1 - \frac{4\ \mathrm{Det}\ \rho_p}{(\mathrm{Tr}\ \rho_p)^2}\right]^{1/2}, \quad (16)$$

which follows by solving for $\mathcal{P}$ in Eq. (7). Note that choosing the parametrization $\alpha_1 = \cos\vartheta$, $\alpha_2 = e^{i\delta}\sin\vartheta$ for $|\psi\rangle = \alpha_1|x\rangle + \alpha_2|y\rangle$, Eq. (15) reproduces Eq. (2), whenever $\mathcal{I} = |\alpha|$. On the other hand, using this parametrization, Eq. (14) reads

$$\mathcal{P}^2 = \cos^2 2\vartheta + \mathcal{I}^2 \sin^2 2\vartheta, \quad (17)$$

which was previously derived within Qian-Eberly's framework [cf. Eq. (4)].

We see thus that $\mathcal{P} \geqslant \mathcal{I}$ holds true as a consequence of Eq. (17), regardless of the quantum or classical nature of the field. In Eq. (17) $\mathcal{I}$ is defined in relation to a fiducial *polarization* state $|\psi\rangle = \cos\vartheta|x\rangle + e^{i\delta}\sin\vartheta|y\rangle$. This state bears no connection with WPI, nor with wave-particle duality. Even though Mandel's approach was tailored to deal with wave-particle duality, it may be applied to any two-state system. By applying such an approach to polarization we obtain $\mathcal{P} \geqslant \mathcal{I}$, similarly to the derivation of $\mathcal{I} \geqslant \mathcal{V}$ in Mandel's case. However, these two inequalities convey two very different messages. Indeed, both quantities entering $\mathcal{I} \geqslant \mathcal{V}$ are basis dependent and related to interference, a phenomenon that shows up only under coherent propagation of the involved states [21]. In contrast to polarization, interference is "a property of a propagator of states, not of a state itself" [21]. As for $\mathcal{P} \geqslant \mathcal{I}$, it involves a basis-independent property ($\mathcal{P}$) of a state, and a basis-dependent quantity ($\mathcal{I}$) that conveys information about statistical correlations [cf. Eq. (5)]. These correlations can be either classical, such as $\langle E_p E_q^*\rangle$, or else quantum correlations, such as $\mathrm{Tr}\{\rho \hat{E}_p^{(-)}\hat{E}_q^{(+)}\}$, involving quantized fields $\hat{E}_p^{(+)} \sim \hat{a}$ and $\hat{E}_p^{(-)} \sim \hat{a}^\dagger$. By turning the classical quantity $E_p(\mathbf{x},t)$ into an operator $\hat{E}_p^{(-)}(\mathbf{x},t)$, we quantize the field, not the parameters on which it depends: $\mathbf{x}$, t, frequency, wavelength, polarization, etc. These parameters keep their classical meaning after field quantization and so does inequality $\mathcal{P} \geqslant \mathcal{I}$.

In order to connect $\mathcal{P} \geqslant \mathcal{I}$ with wave-particle duality, we should derive this inequality within the framework of the product space $\mathcal{H}_S \otimes \mathcal{H}_P$, where we can address both polarization and WPI. Note that this product space is not the same as that used by Qian and Eberly. Indeed, these authors addressed states $|\mathbf{E}\rangle = |\mathbf{x}\rangle|E_x\rangle + |\mathbf{y}\rangle|E_y\rangle$ that belong to $\mathcal{H}_{\mathrm{lab}} \otimes \mathcal{H}_{fcn}$. Even though the same mathematical tools can be used to work with both $\mathcal{H}_S \otimes \mathcal{H}_P$ and $\mathcal{H}_{\mathrm{lab}} \otimes \mathcal{H}_{fcn}$, the respective physical content of most results should markedly differ from one another. For example, as Qian and Eberly noted [17], it is possible to define a polarization matrix analogous to $\mathcal{W}_{\mathrm{lab}}$ [cf. Eq. (2)], but taking the trace over $\mathcal{H}_{\mathrm{lab}}$ instead of $\mathcal{H}_{fcn}$. Now, it is not obvious how to operationally interpret such a polarization in $\mathcal{H}_{fcn}$, i.e., how to prescribe a measuring procedure analogous to the one used to fix Stokes parameters in the case of standard polarization. The difficulty stems from the intrinsic nature of the entanglement being present in $|\mathbf{E}\rangle$. Things are different when we actively generate entanglement. This can be achieved by using an entangling device like the one employed by Kagalwala *et al.* [13], or by other means. Kagalwala *et al.* used spatial light modulators to produce states such as $E(x) = E_e\psi_e(x) + E_o\psi_o(x)$, in which polarization and parity become entangled. In cases like this, we can operationally define quantities such as polarization, which appear as a consequence of ignoring one of the involved degrees of freedom. In what follows, we will deal with an entangling tool that allows us to create manifold objects in product space $\mathcal{H}_S \otimes \mathcal{H}_P$. All quantities will have an operationally well-fixed procedure upon which their definitions may rest.

IV. A SOURCE OF PARTIALLY POLARIZED STATES

A device that performs operations on the product space $\mathcal{H}_S \otimes \mathcal{H}_P$ and that is suitable for our present purposes, is a Mach-Zehnder array with a configuration like that shown in Fig. 1. In order to illustrate the techniques employed when using this device, let us show how we can generate, out of a pure state in $\mathcal{H}_S \otimes \mathcal{H}_P$, a partially polarized state ρ_p. We denote by $|h\rangle,|v\rangle$ horizontally and vertically polarized light, respectively. This light can be classical or quantal. As for the spatial DOF, we also have a binary alternative $|R\rangle,|L\rangle$, which correspond to moving to the right or to the left, respectively. Following [18], we determine the unitary operators V_1,V_2,V_R,V_L, which act on $\mathcal{H}_S \otimes \mathcal{H}_P$ and produce the desired state ρ_p of Eq. (7) from, say, the input state $|Rv\rangle$. To this end, we "purify" ρ_p by seeking a pure state $|\psi\rangle \in \mathcal{H}_S \otimes \mathcal{H}_P$, such that $\mathrm{Tr}_S\,|\psi\rangle\langle\psi| = \rho_p$. This is always possible by Schmidt decomposition, which requires that we find the eigenvectors $|\mathbf{n}_\pm\rangle$ and eigenvalues $\lambda_\pm$ of ρ_p. As we saw before, $\lambda_\pm = (1 \pm \mathcal{P})/2$. We can purify ρ_p by defining $|\Phi_+\rangle \in \mathcal{H}_S \otimes \mathcal{H}_P$ as

$$|\Phi_+\rangle = \sqrt{\lambda_+}|R\mathbf{n}_+\rangle + \sqrt{\lambda_-}|L\mathbf{n}_-\rangle. \tag{18}$$

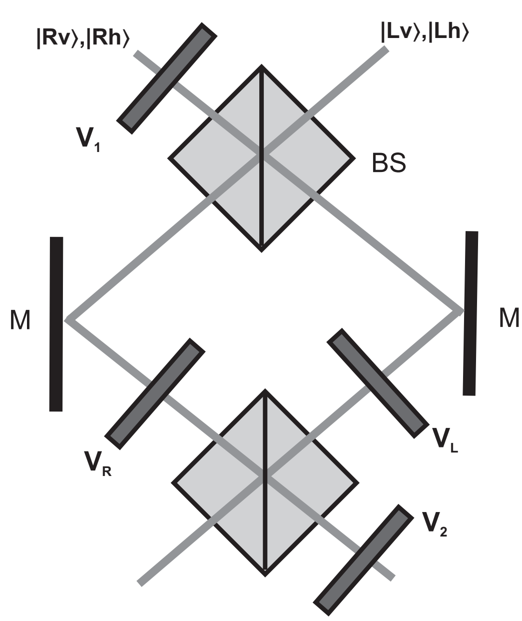

FIG. 1. Mach-Zehnder-like array. The devices labeled V_i ($i = 1,2,R,L$) contain wave plates and phase shifters. BS: 50 : 50 beam splitter; M: mirror. $|Rv\rangle$ denotes a vertically polarized photon or light beam propagating to the right, and similarly for the other cases.

Hence, $\mathrm{Tr}_S\,|\Phi_+\rangle\langle\Phi_+| = \lambda_+|\mathbf{n}_+\rangle\langle\mathbf{n}_+| + \lambda_-|\mathbf{n}_-\rangle\langle\mathbf{n}_-|$. Now, from $|\mathbf{n}_+\rangle\langle\mathbf{n}_+| + |\mathbf{n}_-\rangle\langle\mathbf{n}_-| = \mathbb{1}$ and $|\mathbf{n}_+\rangle\langle\mathbf{n}_+| - |\mathbf{n}_-\rangle\langle\mathbf{n}_-| = \mathbf{n}\cdot\boldsymbol{\sigma}$ it follows that $|\mathbf{n}_\pm\rangle\langle\mathbf{n}_\pm| = (\mathbb{1} \pm \mathbf{n}\cdot\boldsymbol{\sigma})/2$, so that

$$\begin{aligned}\mathrm{Tr}_S\,|\Phi_+\rangle\langle\Phi_+| &= \left(\frac{\lambda_+ + \lambda_-}{2}\right)\mathbb{1} + \left(\frac{\lambda_+ - \lambda_-}{2}\right)\mathbf{n}\cdot\boldsymbol{\sigma}\\ &= \frac{1}{2}(\mathbb{1} + \mathcal{P}\mathbf{n}\cdot\boldsymbol{\sigma}),\end{aligned} \tag{19}$$

as desired. Together with $|\Phi_+\rangle$, the following states define a Bell-like basis in $\mathcal{H}_S \otimes \mathcal{H}_P$:

$$|\Phi_-\rangle = \sqrt{\lambda_-}|R\mathbf{n}_+\rangle - \sqrt{\lambda_+}|L\mathbf{n}_-\rangle, \tag{20}$$

$$|\Psi_+\rangle = \sqrt{\lambda_+}|L\mathbf{n}_+\rangle + \sqrt{\lambda_-}|R\mathbf{n}_-\rangle, \tag{21}$$

$$|\Psi_-\rangle = \sqrt{\lambda_-}|L\mathbf{n}_+\rangle - \sqrt{\lambda_+}|R\mathbf{n}_-\rangle, \tag{22}$$

for which it holds that

$$\mathrm{Tr}_S\,|\Phi_\pm\rangle\langle\Phi_\pm| = \mathrm{Tr}_S\,|\Psi_\pm\rangle\langle\Psi_\pm| = \tfrac{1}{2}(\mathbb{1} \pm \mathcal{P}\mathbf{n}\cdot\boldsymbol{\sigma}).$$

Our next task is to implement the basis transformation $\{|Rv\rangle,|Rh\rangle,|Lv\rangle,|Lh\rangle\} \to \{|\Phi_+\rangle,|\Phi_-\rangle,|\Psi_+\rangle,|\Psi_-\rangle\}$ with the help of our Mach-Zehnder array. The corresponding unitary transformation reads

$$U_{\mathrm{MZ}} = |\Phi_+\rangle\langle Rv| + |\Phi_-\rangle\langle Rh| + |\Psi_+\rangle\langle Lv| + |\Psi_-\rangle\langle Lh|. \tag{23}$$

By rearranging terms we can write U_{MZ} in the suitable form

$$\begin{aligned}U_{\mathrm{MZ}} = &U_{RR}|R\rangle\langle R| + U_{RL}|R\rangle\langle L| + U_{LR}|L\rangle\langle R|\\ &+ U_{LL}|L\rangle\langle L|,\end{aligned} \tag{24}$$

where the operators $U_{ij} \in U(2)$ act on polarization subspace $\mathcal{H}_p$. They can be implemented with phase shifters and wave plates. In the present case, we seek to determine U_{MZ} so that the output state $|\Psi_f\rangle = U_{\mathrm{MZ}}|Rv\rangle = U_{RR}|Rv\rangle + U_{LR}|Lv\rangle$ satisfies $\mathrm{Tr}_S\,|\Psi_f\rangle\langle\Psi_f| = \rho_p$. It can be shown [22] that this task can be achieved by choosing $V_1 = V_2 = \mathbb{1}$ and

$$\begin{aligned}V_R &= e^{i\phi_R}(|t_R\rangle\langle h| + |s_R\rangle\langle v|),\\ V_L &= e^{i\phi_L}(|t_L\rangle\langle h| + |s_L\rangle\langle v|).\end{aligned} \tag{25}$$

Here, the phases ϕ_R and ϕ_L are arbitrary and $\{|t_i\rangle,|s_i\rangle\}_{i=R,L}$ are two orthonormal bases in $\mathcal{H}_p$. One can readily prove that a possible choice for these bases is

$$\begin{aligned}|t_R\rangle &= -\sqrt{\lambda_+}|\mathbf{n}_-\rangle + i\sqrt{\lambda_-}|\mathbf{n}_+\rangle,\\ |s_R\rangle &= \sqrt{\lambda_-}|\mathbf{n}_-\rangle + i\sqrt{\lambda_+}|\mathbf{n}_+\rangle,\\ |t_L\rangle &= -\sqrt{\lambda_+}|\mathbf{n}_-\rangle - i\sqrt{\lambda_-}|\mathbf{n}_+\rangle,\\ |s_L\rangle &= \sqrt{\lambda_-}|\mathbf{n}_-\rangle - i\sqrt{\lambda_+}|\mathbf{n}_+\rangle.\end{aligned} \tag{26}$$

Once we have fixed the bases, the operators $V_{R,L}$ given by (25) are fixed as well. It turns out that each of them can be implemented with two quarter-wave plates. Such an array can be used as a secondary source of partially polarized states with a prescribed Stokes vector. The primary source—e.g., a gas laser—is assumed to produce (almost) completely coherent and polarized light.

Up to this point, no reference has been made to the degree of indistinguishability $\mathcal{I}$. The production of ρ_p with a prescribed degree of polarization $\mathcal{P}$ has been achieved with a couple of quarter-wave plates on each arm of the Mach-Zehnder interferometer. It is not yet clear whether some WPI has thereby become available. As we shall see, $\mathcal{P}$ can indeed be related to WPI.

V. INDISTINGUISHABILITY AND POLARIZATION

A. Relationship between $\mathcal{P}$ and $\mathcal{I}$ for entangled states

Our Mach-Zehnder device can be used to address path space and polarization space either separately or by entangling them. We can exploit this capability in order to properly refer indistinguishability $\mathcal{I}$ to paths in laboratory space. The degree of polarization $\mathcal{P}$ keeps thereby its meaning while being related to $\mathcal{I}$ via entanglement. Our treatment builds upon the following features. First, given a state ρ_P, the degree of polarization is a property of the state itself that can be calculated as the normalized difference between the largest and the smallest eigenvalue of ρ_P, i.e., $\mathcal{P} = (\lambda_+ - \lambda_-)/(\lambda_+ + \lambda_-)$. Second, if we take an entangled state such as

$$|\Phi_+\rangle = \sqrt{\lambda_+}|\mathbf{m}_+,\mathbf{n}_+\rangle + \sqrt{\lambda_-}|\mathbf{m}_-,\mathbf{n}_-\rangle, \tag{27}$$

with $\{|\mathbf{m}_+\rangle,|\mathbf{m}_-\rangle\}$ and $\{|\mathbf{n}_+\rangle,|\mathbf{n}_-\rangle\}$ being orthonormal bases in $\mathcal{H}_S$ and in $\mathcal{H}_P$, respectively, and then construct the operators $\rho_S := \mathrm{Tr}_P\,|\Phi_+\rangle\langle\Phi_+|$ and $\rho_P := \mathrm{Tr}_S\,|\Phi_+\rangle\langle\Phi_+|$, then these operators have common eigenvalues. Indeed,

$$\begin{aligned}\rho_S &= \mathrm{Tr}_P\,|\Phi_+\rangle\langle\Phi_+|\\ &= \lambda_+|\mathbf{m}_+\rangle\langle\mathbf{m}_+| + \lambda_-|\mathbf{m}_-\rangle\langle\mathbf{m}_-|,\end{aligned} \tag{28}$$

$$\rho_P = \mathrm{Tr}_S\,|\Phi_+\rangle\langle\Phi_+| = \lambda_+|\mathbf{n}_+\rangle\langle\mathbf{n}_+| + \lambda_-|\mathbf{n}_-\rangle\langle\mathbf{n}_-|. \tag{29}$$

As a consequence, ρ_S inherits from ρ_P its degree of polarization, so that

$$\rho_S = \mathrm{Tr}_P\,|\Phi_+\rangle\langle\Phi_+| = \tfrac{1}{2}(\mathbb{1} + \mathcal{P}\mathbf{m}\cdot\boldsymbol{\sigma}). \tag{30}$$

Third, we can relate the above $\mathcal{P}$ with $\mathcal{I}$ by following Mandel's procedure, as we did before to get Eq. (17). However, as we are presently working not in subspace $\mathcal{H}_P$ but in $\mathcal{H}_S$, the parameter $\mathcal{I}$ acquires its original, sensible physical meaning as a measure of indistinguishability in path space.

The last step requires choosing a fiducial state, in terms of which $\mathcal{I}$ is defined. While $\mathcal{P}$ is a state's property shared by ρ_P and ρ_S, the degree of indistinguishability is a property that depends on the fiducial state. For this reason, $\mathcal{I}$ turns to be basis dependent like interference itself [21]. In order to gain some more insight into this point let us return for a moment to the case $\{|\mathbf{m}_+\rangle,|\mathbf{m}_-\rangle\} = \{|R\rangle,|L\rangle\}$ and consider first the fiducial state $|\psi\rangle = \alpha_R|R\rangle + \alpha_L|L\rangle$. Equation (27) then reads

$$|\Phi_+\rangle = \sqrt{\lambda_+}|R\mathbf{n}_+\rangle + \sqrt{\lambda_-}|L\mathbf{n}_-\rangle \tag{31}$$

and

$$\rho_S = \mathrm{Tr}_P\,|\Phi_+\rangle\langle\Phi_+| = \lambda_+|R\rangle\langle R| + \lambda_-|L\rangle\langle L|. \tag{32}$$

If we now apply to this ρ_S Mandel's decomposition in terms of the fiducial state $|\psi\rangle = \alpha_R|R\rangle + \alpha_L|L\rangle$, we get $\mathcal{I} = 0$. Indeed, writing $\rho_S = \mathcal{I}\rho_{ID} + (1 - \mathcal{I}\rho_I)$ with $\rho_{ID} = |\psi\rangle\langle\psi|$ and $\rho_D = |\alpha_R|^2|R\rangle\langle R| + |\alpha_L|^2|L\rangle\langle L|$, we get $|\alpha_R|^2 = \lambda_+$, $|\alpha_L|^2 = \lambda_-$, and $\mathcal{I} = 0$. Of course, this is so because ρ_S in Eq. (32) is already in diagonal form, like ρ_D. Correspondingly, if we revert to $|\Phi_+\rangle$ as given in Eq. (27), so that ρ_S reads as in Eq. (28), then we get $\mathcal{I} = 0$ with respect to a fiducial state $|\psi\rangle = \alpha_+|\mathbf{m}_+\rangle + \alpha_-|\mathbf{m}_-\rangle$. But if we take instead the fiducial state $|\psi\rangle = \alpha_R|R\rangle + \alpha_L|L\rangle$, we obtain $\mathcal{I} \neq 0$. All this is just an expression of the basis-dependent nature of $\mathcal{I}$. Given a matrix ρ, and recalling that $\mathcal{I} = |\rho_{12}|/(\rho_{11}\rho_{22})^{1/2}$, we see that $\mathcal{I} \neq 0$ whenever ρ has nonvanishing coherences.

The relationship between $\mathcal{P}$ and $\mathcal{I}$, as given by Eq. (14) with $\alpha_1 = \alpha_R$, $\alpha_2 = \alpha_L$, acquires a new physical sense within the present framework. Such a relationship follows from submitting ρ_S to Mandel's procedure with a fiducial state $|\psi\rangle = \alpha_R|R\rangle + \alpha_L|L\rangle$. Indeed, as in the general case, Eqs. (28) and (30) imply that Det $\rho_S = \lambda_+\lambda_- = (1 - \mathcal{P}^2)/4$ and Tr $\rho_S = \lambda_+ + \lambda_-$. Hence,

$$\mathcal{P}^2 = 1 - \frac{4\ \mathrm{Det}\ \rho_S}{(\mathrm{Tr}\ \rho_S)^2}, \tag{33}$$

which is a manifestly basis-independent relationship. Inserting now $\rho_S = \mathcal{I}\rho_{ID} + (1 - \mathcal{I})\rho_D$ in the above equation we get

$$\mathcal{P}^2 = \cos^2\xi + \mathcal{I}^2\sin^2\xi, \tag{34}$$

where we have used the parametrization $|\psi\rangle = \cos(\xi/2)|R\rangle + e^{i\phi}\sin(\xi/2)|L\rangle$ for the fiducial state.

As already said, Mandel's decomposition is made within a single Hilbert space, e.g., $\mathcal{H}_S$ or $\mathcal{H}_P$. If the whole setting occurs within a single Hilbert space, then we obtain a relationship like Eq. (17), which formally coincides with Eq. (34). However, the meaning of $\mathcal{I}$ is not the same in the two equations. This is so, because we have invoked entanglement in order to derive Eq. (34). In Eq. (17), $\mathcal{I}$ relates to statistical correlations between field components [cf. Eq. (5)]. In Eq. (34), $\mathcal{I}$ refers to the indistinguishability of spatial paths. In the case of photons, for instance, the $\mathcal{I}$ entering Eq. (34) has its original meaning [2]: It is the probability to find a photon in state ρ_{ID}, when the photon has been prepared in state ρ_S. Note that ρ_S and ρ_{ID} are independent from one another. We can prepare the system to be in a state ρ_S and then interrogate it about its being in a state ρ_{ID}. Equation (34) relates preparation and subsequent interrogation through the corresponding parameters $\mathcal{P}$ and $\mathcal{I}$, which depend on what we decide to do with the path degree of freedom via our choice of $\lambda_\pm$ and $|\mathbf{m}_\pm\rangle$. Our Mach-Zehnder array can be used for fixing $\mathcal{P}$, as well as for asking the system about its being in ρ_{ID}.

For a photon, $\mathcal{I} = 1$ means that we do not know along which arm of the interferometer the photon moves. From Eq. (34) we see that $\mathcal{I} = 1$ implies $\mathcal{P} = 1$, so that complete lack of WPI is tantamount to full polarization. This is so because for $\mathcal{I} = 1$ Mandel's decomposition $\rho = \mathcal{I}\rho_{ID} + (1 - \mathcal{I})\rho_D$ reduces to $\rho = \rho_{ID} = |\psi\rangle\langle\psi|$, which means that the system has been prepared in a pure (path) state, and hence also in a pure polarization state. Reciprocally, provided $\xi \neq 0,\pi$, we see that $\mathcal{P} = 1$ implies $\mathcal{I} = 1$ for similar reasons. On the other hand, both $\xi = 0$ and $\xi = \pi$ give $\mathcal{P} = 1$. This is consistent with the fact that the fiducial state is $|\psi\rangle = |R\rangle$ and $|\psi\rangle = e^{i\phi}|L\rangle$, respectively, so that $\rho_{ID} = \rho_D = |i\rangle\langle i|$, with $i = R$ or L. Thus, we could have $\rho_S = \rho_{ID}$ only if ρ_S was in a pure state, and so also ρ_P. On the other hand, from $\mathcal{P} < 1$ it follows that $\mathcal{I} < 1$. In this case some WPI becomes available. $\mathcal{P} < 1$ can be achieved by changing the orientation of the plates in our Mach-Zehnder array. Because by these means we acquire some WPI, the array is working as a which-way marker. Note that for $\xi = \pi/2$ and ϕ arbitrary we have $\mathcal{P} = \mathcal{I}$, so that the degrees of polarization and indistinguishability are numerically the same in this case. This makes $\xi = \pi/2$ the best choice for interrogating the system about its degree of polarization. As we will show below, our Mach-Zehnder device gives us full control over the values $\lambda_\pm$ and $|\mathbf{m}_\pm\rangle$ entering Eq. (28). That is, we can produce any desired $\rho_S = \sum_{i,j=R,L} \rho_{i,j} |i\rangle\langle j|$. In such a case, we have full control over the values $\alpha_R = \cos(\xi/2)$, $\alpha_L = e^{i\phi}\sin(\xi/2)$, and $\mathcal{I}$, because they are fixed by the $\rho_{i,j}$ through $\mathcal{I} = |\rho_{12}|/(\rho_{11}\rho_{22})^{1/2}$, $\sin\xi = 2\sqrt{\rho_{11}\rho_{22}}$, and $\phi = \arg\rho_{12}$. Analogous statements hold for $\rho_p = \lambda_+|\mathbf{n}_+\rangle\langle\mathbf{n}_+| + \lambda_-|\mathbf{n}_-\rangle\langle\mathbf{n}_-|$. In particular, our Mach-Zehnder array can produce any state like $|\Phi_+\rangle$ in Eq. (27). It represents an alternative, more versatile entangling array than the one employed in [13]. Both devices allow us to manipulate $\mathcal{P}$ by ignoring a degree of freedom to which polarization has been entangled. This is just an instance of the interplay between a system and its environment, when the latter appears in its most simple form, namely, as a second system to which the first becomes entangled. Dealing with the system of interest alone generally requires changing from pure to mixed states, and from unitary to nonunitary evolutions.

B. Determination of the required unitary transformations

Now we show how our Mach-Zehnder array can produce states like that of Eq. (27). Such a task can be performed by implementing a unitary transformation U_{MZ} that produces $|\Phi_+\rangle$ out of, say, $|Lv\rangle$. We can implement, for example, the following unitary transformation:

$$|Rv\rangle \to |\Psi_+\rangle = \sqrt{\lambda_+}|\mathbf{m}_+,\mathbf{n}_-\rangle + \sqrt{\lambda_-}|\mathbf{m}_-,\mathbf{n}_+\rangle, \tag{35}$$

$$|Rh\rangle \to |\Psi_-\rangle = \sqrt{\lambda_-}|\mathbf{m}_+,\mathbf{n}_-\rangle - \sqrt{\lambda_+}|\mathbf{m}_-,\mathbf{n}_+\rangle, \tag{36}$$

$$|Lv\rangle \to |\Phi_+\rangle = \sqrt{\lambda_+}|\mathbf{m}_+,\mathbf{n}_+\rangle + \sqrt{\lambda_-}|\mathbf{m}_-,\mathbf{n}_-\rangle, \tag{37}$$

$$|Lh\rangle \to |\Phi_-\rangle = \sqrt{\lambda_-}|\mathbf{m}_+,\mathbf{n}_+\rangle - \sqrt{\lambda_+}|\mathbf{m}_-,\mathbf{n}_-\rangle, \tag{38}$$

with $|\mathbf{m}_\pm\rangle \in \mathcal{H}_S$ given by

$$|\mathbf{m}_+\rangle = \cos\theta_m|R\rangle + e^{i\varphi_m}\sin\theta_m|L\rangle, \tag{39}$$

$$|\mathbf{m}_-\rangle = -\sin\theta_m|R\rangle + e^{i\varphi_m}\cos\theta_m|L\rangle, \tag{40}$$

and similar expressions for $|\mathbf{n}_\pm\rangle \in \mathcal{H}_P$, with $|R\rangle \to |h\rangle$ and $|L\rangle \to |v\rangle$.

We begin by writing U_{MZ} in the form given by Eq. (24). In order to fix the operators V_i (see Fig. 1), we must diagonalize $U_{RR}^\dagger U_{RR}$, $U_{RR}U_{RR}^\dagger$, $U_{LL}^\dagger U_{LL}$, and $U_{LL}U_{LL}^\dagger$ (cf. [18]). It can be shown that $U_{RR}^\dagger U_{RR}$ and $U_{LL}U_{LL}^\dagger$ are unitarily equivalent and their common eigenvalues can be parametrized as $\cos^2\vartheta$ and $\cos^2\theta$, respectively. As for the eigenvectors, the notation is as follows: $U_{RR}^\dagger U_{RR} \to |\psi_i\rangle$, $U_{RR}U_{RR}^\dagger \to |\overline{\psi}_i\rangle$, $U_{LL}^\dagger U_{LL} \to |\chi_i\rangle$, and $U_{LL}U_{LL}^\dagger \to |\overline{\chi}_i\rangle$ ($i = 1,2$). It can be proved [18] that the U_{ij} are given by

$$U_{RR} = |\overline{\psi}_1\rangle\cos\vartheta\langle\psi_1| + |\overline{\psi}_2\rangle\cos\theta\langle\psi_2|, \tag{41}$$

$$U_{LL} = |\overline{\chi}_1\rangle\cos\vartheta\langle\chi_1| + |\overline{\chi}_2\rangle\cos\theta\langle\chi_2|, \tag{42}$$

$$iU_{RL} = |\overline{\psi}_1\rangle\sin\vartheta\langle\chi_1| + |\overline{\psi}_2\rangle\sin\theta\langle\chi_2|, \tag{43}$$

$$iU_{LR} = |\overline{\chi}_1\rangle\sin\vartheta\langle\psi_1| + |\overline{\chi}_2\rangle\sin\theta\langle\psi_2|. \tag{44}$$

The operators V_i, in turn, are given by the following expressions, which entail some arbitrariness in the choice of the phase factors [18]:

$$V_1 = i|\chi_1\rangle\langle\psi_1| + i|\chi_2\rangle\langle\psi_2|, \tag{45}$$

$$V_2 = -i|\overline{\psi}_1\rangle\langle\overline{\chi}_1| - i|\overline{\psi}_2\rangle\langle\overline{\chi}_2|, \tag{46}$$

$$V_R = |\overline{\chi}_1\rangle e^{i\vartheta}\langle\chi_1| + |\overline{\chi}_2\rangle e^{i\theta}\langle\chi_2|, \tag{47}$$

$$V_L = |\overline{\chi}_1\rangle e^{-i\vartheta}\langle\chi_1| + |\overline{\chi}_2\rangle e^{-i\theta}\langle\chi_2|. \tag{48}$$

By applying the above procedure to the present case we readily obtain

$$V_1 = -\mathcal{P}\sigma_3 + \sqrt{1-\mathcal{P}^2}\sigma_1, \tag{49}$$

$$V_2 = -e^{-i\varphi_m}\mathbb{1}, \tag{50}$$

$$V_R = e^{i(\varphi_m+\theta_m)}[|\mathbf{n}_-\rangle(-\sqrt{\lambda_+}\langle h| + \sqrt{\lambda_-}\langle v|) - i|\mathbf{n}_+\rangle(\sqrt{\lambda_-}\langle h| + \sqrt{\lambda_+}\langle v|)], \tag{51}$$

$$V_L = e^{i(\varphi_m-\theta_m)}[|\mathbf{n}_-\rangle(-\sqrt{\lambda_+}\langle h| + \sqrt{\lambda_-}\langle v|) + i|\mathbf{n}_+\rangle(\sqrt{\lambda_-}\langle h| + \sqrt{\lambda_+}\langle v|)]. \tag{52}$$

These operators can be implemented by setting phase shifters and waves plates on the Mach-Zehnder array. We defer to the next section, where we address a more general case, the detailed description of the necessary optical elements.

VI. SCHMIDT SPHERE AND GEOMETRIC PHASE FOR ENTANGLED STATES

A state like $|\Phi_+\rangle$ in Eq. (27) is a coherent superposition of two states whose relative weight is given by $\lambda_\pm$ and whose relative phase is zero. A more general case occurs when we consider a nonvanishing relative phase. To deal with such a case, we address the transformation

$$|Rv\rangle \to |\Psi_+\rangle = e^{-i\beta/2}\cos\xi|\mathbf{m}_+,\mathbf{n}_-\rangle + e^{i\beta/2}\sin\xi|\mathbf{m}_-,\mathbf{n}_+\rangle, \tag{53}$$

$$|Rh\rangle \to |\Psi_-\rangle = e^{-i\beta/2}\cos\xi|\mathbf{m}_+,\mathbf{n}_-\rangle - e^{i\beta/2}\sin\xi|\mathbf{m}_-,\mathbf{n}_+\rangle, \tag{54}$$

$$|Lv\rangle \to |\Phi_+\rangle = e^{-i\beta/2}\cos\xi|\mathbf{m}_+,\mathbf{n}_+\rangle + e^{i\beta/2}\sin\xi|\mathbf{m}_-,\mathbf{n}_-\rangle, \tag{55}$$

$$|Lh\rangle \to |\Phi_-\rangle = e^{-i\beta/2}\cos\xi|\mathbf{m}_+,\mathbf{n}_+\rangle - e^{i\beta/2}\sin\xi|\mathbf{m}_-,\mathbf{n}_-\rangle, \tag{56}$$

where we have set $\cos\xi = \sqrt{\lambda_+}$ and $\sin\xi = \sqrt{\lambda_-}$. Note that ξ measures both the degree of entanglement and the degree of polarization: $\mathcal{P} = \lambda_+ - \lambda_- = \cos(2\xi)$. Up to a global phase, the above states are of the form

$$|\Phi\rangle = \cos(\alpha/2)|\mathbf{m}_\pm,\mathbf{n}_\pm\rangle \pm e^{i\beta}\sin(\alpha/2)|\mathbf{m}_\pm,\mathbf{n}_\pm\rangle. \tag{57}$$

The angles α and β parametrize the so-called Schmidt sphere, where points have coordinates ($\sin\alpha\cos\beta$, $\sin\alpha\sin\beta$, $\cos\alpha$). The Schmidt sphere is analogous to the Poincaré and the Bloch spheres. Our Mach-Zehnder array provides us with full control over states like the above $|\Phi\rangle \in \mathcal{H}_P \otimes \mathcal{H}_S$. By keeping fixed $|\mathbf{m}_\pm\rangle$ and $|\mathbf{n}_\pm\rangle$ while varying α and β, we can make $|\Phi\rangle$ trace out some prescribed curve on the Schmidt sphere. This curve may give rise to a geometric phase [19] that can be tested using the present array. So, in order to implement the desired basis transformation we proceed as we did before and obtain the following V_i operators:

$$V_1 = -\mathcal{P}\sigma_3 + \sqrt{1-\mathcal{P}^2}\sigma_1, \tag{58}$$

$$V_2 = -e^{-i\varphi_m}\mathbb{1}, \tag{59}$$

$$V_R = e^{i(\varphi_m+\theta_m)}[e^{i\beta/2}|\mathbf{n}_-\rangle(-\cos\xi\langle h| + \sin\xi\langle v|) - ie^{-i\beta/2}|\mathbf{n}_+\rangle(\sin\xi\langle h| + \cos\xi\langle v|)], \tag{60}$$

$$V_L = e^{i(\varphi_m-\theta_m)}[e^{i\beta/2}|\mathbf{n}_-\rangle(-\cos\xi\langle h| + \sin\xi\langle v|) + ie^{-i\beta/2}|\mathbf{n}_+\rangle(\sin\xi\langle h| + \cos\xi\langle v|)]. \tag{61}$$

Our next task is to implement the above operators with the help of phase shifters and wave plates. As we can see, V_2 requires just a phase shifter. A commonly used one is a tilted glass plate. Alternatively, one can displace a mirror in the array with a piezoelectric transducer. Depending on the required accuracy, more sophisticated phase shifters could be used [23]. As for V_1, remembering that the action of a half-wave plate is given by $H(\gamma) = -i[\sigma_1\sin(2\gamma) + \sigma_3\cos(2\gamma)]$, it is clear that $V_1 = e^{i\pi/2}H(\gamma_1)$, with $2\gamma_1 = \cos^{-1}(-\mathcal{P})$; that is, $\gamma_1 = \xi + \pi/2$. In order to implement $V_{R,L}$ we first note that, as long as we are concerned with their first factors, it is only the relative phase $(2\theta_m)$ that matters, because $V_{R,L}$ act on different arms of the interferometer. Hence, a single phase shifter is needed for realizing these two factors. Writing $V_R = e^{i(\varphi_m+\theta_m)}O_R$, $V_L = e^{i(\varphi_m-\theta_m)}O_L$, we focus now on the implementation of $O_{R,L}$. We observe that O_R acts as follows:

$$\begin{aligned} O_R(-\cos\xi|h\rangle + \sin\xi|v\rangle) &= e^{i\beta/2}|\mathbf{n}_-\rangle, \\ O_R(\sin\xi|h\rangle + \cos\xi|v\rangle) &= -ie^{-i\beta/2}|\mathbf{n}_+\rangle, \end{aligned} \tag{62}$$

and similarly O_L [see Eq. (61)]. Setting

$$|\mathbf{n}_+\rangle = \cos\theta_p|h\rangle + e^{i\varphi_p}\sin\theta_p|v\rangle,$$
$$|\mathbf{n}_-\rangle = -\sin\theta_p|h\rangle + e^{i\varphi_p}\cos\theta_p|v\rangle,$$

we can readily show that $O_{R,L}$ can be realized with the following setups:

$$\begin{aligned} O_R &= e^{i[(2\varphi_p+3\pi)/4]}Q\left(\frac{\pi}{4}\right)Q\left(\frac{\pi+2\varphi_p}{4}\right)Q\left(\frac{\varphi_p-\pi-2\theta_p}{2}\right)H\left(\frac{2\pi-2\xi-\beta-2\theta_p+\varphi_p}{4}\right) \\ &\quad\times Q\left(\frac{-\pi-4\xi-2\beta}{4}\right)Q\left(\frac{\pi-4\xi}{4}\right), \\ O_L &= e^{i[(2\varphi_p+\pi)/4]}Q\left(\frac{\pi}{4}\right)Q\left(\frac{\pi+2\varphi_p}{4}\right)Q\left(\frac{\varphi_p-\pi-2\theta_p}{2}\right)H\left(\frac{\pi-2\xi-\beta-2\theta_p+\varphi_p}{4}\right) \\ &\quad\times Q\left(\frac{\pi-4\xi-2\beta}{4}\right)Q\left(\frac{\pi-4\xi}{4}\right). \end{aligned} \tag{63}$$

Hence, in order to implement the desired transformations we need to supply our Mach-Zehnder interferometer with the following elements: six wave plates on each arm of the interferometer, a phase shifter on one of these arms, a half-wave plate plus one phase shifter on the R-input arm, and an additional phase shifter on the R-output arm. With such an array we can test, for instance, the geometric phase Φ_g for entangled states. This phase has been predicted to generally differ from the sum of the phases acquired by each of the two entangled states [19]. In particular, Φ_g depends nontrivially on the relative phase β [see Eq. (57)], so that two states, while sharing the same degree of entanglement, might acquire two different geometric phases [19]. If we are interested in exploring cases for which $\beta = 0$, then the required number of retarders reduces to four on each arm. Indeed, for $\beta = 0$ we get

$$\begin{aligned} O_R &= e^{i[(2\varphi_p+3\pi)/4]}Q\left(\frac{\pi}{4}\right)Q\left(\frac{2\varphi_p+\pi}{4}\right)Q\left(\frac{\varphi_p-2\theta_p}{2}+\frac{\pi}{2}\right)H\left(\frac{\varphi_p-2\xi-2\theta_p}{4}+\frac{\pi}{2}\right), \\ O_L &= e^{i[(2\varphi_p+\pi)/4]}Q\left(\frac{\pi}{4}\right)Q\left(\frac{2\varphi_p+\pi}{4}\right)Q\left(\frac{\varphi_p-2\theta_p}{2}\right)H\left(\frac{\varphi_p-2\xi-2\theta_p}{4}\right). \end{aligned} \tag{64}$$

Our Mach-Zehnder array can be used to test more general predictions than those in [19]. The latter—when formulated in terms of entangled spins—are restricted to time-independent Hamiltonians. However, the time-dependent case [24] could be equally well tested with the present array. Furthermore, our arrangement provides us with great versatility when testing, e.g., how the strength of the coupling between the entangled subsystems affects Berry's phase [25], or the high sensitivity of this phase against minute variations of the input state [26], which follows from the nonlinear behavior of Φ_g. Finally, it is worth noting that when we focus on some particular test, the number of wave plates in the array can be drastically reduced, as has been reported elsewhere [22].

VII. COMPLEMENTARITY RELATIONS AND BELL'S MEASURE

A. Distinguishability and visibility

Let us now address the mutual constraint between complementary quantities such as fringe visibility and which-way information. As already said, there is a fundamental difference between the two following constraints: $\mathcal{W}^2 + \mathcal{V}^2 \leqslant 1$ and $\mathcal{D}^2 + \mathcal{V}^2 \leqslant 1$. While the predictability $\mathcal{W}$ refers to our *a priori* which-way knowledge, $\mathcal{D}$ refers to the which-way information being stored in a which-way marker [4]. In our case, $\mathcal{D}$ measures the which-way information being stored in the polarization degree of freedom. Polarization is thus our auxiliary physical system. Once we have coupled it to the path degree of freedom, we can use it as a which-way marker. Our array can be used to test the constraint between $\mathcal{V}$ and $\mathcal{D}$. The simplest case is that of a pure state, for which

$$\mathcal{D}^2 + \mathcal{V}^2 = 1, \tag{65}$$

as shown in [4]. In order to test this prediction, we first note that $\mathcal{D}$ is given by $\mathcal{D} = \mathrm{Tr}_P\,|\rho_p^{(R)} - \rho_p^{(L)}|/2$. To derive this expression, one considers the output polarization state $\rho_p^{(\mathrm{out})} = \mathrm{Tr}_S\,\rho^{(\mathrm{out})} = \mathrm{Tr}_S(U_{\mathrm{MZ}}\rho^{(\mathrm{in})}U_{\mathrm{MZ}}^\dagger)$. Following [4], it can be shown that $\rho_p^{(\mathrm{out})}$ may be written as

$$\rho_p^{(\mathrm{out})} = \tfrac{1}{2}\big(\rho_p^{(R)} + \rho_p^{(L)}\big), \tag{66}$$

with $\rho_p^{(i)} = V_i\rho^{(\mathrm{in})}V_i^\dagger$, $i = R,L$. Given an initial state $\rho^{(\mathrm{in})}$, one can straightforwardly evaluate the eigenvalues $\lambda_{i=1,2}$ of the matrix $|\rho_p^{(R)} - \rho_p^{(L)}|$, from which it follows that $\mathcal{D} = (\lambda_1 + \lambda_2)/2$. The visibility $\mathcal{V}$, on the other hand, can be shown [4] to be given by

$$\mathcal{V} = \big|\,\mathrm{Tr}_P\,\big(\rho_p^{(\mathrm{in})}V_L^\dagger V_R\big)\big|. \tag{67}$$

We have numerically verified that $\mathcal{D}^2 + \mathcal{V}^2 = 1$ holds true for different initial pure states such as, e.g., $\rho^{(\mathrm{in})} = |Rv\rangle\langle Rv|$, and for different values of the parameters fixing U_{MZ}. Experiments could then be performed that complement recently reported tests of the above constraint [7,27].

B. Bell's measure

There is yet another way to assess the extent to which a system is capable of interfering. Traditionally, it is the degree of coherence that has been associated with the ability to interfere. Now, a diminished coherence is not exclusively caused by underlying fluctuations that originate at the source or during propagation. As already stressed, the degree of coherence can also be diminished because the observed degree of freedom is entangled with another, unobserved one. In such a case, it might happen that the degree of coherence, e.g., $\mathcal{P}$ in the case of polarization, does not convey all the information about the "accessible coherence" which is, so to say, being stored in other, unobserved degrees of freedom. Kagalwala *et al.* [13] used Bell's measure $\mathcal{B}_{\max}$ to diagnose how much accessible coherence might be carried along by a degree of freedom. It might occur that, e.g., $\mathcal{P} = 0$ for some light beam, even though there is no randomness affecting the beam. $\mathcal{B}_{\max}$ should be better suited than the degree of coherence to assess the capability to produce interference. Bell's measure $\mathcal{B}_{\max}$ is defined in terms of the well-known Clauser-Horne-Shimony-Holt (CHSH) expression [28]

$$B = |C(\theta_1,\theta_2) + C(\theta_1,\theta_2') + C(\theta_1',\theta_2) - C(\theta_1',\theta_2')|. \tag{68}$$

The correlation functions $C(\theta_1,\theta_2)$ entering B are defined in our case, i.e., for polarization-path states, as $C(\theta_1,\theta_2) = P_{Rh} - P_{Rv} - P_{Lh} + P_{Lv}$. Here, $P_{Rh}(\theta_1,\theta_2)$ means the probability of measuring a horizontally polarized beam moving along the R direction. The angles $\theta_{i=1,2}$ are the orientation angles of the corresponding measuring devices. Similar definitions apply for the other P's. These probabilities, and hence B, depend on the entangled state $|\Psi\rangle$ being considered. Generally, $|\Psi\rangle$ will depend on one or more parameters, and so will B. Bell's measure $\mathcal{B}_{\max}$ is defined by maximizing B over these parameters. Now, analogously to the degree of polarization, one can define a degree of coherence $\mathcal{S}$ associated with the path degree of freedom. Let us focus on a coherent beam $|\Psi\rangle$ and consider the global coherency matrix $G = |\Psi\rangle\langle\Psi|$. We define $G_P := \mathrm{Tr}_S\,|\Psi\rangle\langle\Psi|$ and $G_S := \mathrm{Tr}_P\,|\Psi\rangle\langle\Psi|$ and denote their common eigenvalues by $\mu_{i=1,2}$. The two degrees of coherence being considered are $\mathcal{P} = |\mu_1 - \mu_2| = \mathcal{S}$. In the present case, it can be proved [13] that $4\mathcal{P}^2 + \mathcal{B}_{\max}^2 = 4\mathcal{S}^2 + \mathcal{B}_{\max}^2 = 8$. More generally, for partially coherent beams one gets $4\mathcal{P}^2 + \mathcal{B}_{\max}^2 \leqslant 8$ and $4\mathcal{S}^2 + \mathcal{B}_{\max}^2 \leqslant 8$. These results led Kagalwala *et al.* [13] to introduce "degrees of accessible coherence" for each degree of freedom, which applied to our case read

$$\mathcal{C}_p = \frac{\mathcal{P}^2}{2} + \left(\frac{\mathcal{B}_{\max}}{2\sqrt{2}}\right)^2, \quad \mathcal{C}_s = \frac{\mathcal{S}^2}{2} + \left(\frac{\mathcal{B}_{\max}}{2\sqrt{2}}\right)^2. \tag{69}$$

Kagalwala *et al.* [13] dealt with states that are entangled in polarization and spatial parity. The latter is a binary degree of freedom that refers to the even $[\psi_e(x) = \psi_e(-x)]$ and odd $[\psi_o(x) = -\psi_o(-x)]$ components of a scalar function $\psi(x)$. Experimentally, spatial parity can be manipulated with a spatial light modulator (SLM). As for its mathematical description, one can apply the same tools used for polarization, i.e., Jones vectors, Mueller matrices, etc. Because spatial parity and the path DOF are formally the same, the results found in [13] apply in our case as well. It is worth discussing in some detail the results reported in [13], as they could be tested using independent experimental techniques, like the ones offered by our Mach-Zehnder array. Besides, as we will see later, there are some results in [13] that need modification.

The following case is well suited to be addressed with our Mach-Zehnder device. Using a polarization-sensitive SLM (PS-SLM), Kagalwala *el al.* introduced a phase $\varphi/2$ into the h component, leaving the v component unaffected. Thus, submitting a direct-product state $|\Psi\rangle_{\mathrm{dp}} = |\psi\rangle_p \otimes |\phi\rangle_{\mathrm{par}}$ to the action of a PS-SLM, one generates an entangled state $|\Psi\rangle_e = U_{\text{PS-SLM}}|\Psi\rangle_{\mathrm{dp}}$. In [13], the case $|\psi\rangle_p = (1,1)_p^T/\sqrt{2}$, $|\phi\rangle_{\mathrm{par}} = (1,0)_s^T$ was considered. The state reported in [13] is $|\Psi\rangle_e = (\cos(\varphi/2), -i\sin(\varphi/2),1,0)^T/\sqrt{2}$. Accordingly, the action of the PS-SLM should be given by

$$R_{\varphi/2} = \begin{pmatrix} \cos(\varphi/2) & -i\,\sin(\varphi/2) \\ -i\,\sin(\varphi/2) & \cos(\varphi/2) \end{pmatrix} \tag{70}$$

attached to the h component and $\mathbb{1}_{\text{par}}$ to the v component. Indeed, this gives

$$U_{\text{PS-SLM}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_p\otimes\begin{pmatrix}1\\0\end{pmatrix}_s\right] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_p\otimes R_{\varphi/2}\begin{pmatrix}1\\0\end{pmatrix}_s+\begin{pmatrix}0\\1\end{pmatrix}_p\otimes\begin{pmatrix}1\\0\end{pmatrix}_s\right] \quad (71)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos(\varphi/2)\\ -i\,\sin(\varphi/2)\\ 1\\ 0\end{pmatrix} = |\Psi\rangle_e\,. \quad (72)$$

The correlation function for this $|\Psi\rangle_e$ reads

$$C(\theta_1,\theta_2,\varphi) = \sin\frac{\varphi}{2}\sin\left(\theta_1-\theta_2-\frac{\varphi}{2}\right) - \left(1-\sin\frac{\varphi}{2}\right)\cos\left(\theta_2+\frac{\varphi}{2}\right)\sin\theta_1. \quad (73)$$

Following [13], we focus on the states having $\varphi = 0,\ \pi/2$, and π. For them, we obtain

$$\begin{aligned} C(\theta_1,\theta_2,0) &= -\sin\theta_1\cos\theta_2,\\ C(\theta_1,\theta_2,\pi) &= -\cos(\theta_1-\theta_2),\\ C(\theta_1,\theta_2,\pi/2) &= \sqrt{\frac{1}{2}}\sin\left(\theta_1-\theta_2-\frac{\pi}{4}\right) - \left(1-\sqrt{\frac{1}{2}}\right)\cos\left(\theta_2+\frac{\pi}{4}\right)\sin\theta_1. \end{aligned} \quad (74)$$

Measured values of the above correlation functions have been reported in [13], together with values of the global agreement between experimental data and Eqs. (74). These are 0.936, 0.944, and 0.903, respectively, in the root-mean-square sense [13]. However, we note that $C(\theta_1,\theta_2,0)$ has the wrong sign, and plotting $C(\theta_1,\theta_2,\pi/2)$ we observe some salient differences between this plot (see Fig. 2), and the data plotted

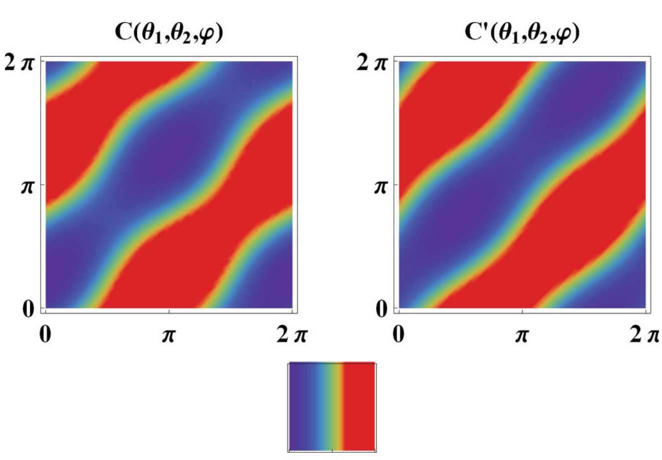

FIG. 2. (Color online) Correlation functions $C(\theta_1,\theta_2,\varphi)$ and $C'(\theta_1,\theta_2,\varphi)$. The latter appears to outperform the former in fitting recently reported experimental data (see Ref. [13]).

in [13] [see Fig. 2(b) there]. Considering instead the correlation function

$$C'(\theta_1,\theta_2,\varphi) = \sin\left(\theta_2+\frac{\varphi}{2}\right)\sin\theta_1 - \cos\left(\theta_2+\frac{\varphi}{2}\right)\cos\theta_1\sin\frac{\varphi}{2}, \quad (75)$$

we get

$$\begin{aligned} C'(\theta_1,\theta_2,0) &= \sin\theta_1\cos\theta_2,\\ C'(\theta_1,\theta_2,\pi) &= -\cos(\theta_1-\theta_2),\\ C'(\theta_1,\theta_2,\pi/2) &= \sin\left(\theta_2+\frac{\pi}{4}\right)\sin\theta_1 - \sqrt{\frac{1}{2}}\cos\left(\theta_2+\frac{\pi}{4}\right)\cos\theta_1. \end{aligned} \quad (76)$$

By plotting $C'(\theta_1,\theta_2,\pi/2)$ (see Fig. 2), we see that it better fits the experimental values, as compared to $C(\theta_1,\theta_2,\pi/2)$. Thus, $C'(\theta_1,\theta_2,\varphi)$ appears to be the correct result. This is further substantiated by the fact that $C(\theta_1,\theta_2,\varphi)$ does not correctly reduce to the well-known expressions for the singlet state, used in Bell-CHSH inequalities [29], while $C'(\theta_1,\theta_2,\varphi)$ does reduce to such expressions. The problem seems to lie on the assumption that the action of the PS-SLM is given by the unitary operator $U_{\text{PS-SLM}} = |h\rangle\langle h|\otimes R_{\varphi/2} + |h\rangle\langle h|\otimes\mathbb{1}_{\text{par}}$, with $R_{\varphi/2}$ given by Eq. (70). If instead of this $R_{\varphi/2}$ we use

$$R'_{\varphi/2} = \begin{pmatrix}\cos(\varphi/2) & \sin(\varphi/2)\\ -\sin(\varphi/2) & \cos(\varphi/2)\end{pmatrix}, \quad (77)$$

then we obtain results in accordance with the reported experimental data. Indeed, the corresponding Jones vector at the output of the PS-SLM now reads $|\Psi\rangle'_e = (1/\sqrt{2})(\cos(\varphi/2), -\sin(\varphi/2),1,0)^T$ and the correlation function that belongs to this vector is $C'(\theta_1,\theta_2,\varphi)$. It is thus not $R_{\varphi/2}$ but $R'_{\varphi/2}$ the operation we should seek to implement with our Mach-Zehnder array, if we aim at testing features like those in [13] with an independent experimental technique.

Now, using either $C(\theta_1,\theta_2,\varphi)$ or $C'(\theta_1,\theta_2,\varphi)$ in Eq. (68), we obtain the same value for $\mathcal{B}_{\max}$, namely,

$$\mathcal{B}_{\max}(\varphi) = 2\sqrt{1+\sin^2(\varphi/2)}. \quad (78)$$

Experimental values of $\mathcal{B}_{\max}(\varphi)$ were also reported in [13] for the range $\varphi\in[0,2\pi]$, showing good agreement with Eq. (78). The simple expression for $\mathcal{B}_{\max}(\varphi)$ above can be derived by applying the Popescu-Rohrlich theorem [30]. This theorem generalizes Cirel'son's bound [31] $\mathcal{B}_{\max} = 2\sqrt{2}$ by addressing general bipartite states of two-level systems. By Schmidt decomposition, these states can be written in the form

$$|\Psi\rangle = \alpha|\mathbf{m}_+,\mathbf{n}_+\rangle + \beta|\mathbf{m}_-,\mathbf{n}_-\rangle, \quad (79)$$

where α and β may be chosen real, with $\alpha^2+\beta^2 = 1$. The Popescu-Rohrlich theorem then establishes that

$$\mathcal{B}_{\max} = 2\sqrt{1+4\alpha^2\beta^2}. \quad (80)$$

Thus, given a state $|\Psi\rangle$, in order to get $\mathcal{B}_{\max}$ we need only to construct the correlation function $G_P = \text{Tr}_S\,|\Psi\rangle\langle\Psi|$

or else $G_S = \text{Tr}_p |\Psi\rangle\langle\Psi|$, and then calculate their common eigenvalues $\lambda_{1,2}$. The required Schmidt decomposition is then given by $|\Psi\rangle = \sqrt{\lambda_1}|\mathbf{m}_+,\mathbf{n}_+\rangle + \sqrt{\lambda_2}|\mathbf{m}_-,\mathbf{n}_-\rangle$. Doing this for both $|\Psi\rangle_e$ and $|\Psi\rangle_e'$ we obtain the same result, Eq. (78).

In order to experimentally test these and other predictions, we may use our Mach-Zehnder array. We can produce states of the form $|\Psi\rangle_e'$ by the procedure we used before. Indeed, this state reads

$$|\Psi\rangle_e' = \frac{1}{\sqrt{2}}|\mathbf{m}_1,h\rangle + \frac{1}{\sqrt{2}}|\mathbf{m}_2,v\rangle, \quad (81)$$

and is thus of the form previously treated [see, e.g., Eq. (35)], with $|\mathbf{m}_1\rangle = \cos(\varphi/2)|R\rangle - \sin(\varphi/2)|L\rangle$ and $|\mathbf{m}_2\rangle = |R\rangle$. Hence, we need only make $|\Psi\rangle_e'$ a member of an orthonormal basis of $\mathcal{H}_S \otimes \mathcal{H}_P$ and then proceed as we did before. That is, we fix U_{MZ} as we did when dealing with the basis transformation $\{|Rv\rangle,|Rh\rangle,|Lv\rangle,|Lh\rangle\} \to \{|\Phi_+\rangle,|\Phi_-\rangle,|\Psi_+\rangle,|\Psi_-\rangle\}$. In this way we can address issues like those addressed in [13], but with an increased versatility. For instance, mounting our array so that it performs the transformation U_{MZ} that we considered before [Eq. (35)] we can produce the state $|\Psi_+\rangle = \sqrt{\lambda_+}|\mathbf{m}_+,\mathbf{n}_-\rangle + \sqrt{\lambda_-}|\mathbf{m}_-,\mathbf{n}_+\rangle$, with $\lambda_\pm = (1 \pm \mathcal{P})/2$. This gives

$$\mathcal{B}_{\max}(\mathcal{P}) = 2\sqrt{2 - \mathcal{P}^2}. \quad (82)$$

Hence, Csirel'son's bound is reached when the marginal state is totally unpolarized. This is as expected, because for $\mathcal{P} = 0$ the bipartite state is maximally entangled ($\lambda_\pm = 1/2$). On the other hand, the classical bound $B = 2$ is attained for a totally polarized state ($\mathcal{P} = 1$). This is a direct-product state $|\Psi_+\rangle = |\mathbf{m}_+\rangle|\mathbf{n}_-\rangle$, as expected, because $\mathcal{P} = 1$ implies that ρ_p is a projector, i.e., a pure state, and a marginal state cannot be pure and part of an entangled state at the same time. As for the degrees of accessible coherence $\mathcal{C}_p$ and $\mathcal{C}_s$ [see Eq. (69)], they are unity. Indeed, $\mathcal{C}_p = \mathcal{P}^2/2 + 4(2 - \mathcal{P}^2)/8 = 1$, and similarly for $\mathcal{C}_s$. This is in accordance with the definitions of $\mathcal{C}_{p,s}$, which satisfy $\mathcal{C}_p = \mathcal{C}_s = 1$ for coherent beams [13]. Here again, our Mach-Zehnder array proves useful to test these theoretical predictions.

VIII. SUMMARY AND CONCLUSIONS

We have discussed the connection between the degree of polarization, indistinguishability, and entanglement. Since relatively recent times, it has been realized that both polarization and indistinguishability are intimately connected to coherence [2,32]. Mandel addressed this issue by considering the two photon paths of a Young's interferometer [2]. He made precise the connection between the degree of indistinguishability $\mathcal{I}$ and coherence, by showing that $\mathcal{I}$ equals the degree of coherence. Although Mandel considered states such as $|\psi\rangle = \alpha|1\rangle_1|0\rangle_2 + \beta|0\rangle_1|1\rangle_2$ and explicitly pointed out that these are entangled states, we have seen that his findings do not really invoke entanglement, nor the quantized nature of the field. The quantities he constructed belong to a two-dimensional Hilbert space with basis $\{|1\rangle \equiv |1\rangle_1|0\rangle_2, |2\rangle \equiv |0\rangle_1|1\rangle_2\}$, which is just a subspace of the four-dimensional, direct-product space $\text{Span}\{|0\rangle_1,|1\rangle_1\} \otimes \text{Span}\{|0\rangle_2,|1\rangle_2\}$. Formally, Mandel's results hold true for any two-dimensional Hilbert space, in particular for polarization space, the one addressed by Qian and Eberly [17]. These authors highlight the intrinsically entangled nature of a field such as $\mathbf{E} = \mathbf{x}E_x + \mathbf{y}E_y$, which belongs to a direct-product space made of laboratory space and the space of statistical functions. The degree of polarization $\mathcal{P}$ appears here as a measure of the degree of separability of the state. The standard polarization (or coherence) matrix is constructed within a two-dimensional Hilbert space that is obtained by tracing over the statistical functions. As we have shown in the present work, applying Mandel's decomposition brings $\mathcal{I}$ into play, and a formal connection between $\mathcal{P}$ and $\mathcal{I}$ can be obtained. However, $\mathcal{I}$ has in this context a different meaning as in Mandel's framework. In Qian-Eberly's framework $\mathcal{I}$ measures the magnitude of cross correlations between field components, instead of being connected to indistinguishability between paths in space. As a third approach, we have the one due to Kagalwala *et al.* [13], who also addressed polarization and entanglement. However, while Qian and Mandel resort to the intrinsic entanglement of a field, Kagalwala *et al.* use an entangling device to produce fields such as $E(x) = E_e\psi_e(x) + E_o\psi_o(x)$. With the help of such a device they could explore several issues that the present work also addresses. In our case, we propose using an entangling device, a Mach-Zehnder-like array, that has a greater versatility than the one used in [13]. Among other things, it serves to give a sound physical sense to Lahiri's inequality [1]: $\mathcal{P} \geqslant \mathcal{I}$. As we have seen, this inequality was originally derived following Mandel's procedure and working within a single Hilbert space, namely, polarization space. A proper interpretation in terms of indistinguishability or wave-particle duality was thus inappropriate. It is only by entangling polarization with a second degree of freedom, namely, path space, that a sound physical interpretation can be given to the constraint $\mathcal{P} \geqslant \mathcal{I}$. The present work allows us to operationally prescribe how to test such an inequality.

Our Mach-Zehnder-like array serves as a tool for testing several properties that intertwine the classical and the quantum domain. Possible experiments could use either a classical or a quantum source of light. Polarization and path (momentum) are attributes—i.e., degrees of freedom—that can be equally well assigned to both a beam of photons and a classical (paraxial) beam of light. By entangling these two degrees of freedom several features show up, e.g., constraints of complementary quantities, geometric phases, partial coherence without randomness, etc. All these features could be tested with the proposed array, which provides us with increased capabilities as compared to recently reported experiments. Besides these experimental advantages, the setup we have discussed here serves as a theoretical tool for studying manifold properties of entangled degrees of freedom, irrespective of their classical or quantal nature.

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