Gaussian local unitary equivalence of n-mode Gaussian states and Gaussian transformations by local operations with classical communication

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We derive necessary and sufficient conditions for arbitrary multimode (pure or mixed) Gaussian states to be equivalent under Gaussian local unitary operations. To do so, we introduce a standard form for Gaussian states, which has the properties that (i) every state can be transformed into its standard form via Gaussian local unitaries and (ii) it is unique and (iii) it can be easily computed. Thus, two states are equivalent under Gaussian local unitaries if and only if their standard forms coincide. We explicitly derive the standard form for two- and three-mode Gaussian pure states. We then investigate transformations between these classes by means of Gaussian local operations assisted by classical communication. For three-mode pure states, we identify a global property that cannot be created but only destroyed by local operations. This implies that the highly entangled family of symmetric three-mode Gaussian states is not sufficient to generate all three-mode Gaussian states by local Gaussian operations.

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I. INTRODUCTION

Since most applications of quantum information rest upon the subtle properties of multipartite quantum systems, the qualification and quantification of multipartite entanglement is a central task of quantum information theory. Whereas the bipartite case for finite- as well as for certain infinite-dimensional systems is well understood, many questions are still open in the multipartite setting [1].

The set of *Gaussian states* still plays a major role in current experiments dealing with continuous quantum variables, as it comprises those states that are processed in most experiments. This, and the mathematical simplicity of those states, which can be fully characterized by the finite set of first and second moments, are the reasons why mainly Gaussian states have been investigated in the context of continuous-variable (CV) quantum information [2].

Regarding the entanglement properties of Gaussian states, it has been shown that in the finite-dimensional case a separable state has positive partial transpose and that there exist entangled states with positive partial transpose [3]. However, for party A possessing one mode and party B arbitrarily many, it has been shown that partial transposition leads to a necessary and sufficient condition for separability [3]. For the general bipartite case, i.e., when both parties possess an arbitrary number of modes, efficiently testable necessary and sufficient conditions of separability have been derived [4,5]. In contrast to the case of finite-dimensional Hilbert spaces, the question of which states can be distilled to pure entanglement has been solved for bipartite Gaussian states. In fact, is was shown [6] that a bipartite Gaussian state is distillable if and only if (iff) its partial transpose is not positive semidefinite. In Refs. [7–10] the problem of manipulation of Gaussian states has been studied. In particular, in [9,10] the most general operations transforming Gaussian states to Gaussian states were studied. These operations are called Gaussian operations. In [10] it has been proven that it is not possible to distill Gaussian states using Gaussian operations (see also [8,9]).

The knowledge about entanglement in the multipartite setting is still far from complete, although a large number of (mostly) partial results has been obtained. The generation of pure multipartite entangled Gaussian states was discussed in [11]. A classification of multipartite entanglement classes of arbitrary three-mode Gaussian states has been presented in [12]. Practical criteria for the certification of genuine multipartite entanglement were derived in [13]. A general solution to the multipartite separability problem in the Gaussian case was provided by the Gaussian entanglement witnesses and related semidefinite programs studied in [5]. A large number of quantum optical experiments demonstrating multimode entanglement in increasingly large systems [14–20], culminating in 10 000-mode (time-bin) entanglement reported in [21]. Moreover, several standard entanglement measures have been adapted to the Gaussian setting (such as, e.g., robustness [10], obtainable from a semidefinite program as described in [5]) or Gaussian localizable entanglement [22]) and notions such as Greenberger-Horne-Zeilinger- (GHZ-) like states [23], maximal entanglement (as quantified by bipartite entanglement) [24], and monogamy of entanglement [25] have been specialized to the Gaussian setting.

Despite these advances, the study of multipartite entanglement is still in an early stage. One method to gain more insight into the entanglement properties of multipartite states is to investigate their interconvertibility. An important fine-grained classification of multipartite entangled states sorts them according to convertibility by local unitaries, leading to the notion of local unitary (LU) equivalence [26–29]. Clearly, two LU-equivalent states possess the same amount of entanglement and are equivalent as a nonlocal resource. LU equivalence leads to a very detailed classification of multipartite states with a continuum of inequivalent classes. A more coarse-grained (and therefore often more insightful) picture emerges if a larger class of transformations is allowed. Especially useful for entanglement classification is to allow for non-trace-preserving operations [(partial) measurements] and classical communication between parties, which leads to the set of stochastic local operations and classical communication (SLOCC) [30]. SLOCC play an important role in entanglement theory [31–37]. SLOCC convertibility gives rise to fewer equivalence classes than LU equivalence and in some cases only finitely many [31–33] SLOCC classes exist.

For Gaussian states, it is reasonable to consider convertibility under Gaussian operations. Conversion (of mixed states) under trace-preserving local Gaussian operations (not necessarily unitary) was investigated for the two-mode case in [38] and for the general bi- and tripartite setting in [39,40], while transformation under trace-nonpreserving local Gaussian operations has been investigated in [41] for pure bipartite states. The equivalence of Gaussian states under Gaussian local unitaries (GLUs) was studied for the (mixed) bipartite setting in [42] and for more parties in [25,40,43,44]. In [44] and [43] standard forms for "generic" n-mode mixed and pure states were introduced. The case of pure three-mode states has been studied in detail in [23]. There, it is shown (for generic pure Gaussian states) that the GLU equivalence classes are characterized by three positive numbers (related to local purities) and a simple standard form was derived.

The aim of this paper is to derive a standard form for arbitrary Gaussian states which has the properties that (i) every state can be transformed into its standard form via Gaussian local unitaries, (ii) it is unique, and (iii) it can be easily computed. Due to these properties, the solution to the Gaussian LU equivalence problem follows easily. We then focus on pure Gaussian three-mode states and show that any such state is characterized by the three local purities. The standard form of those states is used to investigate the manipulation of those states using GLOCC. We show that the completely symmetric states, which are sometimes referred to as maximally entangled states, *cannot* be used to obtain an arbitrary state via GLOCC.

The remainder of the paper is organized as follows. In Sec. II we briefly review the basic concepts and results on Gaussian states needed later. In Sec. III we present a standard form for arbitrary (pure and mixed) n-mode Gaussian states, where all modes are spatially separated, and derive the necessary and sufficient conditions for Gaussian states to be equivalent under Gaussian local unitaries. As we will show, this criterion can efficiently be applied, since it involves only the computation of the singular-value decomposition of 2×2 matrices, independently of the system size. We will then demonstrate our methods by considering first the simplest case of two modes, where we show that our standard form coincides with the one presented in [45,46]. In Sec. III B, we investigate the different GLU equivalence classes of three-mode Gaussian states. We show that *any* pure three-mode Gaussian state is GLU-equivalent to a state with no correlations between the X and P quadratures and that an arbitrary three-mode pure Gaussian state is (up to GLUs) uniquely characterized via the three local purities, i.e., by the bipartite entanglement between each single mode and the remaining two modes. This reproduces the results of [23] but shows that they apply to all three-mode states (not only a subset of generic states). In order to obtain more insight into the entanglement properties of three-mode states, we consider in Sec. V the more general set of Gaussian local operations assisted by classical communication. In particular, we show that it is not possible to obtain from the symmetric Gaussian pure three-mode states (which are sometimes referred to as maximally entangled states or continuous-variables analogs of both GHZ and W states ("CV GHZ/W-states", for short) [11,23,25]), all pure three-mode state via GLOCC. This implies that those states are not, as the two-mode squeezed states are in the bipartite case, sufficient to obtain deterministically any other state via local Gaussian operations (and thus not a Gaussian analog of the maximally entangled set introduced in [36]). In contrast, we finally present a class of states from which, in particular, all symmetric states can be obtained via GLOCC. Hence, this class of states might be called more entangled than the symmetric one.

II. PRELIMINARIES

We summarize here some results concerning Gaussian states and introduce our notation. We consider systems composed of n modes, i.e., n distinguishable infinite-dimensional subsystems, each with Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. To each mode $k = 1,\ldots,n$ belong two canonical observables X_k, P_k which obey the commutation relation $[X_k, P_k] = i$. Defining $R_{2k-1} = X_k, R_{2k} = P_k$, these relations are summarized as $[R_l, R_m] = -i J_{lm}$, using the antisymmetric $2n \times 2n$ matrix

$$J \equiv \oplus_{k=1}^{n} J_1, \quad J_1 \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{1}$$

where here and in the following $\oplus$ denotes the direct sum. Let us denote the unitary displacement operator by

$$D(x) = e^{i\sum_k (q_k X_k + p_k P_k)} \equiv e^{i x\cdot R}, \tag{2}$$

where $x = (q_1, p_1, \ldots, q_n, p_n) \in \mathbb{R}^{2n}$. Using this notation, the characteristic function of a state ρ is defined as

$$\chi_\rho(x) = \mathrm{tr}[\rho D(x)]. \tag{3}$$

Gaussian states are those states for which χ is a Gaussian multivariant function of the phase space coordinates x [47], i.e.,

$$\chi_\rho(x) = e^{-(1/4)x^T \gamma x - i d^T x}. \tag{4}$$

Here, γ is a real, symmetric, strictly positive $2n \times 2n$ matrix, the *covariance matrix* (CM), and $d \in \mathbb{R}^{2n}$ is a real vector, the *displacement*. A Gaussian state is completely determined by γ and d. Note that both γ and d are directly measurable quantities, as their elements γ_{kl} and d_k are determined by the expectation values and variances of the operators R_k, via

$$d_k = \mathrm{tr}(\rho R_k), \tag{5}$$

$$\gamma_{kl} = 2\mathrm{Re}\{\mathrm{tr}[\rho(R_k - d_k)(R_l - d_l)]\}. \tag{6}$$

The displacement of a (known) state can always be adjusted to $d = 0$ by a sequence of local unitary operators applied to individual modes.[1] Thus, the first moments are irrelevant for both the study of GLU equivalence classes and the entanglement contained in the state and will therefore be set to zero.

[1]The unitaries are generated by linear Hamiltonians.

Not all real, symmetric, positive matrices γ correspond to the CM of a physical state; they also have to satisfy the uncertainty principle. There are several equivalent ways to characterize valid CMs, which are all useful in the following. Before we summarize them in Lemma 1 let us recall that a (real) linear transformation S on phase space is called *symplectic* if it preserves J, i.e., if $SJS^T = J$ holds. The group of real symplectic $2n \times 2n$ matrices is denoted by $Sp_{2n}(\mathbb{R})$. Let us now state the conditions for a matrix to be a valid CM.

Lemma 1 (covariance matrices). A real, symmetric, and positive $2n \times 2n$ matrix γ is the CM of a physical state iff one of the following equivalent conditions holds:

$$\gamma + J\gamma^{-1}J \geqslant 0, \tag{7a}$$

$$\gamma - iJ \geqslant 0, \tag{7b}$$

$$\gamma = S^T(D \oplus D)S, \tag{7c}$$

for S symplectic and $D \geqslant \mathbb{1}$ diagonal. The CM γ describes a *pure* state iff equality holds in Eq. (7a) or, equivalently, iff $D = \mathbb{1}$ in Eq. (7c), i.e., iff $\det \gamma = 1$.

The proofs of these statements can be found in [3,47,48], respectively. As an example of a valid CM, let us recall that the CM of an arbitrary pure two-mode state (1×1 case) γ can be written as [49]

$$\gamma = (S_1 \oplus S_2)\begin{pmatrix} \cosh r\mathbb{1} & \sinh r\sigma_z \\ \sinh r\sigma_z & \cosh r\mathbb{1}\end{pmatrix}\left(S_1^T \oplus S_2^T\right). \tag{8}$$

Here and in the following $S_{1,2}$ are local symplectic matrices, $r \geqslant 0$, and $\sigma_x,\sigma_y,\sigma_z$ denote the Pauli operators. The parameter r contains all information about the entanglement of the state, whereas S_1 and S_2 contain information about local squeezing.[2] An example of a pure state would be the two-mode squeezed (TMS) state, whose CM is given by Eq. (8) with $S_1 = S_2 = \mathbb{1}$.

Whenever we consider a bipartite splitting of the state (n modes at one side and m modes at the other, which we call the $n \times m$ case in the following) we might write the CM in the index-free block form

$$\gamma = \begin{pmatrix} A & C \\ C^T & B\end{pmatrix}. \tag{9}$$

Here A, B, and C are $2n \times 2n$, $2m \times 2m$, and $2n \times 2m$ matrices, respectively. Note that A (B) is the CM corresponding to the reduced state of the first (second) system, respectively. The correlations between both systems are described by the matrix C, which vanishes for product states.

Since we are interested in Gaussian local unitary equivalence classes in this paper, we also review here how the CM γ (and the displacement d) of a Gaussian state ρ change under the evolution of a Gaussian unitary operator U. As can be easily verified, a unitary operator transforms any Gaussian state into a Gaussian state (i.e., describes a Gaussian operation) iff there exists a symplectic matrix S and a real vector $r \in \mathbb{R}^{2n}$, such that $U^\dagger R U = SR + r$. Discarding the irrelevant displacement, the CM transforms according to[3]

$$\gamma' = S\gamma S^T. \tag{10}$$

The most general $S \in Sp_{2n}(\mathbb{R})$ can be written as $S = O_1 D O_2$, where $O_{1,2}$ are real orthogonal and symplectic matrices and $D = \mathrm{diag}(r_1,\dots,r_n,1/r_1,\dots,1/r_n)$, with $r_i \in \mathbb{R}^+$ [50]; for $D = \mathbb{1}$, S is called a passive operation; otherwise it is called active. Apart from describing Gaussian unitary operations, symplectic matrices can also be used to derive a simple normal form (Williamson normal form) for arbitrary CM; see Eq. (7c). The eigenvalues d_i of D are called the symplectic eigenvalues of γ and are $\geqslant 1$. They are related to the purity of the corresponding Gaussian state ρ, since $\mathrm{tr}(\rho^2)$ is given by [51]

$$\mathrm{tr}(\rho^2) = |\gamma|^{-1/2} = \prod_{i=1}^{n} d_i^{-1}, \tag{11}$$

where here and in the following $|\cdot|$ denotes the determinant. This can be easily verified by noting that $|\gamma| = |\gamma J| = |S^{-1} \oplus_{i=1}^n d_i \mathbb{1}(S^T)^{-1}J| = |\oplus_{i=1}^n d_i\mathbb{1}| = \prod d_i^2$. The purity can be utilized to quantify the entanglement contained in pure states. For instance, the quantity

$$P(|\Psi\rangle) = \mathrm{tr}\big(\rho_{\mathrm{red}}^2\big)^{-2}, \tag{12}$$

where ρ_{red} denotes the reduced density operator of either system A or B of the pure state $|\Psi\rangle$, increases the more entangled $|\Psi\rangle$ is. Using the block form of the CM, γ [see Eq. (9)], $P(|\Psi\rangle)$ is given by $|A| = |B|$.

III. GLU EQUIVALENCE AND STANDARD FORM

We consider an arbitrary n-mode Gaussian state (pure or mixed) with CM γ and assume a partition of one mode per site. We first derive a standard form of γ, $S(\gamma)$, which we show to be unique and easily computable and to which each CM can be mapped via GLUs. Two states are called GLU equivalent if their density matrices can be transformed into each other by Gaussian local unitaries. Thus two Gaussian states with CM γ (Γ) are GLU equivalent iff their CMs can be transformed into each other by a local symplectic transformation. Due to the fact that the standard form, which we introduce here, is unique it easily follows that two Gaussian states are GLU equivalent iff their standard forms coincide.

We denote in the following by γ_{jk} the 2×2 matrix describing the covariances between mode j and k. As mentioned before any 2×2 real symplectic matrix can be written as $O_1\mathrm{diag}(r,1/r)O_2$, with $r \in \mathbb{R}$ and O_i real orthogonal. The standard form is reached in two steps. First, we apply to each mode j the active GLU that symplectically diagonalizes γ_{jj}, i.e., $S(\gamma)_{jj} = \lambda_j \mathbb{1}$. This leaves still the freedom to apply local

[2] Given a CM γ in its block form (9), one can readily find its pure-state standard form using the following procedure: We have $S_k = O_k D_k O_k'$, where O_k, O_k' are rotations and $D_k = \mathrm{diag}(e^{r_k}, e^{-r_k})$. The six matrices are determined as follows: $O_{1(2)}$ diagonalize A (B). The rotations O_k' realize the singular-value decomposition of $D_1^{-1}O_1^T C O_2 D_2^{-1}$. The two-mode squeezing parameter r is given by $\cosh r = \sqrt{\det(A)}$, while the squeezing parameters r_1, r_2 of S_k can be calculated by the traces of A and B, respectively: $\cosh 2r_1 = (\mathrm{tr} A)/(2\cosh r)$ and $\cosh 2r_2 = (\mathrm{tr} B)/(2\cosh r)$.

[3] One could also use the Heisenberg equation, i.e., $dA/dt = i[H,A]$ for the mode operators (X_k, P_k) to obtain this result.

passive operations S_j^p to each mode j, which are given by $O_j \in \mathrm{SO}(2)$. In the second step, we fix the $O_j = \exp(i\alpha_j\sigma_y)$ by considering the off-diagonal blocks γ_{jk}, $j<k$, in turn (row by row, from left to right). First consider γ_{12} and determine its singular values; if they both are zero, continue with the next block; if they are nonzero but degenerate, then γ_{12}, obeying $\gamma_{12}\gamma_{12}^T \propto \mathbb{1}$ and $|\gamma_{12}|>0$ is proportional to a real special orthogonal matrix O which we write without loss of generality as $O = e^{i\alpha\sigma_y} \in \mathrm{SO}(2)$. We fix $\alpha_2 = \alpha + \alpha_1$ (with α_1 being determined subsequently); if they are nondegenerate and add to zero, then $\gamma_{12} \propto \sigma_z O$, with $O = e^{i\alpha\sigma_y} \in \mathrm{SO}(2)$ and we fix $\alpha_2 = \alpha - \alpha_1$ (we refer to the two cases where γ_{ij} is orthogonal as "degenerate"); otherwise, we fix both α_1,α_2 such that O_1,O_2 are the unique matrices $\in\mathrm{SO}(2)$ such that $O_1\gamma_{12}O_2^T = \mathrm{diag}(d_{12},d'_{12})$, with $d_{12} \geqslant |d'_{12}|$.[4] In all four cases $S(\gamma)_{12}$ is diagonal. Now treat γ_{13} (and then all subsequent γ_{jk}) in the same manner. If α_j has already been determined in a previous step, then for nondegenerate singular values of γ_{jk} we fix α_k by diagonalizing $\gamma_{jk}^T\gamma_{jk}$. In this manner, all α_j will be uniquely determined except in the case that (for some j) all γ_{jk} are zero (in which case the mode j factorizes and we set $\alpha_j = 0$) or that for each j there is exactly one nonvanishing degenerate γ_{jk} (in this case we set the undetermined $\alpha_j = 0$). Any n-mode CM is transformed to its standard form $S(\gamma)$ by applying the n local active and n local passive unitaries as described above, and we have the following theorem.

Theorem 1 (*criterion for GLU equivalence*). Any CM γ can be transformed into its standard form $S(\gamma)$ by Gaussian local unitaries. Two CMs γ and Γ are GLU equivalent if and only if $S(\gamma) = S(\Gamma)$.

Note that this criterion for GLU equivalence is valid for both mixed and pure states. Let us mention here that an essentially identical form for n-partite n-mode Gaussian states was introduced in [44] and that the $n\times n\times n$ case was discussed in [40]. However, the question whether this is a unique standard form (which is essential for Theorem III) was discussed only for generic states. Let us close this discussion with a remark on the relation of LU and GLU equivalence before using the GLU criterion to derive the different GLU classes of two-mode and three-mode states.

When studying GLU equivalence, we restrict the allowed operations to a very small subset of all local unitaries. Hence, in general, two LU-equivalent states are not GLU equivalent. However, for pure Gaussian states in a number of relevant cases the two notions coincide. Note that, in particular, if two pure states are LU equivalent, then the Schmidt coefficients of these states across any bipartition must be the same. If we can show that the GLU classes of Gaussian states are uniquely characterized by their Schmidt coefficients across all bipartitions, then it follows that for those Gaussian states LU equivalence implies GLU equivalence. This is actually the case for pure bipartite Gaussian states, as implied by the results of [41]: every pure $n\times m$ Gaussian state $|\psi\rangle$ is GLU equivalent to $\min\{n,m\}$ two-mode squeezed states $|\psi_{\mathrm{tms}}(r_j)\rangle$ with squeezing parameters $r_1 \geqslant r_2 \geqslant \cdots \geqslant r_{\min\{n,m\}} \geqslant 0$, which fixes the Schmidt coefficients $\lambda_{l_1,\ldots,l_n} = \prod_{j=1}^n \frac{\tanh^{2l_j} r_j}{\cosh r_j}$ where $l_j \in \mathbb{N}$. Thus, if two pure bipartite Gaussian states are LU equivalent they have the same standard form $\otimes_{j=1}^n |\psi_{\mathrm{tms}}(r_j)\rangle$ and therefore are also GLU equivalent. As we show in Sec. III B below, the same implication also holds for pure $1\times 1\times 1$ Gaussian states.

[4] O_1 diagonalizes $\gamma_{12}\gamma_{12}^T$ and O_2 diagonalizes $\gamma_{12}^T\gamma_{12}$.

A. 1 × 1 case

Let us first consider the simplest case of two-mode Gaussian states. First we apply active transformations to map the reduced states γ_{ii} to thermal states $\lambda_i\mathbb{1}$. Since the state is pure the reduced states must be identical, i.e., $\lambda_1 = \lambda_2 = \lambda$. According to the algorithm above we apply next the orthogonal matrices O_1,O_2 such that $O_1\gamma_{12}O_2^T = D$, where D is diagonal. Thus, the standard form $S(\gamma)$ is

$$S(\gamma) = \begin{pmatrix} \lambda\mathbb{1} & D \\ D & \lambda\mathbb{1} \end{pmatrix}. \tag{13}$$

Next, we show that the standard form introduced here coincides in the case of pure two-mode states with the form [41]

$$\gamma = \begin{pmatrix} \cosh(r)\mathbb{1} & \sinh(r)\sigma_z \\ \sinh(r)\sigma_z & \cosh(r)\mathbb{1} \end{pmatrix}, \tag{14}$$

with the squeezing parameter r. Note that due to the condition $\gamma J\gamma \geqslant J$, we have $\lambda \geqslant 1$. Imposing now the condition that γ corresponds to a pure state, i.e., $\gamma J\gamma = J$, we find $\lambda^2\mathbb{1} + \tilde{D}D = \mathbb{1}$ and $\lambda\{J,D\} = 0$, where here and in the following $\{A,B\} = AB + BA$ denotes the anticommutator between any operators A and B and $\tilde{D} = \sigma_x D\sigma_x$. Since $\lambda \geqslant 1$ must be fulfilled by any CM, it must hold that $\{J,D\} = J(D+\tilde{D}) = 0$, which implies that $D = \bar{\lambda}\sigma_z$, for some real $\bar{\lambda}$. Due to the first condition we get then $\lambda^2 - \bar{\lambda}^2 = 1$, which implies that we can choose $\bar{\lambda} = \sinh(r)$ and $\lambda = \cosh(r)$, for some $r \in \mathbb{R}$. Thus the standard form coincides with Eq. (14).

B. 1 × 1 × 1 case

In this section we identify the different GLU classes of three-mode Gaussian states. First we explicitly provide the general standard form of Theorem III for the three-mode case. Then we show that it considerably simplifies for pure states and prove an exhaustive parametrization of the pure three-mode states.

1. Standard form: Mixed states

In this section we derive the standard form for an arbitrary $1\times 1\times 1$ Gaussian state. It is convenient to introduce (index-free) notation for the nine 2×2 blocks of γ by defining the matrix γ as

$$\gamma \equiv \begin{pmatrix} A & K & L \\ K^T & B & M \\ L^T & M^T & C \end{pmatrix}. \tag{15}$$

The basis chosen here will be called mode ordered, as indices referring to the same mode (A, B or C) are grouped. Sometimes the *quadrature-ordered* basis is used. This is a permutation in which first all the indices referring to X quadratures appear, followed by those referring to P.

As before, we first choose the active transformations to map the reduced states into thermal states. Using the same notation as before, the real orthogonal matrices O_i for $i = 1,2,3$ are then used to map the off-diagonal matrices into diagonal matrices. If the singular values of all off-diagonal blocks are nondegenerate, we use O_1 and O_2 to map K into a diagonal matrix with sorted entries in the diagonal, i.e., $O_1 K O_2^T \equiv \mathrm{diag}(d_{12}^+, d_{12}^-)$, with $d_{12}^+ \geqslant |d_{12}^-|$. O_3 is used to map L into the form $O D_{13}$, for diagonal D_{13} and some matrix $O \in \mathrm{SO}(2)$. Thus, the standard form is given by

$$\gamma_s = \begin{pmatrix} \lambda_1 \mathbb{1} & D_{12} & O D_{13} \\ D_{12} & \lambda_2 \mathbb{1} & M \\ D_{13} O^T & M^T & \lambda_3 \mathbb{1} \end{pmatrix}, \tag{16}$$

where D_{12} and D_{13} are diagonal and $O \in \mathrm{SO}(2)$. Hence, the number of free parameters in Eq. (16) is 12. In the case of degeneracy, more of the off-diagonal blocks can be made diagonal, as explained above. Due to Theorem III we know that two states are GLU equivalent iff their standard forms [Eq. (16)] coincide.

2. Standard form: Pure states

If we specialize to *pure* states, the CM must fulfill additional constraints and the number of free parameters is greatly reduced. We then have $\gamma J \gamma = J$, i.e., γ is a symplectic matrix. Taking into account that γ is symmetric, we have $\gamma = S S^T$ for a symplectic matrix $S = O D O'$. The number of real parameters describing a pure n-mode state is therefore $n^2 + n$. Since the GLU, i.e., the local (single-mode) symplectic operations are parametrized by $3n$ parameters, one would expect an $(n^2 - 2n)$-parameter standard form. Hence, for the three-mode Gaussian states considered here, one would expect three free parameters. In order to derive the parametrization we first show in the following theorem that pure three-mode Gaussian states are of a particularly simple form.

Theorem 2 ($1 \times 1 \times 1$ *pure state is xp block diagonal*). Any pure $1 \times 1 \times 1$ Gaussian state is GLU equivalent to a state whose CM γ, as given in Eq. (15), has the property that *all* the submatrices A,B,C,K,L,M are diagonal. That is, in the xp-ordered basis we have

$$\gamma_s = \gamma_x \oplus \gamma_x^{-1}, \tag{17}$$

where
$$\gamma_x = \begin{pmatrix} \lambda_1 & d_{12}^+ & d_{13}^+ \\ d_{12}^+ & \lambda_2 & d_{23}^+ \\ d_{13}^+ & d_{23}^+ & \lambda_3 \end{pmatrix}, \tag{18}$$

with λ_i denoting the local purities and $d_{ij}^+ \in \mathbb{R}$.

Proof. In Appendix A we show that the necessary condition for γ to correspond to a pure state, $\gamma J \gamma = J$, implies that all submatrices K,L,M have to be diagonal. This implies that pure three-mode states can always be brought into a form in which correlations exist only among the X quadratures and among the P quadratures, respectively. That is, the CM is *xp block diagonal* in the standard form, i.e., $\gamma = \gamma_x \oplus \gamma_p$ (in the xp-ordered basis). Using then that the state is pure, which implies the condition $\tilde{J} \gamma \tilde{J}^T \gamma = \mathbb{1}$, where $\tilde{J} = [0_n, -\mathbb{1}_n; \mathbb{1}_n, 0_m]$ is J in the xp basis, and 0_n ($\mathbb{1}_n$) denote the $n \times n$ zero (identity) matrix, respectively, it is easy to see that for pure states $\gamma_p = \gamma_x^{-1}$, which proves the statement. ■

Since the positive real and symmetric matrix γ_x can always be written as $\gamma_x = O D O^T$ for O orthogonal and D real and diagonal, six free parameters are required to characterize γ_x. Since in the standard form both γ_x and γ_p must have the same diagonal elements, this yields three constraining equations, leaving three parameters characterizing the equivalence classes. We derive in the next section the conditions on those parameters so as to correspond to a valid CM of a pure state.

3. Parametrization of pure 1 × 1 × 1 states

As we have just seen, an arbitrary pure three-mode state can be written as

$$\gamma = \begin{pmatrix} \lambda_1 \mathbb{1} & D_{12} & D_{13} \\ D_{12} & \lambda_2 \mathbb{1} & D_{23} \\ D_{13} & D_{23} & \lambda_3 \mathbb{1} \end{pmatrix}, \tag{19}$$

where D_{ij} is diagonal. Due to the condition $\gamma \geqslant iJ$ [see Eq. (7b)], we have $\lambda_i \geqslant 1 \; \forall \, i$.

In this section we derive the conditions for γ corresponding to a pure state and show that the CM can be fully parametrized by the three local-mixedness parameters λ_j. Recall that γ is pure iff $\gamma \geqslant 0$ and $\gamma J \gamma = J$. We first derive the necessary and sufficient conditions for a matrix γ, as given in Eq. (19) with $\lambda_i \geqslant 1$ to fulfill $\gamma J \gamma = J$ (see Lemma 2). After that, we derive the condition for such a matrix to be positive (see Lemma 4).

Lemma 2. A matrix γ, as given in Eq. (19) with $\lambda_i \geqslant 1$, fulfills $\gamma J \gamma = J$ iff the entries of the diagonal matrices $D_{ij} = \mathrm{diag}(d_{ij}^+, d_{ij}^-)$ are given (up to GLUs) by

$$d_{ij}^{\pm} = \frac{1}{4\sqrt{\lambda_i \lambda_j}} (\sqrt{a_{ij}} \pm \sqrt{b_{ij}}), \tag{20}$$

with

$$a_{ij} = [(\lambda_i - \lambda_j)^2 - (\lambda_k - 1)^2][(\lambda_i - \lambda_j)^2 - (\lambda_k + 1)^2]$$
$$b_{ij} = [(\lambda_i + \lambda_j)^2 - (\lambda_k - 1)^2][(\lambda_i + \lambda_j)^2 - (\lambda_k + 1)^2],$$

where $i \neq j$ and $k \neq i,j$ refers to the third index.

Proof. It is straightforward to show that the condition $\gamma J \gamma = J$ is equivalent to the following set of equations:

$$\lambda_1^2 + |D_{12}| + |D_{13}| = 1, \tag{21a}$$
$$\lambda_2^2 + |D_{12}| + |D_{23}| = 1, \tag{21b}$$
$$\lambda_3^2 + |D_{13}| + |D_{23}| = 1, \tag{21c}$$
$$\lambda_1 D_{12} + \lambda_2 \tilde{D}_{12} + \tilde{D}_{13} \odot D_{23} = 0, \tag{21d}$$
$$\lambda_1 D_{13} + \lambda_3 \tilde{D}_{13} + \tilde{D}_{12} \odot D_{23} = 0, \tag{21e}$$
$$\lambda_2 D_{23} + \lambda_3 \tilde{D}_{23} + \tilde{D}_{12} \odot D_{13} = 0, \tag{21f}$$

where $\odot$ denotes the componentwise multiplication (Hadamard product). Here, we used the notation $D_{ij} = \mathrm{diag}(d_{ij}^+, d_{ij}^-)$, $\tilde{D} = \sigma_x D \sigma_x$ and that $DJ = J\tilde{D}$ (i.e., $\tilde{D} = -JDJ$) for any diagonal matrix D and therefore $DJD = |D|J$. Note that if $D = \mathrm{diag}(a,b)$, then $\tilde{D} = \mathrm{diag}(b,a)$. In Appendix B we show that those conditions (together with $\lambda_j \geqslant 1$) are satisfied iff the entries of the diagonal matrices $D_{ij} = \mathrm{diag}(d_{ij}^+, d_{ij}^-)$ are given (up to GLUs) by $d_{ij}^{\pm}$ as given in the lemma. ■

Note that in [23] it has been stated that a generic state can be written as in Eq. (19), with the entries of the diagonal matrices given in Eq. (20). However, we are aiming here for a complete characterization of three-mode pure states. As we prove below, the results of [23] hold for all pure three-mode Gaussian states.

Clearly a_{ij},b_{ij} must be positive in order to obtain a real CM. This leads to the (mutually exclusive) conditions $|\lambda_i - \lambda_j| \leqslant \lambda_k - 1 \,\forall\, (ijk)$ or $|\lambda_i - \lambda_j| \geqslant \lambda_k + 1 \,\forall (ijk)$. We show now that only the first condition is compatible with the positivity of the reduced CM [at modes (ij)]. To see that, note that for pure three-mode states it follows from Eqs. (21a)–(21c) that for all (ijk) holds $\lambda_k^2 = \lambda_i^2 + \lambda_j^2 + 2|D_{ij}| - 1 = (\lambda_i + \lambda_j + 1)^2 - 2(\lambda_i + \lambda_j + \lambda_i\lambda_j - |D_{ij}| + 1)$. The last term in this expression is strictly negative since due to the fact that the CM of the modes i,j has to be positive, we have $\lambda_i\lambda_j \geqslant \pm|D_{ij}|$, which implies that $\lambda_k < \lambda_i + \lambda_j + 1$. Thus, the conditions

$$\lambda_i + 1 \leqslant \lambda_j + \lambda_k \;\forall\, (ijk) \tag{22}$$

are the necessary and sufficient conditions for a valid pure CM γ to be real. Note that if $\lambda_i \geqslant \lambda_j,\lambda_k$, the conditions in Eq. (22) are equivalent to the condition $\lambda_i \leqslant -1 + \lambda_j + \lambda_k$. For later reference, we also note the simple expression for $|D_{ij}|$ in terms of the λ's:

$$|D_{ij}| = \tfrac{1}{2}\left(\lambda_k^3 + 1 - \lambda_i^2 - \lambda_j^2\right). \tag{23}$$

It remains to impose the condition that $\gamma \geqslant 0$. For this, we use the following lemma (Schur's complement), which is proven for instance in [4].

Lemma 3 (positivity of self-adjoint matrices). A self-adjoint matrix

$$M = \begin{pmatrix} A & C \\ C^\dagger & B \end{pmatrix}, \tag{24}$$

with $B > 0$ is positive if and only if

$$A - C\frac{1}{B}C^\dagger \geqslant 0. \tag{25}$$

Using this lemma we show that any CM γ as given in Eq. (19) is positive when the condition (22) is satisfied, as stated in the following lemma.

Lemma 4. The symmetric matrix γ, as given in Eq. (19) with $\lambda_k \geqslant 1$, for $k \in \{1,2,3\}$ is positive semidefinite if Eq. (22) holds.

Proof. Since $\gamma = \gamma_x \oplus \gamma_x^{-1}$ (see Theorem 2), we have $\gamma > 0$ iff $\gamma_x > 0$. Using now Lemma 3 and the fact that $\lambda_3 > 0$, we know that the 3×3 matrix γ_x is positive iff the 2×2 matrix

$$Y = \begin{pmatrix} \lambda_1 & d_{12}^+ \\ d_{12}^+ & \lambda_2 \end{pmatrix} - \frac{1}{\lambda_3}\begin{pmatrix} d_{13}^+ \\ d_{23}^+ \end{pmatrix}\begin{pmatrix} d_{13}^+ & d_{23}^+ \end{pmatrix} > 0. \tag{26}$$

Note that $Y > 0$ iff $|Y| > 0$ and $\mathrm{tr}(Y) > 0$. Using that $\lambda_k \geqslant 1$ for all k, tedious but elementary calculations (see Appendix C) show that both expressions are positive if the condition (22) holds. ∎

Combining Lemma 2 and Lemma 4 we obtain the following theorem:

Theorem 3. Any CM of a pure three-mode Gaussian state can be written (up to GLUs) as

$$\gamma = \begin{pmatrix} \lambda_1\mathbb{1} & D_{12} & D_{13} \\ D_{12} & \lambda_2\mathbb{1} & D_{23} \\ D_{13} & D_{23} & \lambda_3\mathbb{1} \end{pmatrix}, \tag{27}$$

where $D_{ij} = \mathrm{diag}\,(d_{ij}^+,d_{ij}^-)$, with $d_{ij}^\pm$ given in Eq. (20).

Thus, the nonlocal properties of any pure three-mode Gaussian state are completely characterized by the local-mixedness parameters λ_i, i.e., by the bipartite entanglement shared between each mode with the other two. Recalling our discussion of LU and GLU equivalence at the beginning of this section, we see that (like the pure bipartite Gaussian states) also the pure $1 \times 1 \times 1$ Gaussian states are completely characterized by their Schmidt coefficients across the three different bipartitions (which are in one-to-one relation with the λ_l). Therefore, those states are LU equivalent iff they are GLU equivalent and Theorem 3 also characterizes the LU classes of pure three-mode Gaussian states.

4. Some special cases

Let us briefly consider two special cases, namely, the one where one of the off-diagonal matrices, say, γ_{ij}, is (a) not invertible or (b) proportional to $\mathbb{1}$.

Case (a). The condition $|D_{ij}| = 0$ together with Eq. (23) implies $\lambda_k^2 = \lambda_i^2 + \lambda_j^2 - 1$, and inserting it in Eq. (20) we find that $d_{ij}^- = 0$ and $d_{ij}^+ = \sqrt{(\lambda_i^2 - 1)(\lambda_j^2 - 1)}(\lambda_i\lambda_j)^{-1/2}$. Note that $d_{ij}^+ \neq 0$ (as are $|D_{ik}|,|D_{jk}|$) unless λ_i or λ_j equals 1, in which case the respective mode factorizes and the remaining two would be in a two-mode squeezed state.

Case (b). $D_{ij} \propto \mathbb{1}$ is possible only if $b_{ij} = 0$, which implies that $\lambda_k = \lambda_i + \lambda_j - 1$ (i.e., in particular, $\lambda_k \geqslant \lambda_i,\lambda_j$).[5] Clearly, then $\lambda_k - \lambda_{i(j)} = \lambda_{j(i)} - 1$ and thus $a_{ik} = a_{jk} = 0$. Hence, the remaining two off-diagonal blocks are both proportional to σ_z. It also holds that if $D_{ij} \propto \sigma_z$, which implies that $a_{ij} = 0$ (which fixes $\lambda_k = 1 + |\lambda_i - \lambda_j|$), one of the remaining two off-diagonal blocks is degenerate (and the other proportional to σ_z): If $\lambda_i \geqslant \lambda_j$ then $b_{jk} = 0$ and $a_{ki} = 0$, and otherwise they are reversed. As we see below, these states can all be generated by letting a beam splitter couple one mode of a two-mode system in a two-mode squeezed vacuum state with a third mode in the vacuum.

Another interesting special case is represented by the fully permutation *symmetric* states [13,23], for which the three local mixednesses are identical, i.e., $\lambda_l = \lambda \;\forall\, l$. We denote the CM of a symmetric state in standard form by $\gamma_{\mathrm{sym}}(\lambda)$. These states were sometimes called maximally entangled [23,25] due to their extremal entanglement properties reminiscent of their qubit analogs [33]. For these states the matrices D_{ij} are given by $\mathrm{diag}(d^+,d^-)$ with

$$d^\pm = \frac{1}{4\lambda}[(\lambda^2 - 1) \pm \sqrt{9\lambda^4 - 10\lambda^2 + 1}]. \tag{28}$$

[5]Note that $D_{ij} \propto \mathbb{1}$ implies that the reduced state of modes i and j has a positive semidefinite partial transpose and is therefore separable.

In Sec. V C we investigate which states can be obtained from symmetric states via Gaussian local operations assisted by classical communication.

IV. GENERATION OF THREE-MODE PURE STATES

Let us briefly remark on the generation of pure three-mode states. In [52] a general state-generation scheme for this case is presented. There, a two-mode squeezed state (of modes 1 and 2, with squeezing parameter r) is coupled to mode 3 (in the vacuum state) by a sequence of three beam splitters (BSs) acting on modes (13), (23), and (13), respectively. The transmissivities of the third BS is fixed while those of the first two are adjusted so as to produce the desired local purities.

Note that in the special case in which one of the off-diagonal matrices is degenerate (case (b) above), a simplified scheme suffices: Letting a beam splitter with transmissivity $\cos^2\theta \in [0,1]$ act on part of a two-mode squeezed vacuum (with squeezing parameter $s \geqslant 0$) and a vacuum mode allows generation of all states with degenerate CMs: If λ_1 is the largest local mixedness, then

$$\gamma(s,\theta) = B(\theta)\,[\gamma_{\text{tms}}(s) \oplus \mathbb{1}]\,B(\theta)^T, \tag{29}$$

where

$$B(\theta) = \mathbb{1} \oplus \begin{pmatrix} \cos\theta\,\mathbb{1}_2 & \sin\theta\,\mathbb{1}_2 \\ -\sin\theta\,\mathbb{1}_2 & \cos\theta\,\mathbb{1}_2 \end{pmatrix}$$

and $\gamma(s,\theta)$ then has the three local purities $\lambda_1 = \cosh s, \lambda_2 = \sin^2\theta + \cos^2\theta\cosh s, \lambda_3 = \cos^2\theta + \sin^2\theta\cosh s$, satisfying the characteristic equation $\lambda_1 + 1 = \lambda_2 + \lambda_3$ of case (b) above. And since for any given $\lambda_2,\lambda_3 \geqslant 1$ there is a pair $(s,\theta) \in \mathbb{R}^+ \times [0,2\pi]$ such that the above equations hold, we can generate all degenerate states this way. Since these states are obtained from a two-mode squeezed state by distributing it via a beam splitter among two parties, we also refer to them as *distributed two-mode squeezed states*.

In order to see how the different GLU classes relate to each other we now extend the set of operations from Gaussian local unitaries to Gaussian (stochastic) local operations with classical communication. In particular, this will allow us to investigate whether the GHZ/W states are *maximally entangled* also in the sense that they allow preparation of any other Gaussian state via GLOCC (in the same way as, e.g., the Bell state does for two qubits or certain families of states do in the pure multiqubit setting [36]).

V. GAUSSIAN LOCAL OPERATIONS

LU equivalence leads to a very detailed classification of multipartite states with a continuum of inequivalent classes. A more coarse-grained picture emerges if interconvertibility of states under a larger class of transformations, stochastic local operations and classical communication [30], is studied. SLOCC plays an important role in entanglement theory [31–33]. Two states are said to be SLOCC equivalent if there is a non-zero probability to convert each into the other. Due to the stochastic interconvertibility of all pure bipartite states of equal Schmidt rank [33] there are $d-1$ different kinds of bipartite (pure state) entanglement of d-dimensional systems. In contrast, in the tripartite case, even for three qubits two inequivalent classes have been identified that are not connected by SLOCC transformations [33].

Also in the Gaussian setting, GLU operations can be extended by allowing for local (generalized) measurements, namely, adjoining additional modes (in a pure state) and then performing (partial) Gaussian measurements. However, Gaussian SLOCC have not been investigated since the only Gaussian operators with a bounded inverse are the Gaussian unitaries.[6] Instead, we are interested here in the convertibility of states under Gaussian LOCC (GLOCC). In light of the complicated structure of general LOCC transformations [53] the Gaussian case is remarkably simple: all Gaussian operations can be characterized via the Choi-Jamiołkowski (CJ) isomorphism by an equivalent Gaussian state [9,10,54]. When acting on a Gaussian state with *known* CM, all such transformations can be implemented *deterministically* by teleporting through that state [10]. While teleportation is probabilistic (yielding a random displacement), this can be computed from the measurement outcome and the involved CMs and can then be undone by local unitaries. In particular, this implies that a finite number of communication rounds is enough to implement any GLOCC. Note that the inverse of a GLOCC is not Gaussian, and hence GLOCC does not induce an equivalence relation among Gaussian states but rather gives rise to a partial ordering (see [41] for the bipartite case).

Gaussian operations mapping pure states to pure states ("pure operations") are characterized by a pure CJ-CM Γ and pure operations on a single mode are characterized by a pure 1×1 CM, i.e., by one GLU-invariant parameter r (two-mode squeezing) and two sets of three local parameters (each characterizing a single-mode Gaussian unitary), which describe local unitary pre- and postprocessing of the state. Following the treatment in [41] for the bipartite case we can easily obtain expressions for the output CM of a three-mode state after a general three-mode GLOCC.

A. General GLOCC on three-mode systems

The most general Gaussian operation transforming a three-mode Gaussian state into another corresponds to a six-mode CJ-CM $\Gamma = [\Gamma_1,\Gamma_{12};\Gamma_{12}^T,\Gamma_2]$. Here, the index 2 (1) denotes the three input (output) modes, respectively. According to [10] the output CM γ' is related to the input CM γ by

$$\gamma' = \Gamma_1 - \Gamma_{12}\,(\Gamma_2 + \Lambda\gamma\Lambda)^{-1}\,\Gamma_{12}^T, \tag{30}$$

where $\Lambda = \oplus_{x=1}^3 \sigma_z$. For ease of notation we denote the three diagonal blocks of the input CM γ by A_x, $x = 1,2,3$, and use the convention that indices (x,y,z) in a single equation refer to distinct modes. In the case of pure LOCC transformations,

[6]However, one might note that the Gaussian operator $G_{-c} = e^{-c(X^2+P^2)}$, $c > 0$, has an "inverse" $G_{+c} = e^{+c(X^2+P^2)}$ which is unbounded, but well defined on the image of G_{-c} (i.e., on sufficiently rapidly decaying states). This might be used to argue for the "Gaussian SLOCC equivalence" of, e.g., two-mode squeezed states with different squeezing parameters $r < r'$ (which can be "stochastically" mapped to each other by $G_{\pm c}$ with $e^c = \tanh r'/\tanh r$). However, such unbounded filtering operations have no clear physical implementation and we do not pursue this further here.

the CM Γ is block diagonal, i.e., $\Gamma = \oplus_{x=1}^{3}\Gamma_x$ with

$$\Gamma_x = \begin{pmatrix} \Gamma_{1x} & \Gamma_{12x} \\ \Gamma_{12x}^T & \Gamma_{2x} \end{pmatrix} \tag{31}$$

$$\equiv (S_{1x} \oplus S_{2x})\gamma(r_x)(S_{1x} \oplus S_{2x})^T. \tag{32}$$

Using that S_{1x} describes only local unitary postprocessing (which is irrelevant for GLU-invariant properties) we can without loss of generality take $S_{1x} = \mathbb{1}$. We write the Euler decomposition [50] of $S_{2x} = O_{1x}Q_xO_{2x}$ where O_{1x},O_{2x} are in SO(2) and $Q_x = \mathrm{diag}(q_x,q_x^{-1})$ with $q_x > 0$ is a single-mode squeezing transformation. Since the effect of O_{2x} can be undone by local unitary postprocessing, we set $O_{2x} = \mathbb{1}$, and obtain

$$\Gamma_{1x} = \cosh r_x \mathbb{1}, \tag{33}$$

$$\Gamma_{2x} = \cosh r_x\, O_{1x} Q_x^2 O_{1x}^T, \tag{34}$$

$$\Gamma_{12x} = \sinh r_x \sigma_z Q_x O_{1x}^T, \tag{35}$$

To make the ensuing expressions shorter, we will from now on use the notation $\cosh r \equiv \mathrm{ch} r$ and $\sinh r \equiv \mathrm{sh} r$.

Using the Schur-complement formula for the inverse of a symmetric matrix, this allows us to write the reduced CMs of the output state γ' at mode x in compact form as

$$(\gamma')_{xx} = \Gamma_{1x} - \Gamma_{12x}\,(\Gamma_{1x} + A_x - T_x)^{-1}\,\Gamma_{12x}^T, \tag{36}$$

where we have introduced the auxiliary matrices

$$T_x = (D_{xy}\ \ D_{xz})R_x^{-1}\begin{pmatrix} D_{xy} \\ D_{xz} \end{pmatrix}, \tag{37}$$

$$R_x = \begin{pmatrix} \Gamma_{2y} + A_y & D_{yz} \\ D_{yz} & \Gamma_{2z} + A_z \end{pmatrix}. \tag{38}$$

Note that the identity operation corresponds to the limiting case of an infinitely squeezed CJ-CM Γ (i.e., $r \to \infty$ and $O_1 = X = \mathbb{1}$). Hence, the case of not operating on mode x corresponds to taking the limit $r_x \to \infty$ in the above expressions. Since the expression in terms of the nine pure GLOCC and three CM parameters is rather long and not transparent, we split the general three-mode GLOCC into a sequence of three single-mode transformations. Sometimes we will focus on a simpler family of transformations [which we refer to as (local) TMS filtering], namely, those where $O_{1x} = Q_x = \mathbb{1}\ \forall\, x$ (i.e., no unitary preprocessing), leaving only the three two-mode squeezing parameters free. In the bipartite case, GLOCC of that form are know to suffice to perform all possible transformations between GLU equivalence classes.

Here, our aim is not the complete analysis of the GLOCC transformations of three-mode pure Gaussian states but only an illustration of the usefulness of the GLU classification and standard form. In particular, we use the standard form derived in the previous section to study which pure three-mode states can be transformed into each other by GLOCC. We first provide simple expressions for the single-mode transformations of three-mode states which we then use in the subsequent sections to show that the CV GHZ/W states lack certain properties of maximal entanglement. To this end we first show that there are pure Gaussian three-mode states that cannot be obtained from any γ_{sym} via GLOCC by identifying a qualitative feature that the symmetric states lack and that cannot be generated by GLOCC. Then we identify a one-parameter family of such states [unreachable from $\gamma_{\mathrm{sym}}(\lambda)$] that, in contrast, allows all symmetric states to be reached.

B. GLOCC transformation of 1 × 1 × 1 states

As we have seen before, a three-mode pure Gaussian state is completely characterized by its local-mixedness parameters λ_i. Therefore, we simply write $(\lambda_1,\lambda_2,\lambda_3)$ when referring to the CM given in Eqs. (19) and (20). Here we derive a compact prescription of how the CM of a Gaussian state changes under single-mode GLOCC and, in particular, give expressions for the matrices determining the local-mixedness parameters λ_i.

Let us denote the 1×2 CM of the three-mode Gaussian input state $(\lambda_1,\lambda_2,\lambda_3)$ by

$$\gamma = \begin{pmatrix} A & K \\ K^T & R \end{pmatrix}, \tag{39}$$

where $A = \lambda_i \mathbb{1}$, $R = \left(\begin{smallmatrix} A_j & D_{jk} \\ D_{jk} & A_k \end{smallmatrix}\right)$, with $A_l = \lambda_l \mathbb{1}$, and $K = (D_{ij}\ \ D_{ik})$, where the block A refers to the mode i to be acted upon.

As mentioned above, Gaussian completely positive maps (CPMs) acting on a single mode and mapping pure states to pure states are in one-to-one correspondence to pure two-mode Gaussian states by the CJ isomorphism [10,54–56]; they are GLU equivalent to a two-mode squeezed state and can therefore be completely characterized by the 1×1 CM Γ [see Eq. (8)],

$$\Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_{12} \\ \Gamma_{12}^T & \Gamma_2 \end{pmatrix} \equiv (S_1 \oplus S_2)\gamma(r)(S_1 \oplus S_2)^T, \tag{40}$$

with symplectic S_1,S_2. As discussed in the previous section, we can without loss of generality choose $S_1 = \mathbb{1}$ and $S_2 = O_1^T X^{-1}$ with $X = \mathrm{diag}(x,x^{-1})$.

If the CPM corresponding to Γ acts on mode i of the state with CM γ of Eq. (39), it is transformed to γ' with [41]

$$A' = \Gamma_1 - \Gamma_{12}(\Gamma_2 + \sigma_z A \sigma_z)^{-1}\Gamma_{12}^T, \tag{41}$$

$$R' = R - K^T\sigma_z(\Gamma_2 + \sigma_z A\sigma_z)^{-1}\sigma_z K, \tag{42}$$

$$K' = \Gamma_{12}(\Gamma_2 + \sigma_z A\sigma_z)^{-1}\sigma_z K. \tag{43}$$

To characterize the output state only the three 2×2 diagonal blocks of γ' are of interest. We have

$$A_i' = \mathrm{ch} r\mathbb{1} - \mathrm{sh}^2 r(\mathrm{ch} r\mathbb{1} + \lambda_i X^2)^{-1}, \tag{44}$$

$$A_j' = \lambda_j\mathbb{1} - T_j(\mathrm{ch} r\mathbb{1} + \lambda_i X^2)^{-1}T_j^T, \tag{45}$$

$$A_k' = \lambda_k\mathbb{1} - T_k(\mathrm{ch} r\mathbb{1} + \lambda_i X^2)^{-1}T_k^T, \tag{46}$$

where $T_l = D_{il}O_1X$ for $l = j,k$. Clearly, up to GLUs the final state depends only on the parameters r,x,ϕ, where $O_1 = e^{i\phi\sigma_y}$. Note that these expressions could be obtained from Eq. (36) in the limit $r_y,r_z \to \infty$.

We now use the GLOCC formalism to explore the entanglement properties of certain families of pure three-mode Gaussian states, in particular the symmetric states $\gamma_{\mathrm{sym}}(\lambda)$. For large λ these are highly entangled states and they have been suggested as maximally entangled continuous variable states. We show, however, in the next section that, in contrast to what one might expect, it is *not* possible to prepare by GLOCC an arbitrary pure three-mode Gaussian state from a state $\gamma_{\rm sym}(\lambda)$ no matter how large is λ. In contrast, we study in the final section a different family, and show that it allows preparation, in particular, of all symmetric states.

C. Symmetric initial states

We show now that it is not possible to reach an arbitrary three-mode entangled state via GLOCC from a symmetric three-mode entangled state. To this end, we first show that a state can be generated from a symmetric state $\gamma_{\rm sym}$ by a single-mode GLOCC if and only if it can be generated (possibly from some other symmetric initial state) via a single-mode TMS operation [i.e., $X = O_1 = \mathbb{1}$ in Eqs. (44)–(46)] and that any state $(\lambda_1',\lambda_2',\lambda_2')$ with $\lambda \geqslant \lambda_2' \geqslant \lambda_1'$ can be generated in this way. Then we show that starting with a state $(\lambda_1',\lambda_2',\lambda_2')$ a general measurement on the second mode allows us only to reach states with $|D_{12}| = (\lambda_3'^2 - \lambda_1'^2 - \lambda_2'^2 + 1)/2 \leqslant 0$. After that, we show that performing a general GLOCC on the third mode cannot change the sign of this determinant. Consequently, a pure three-mode Gaussian state with $|D_{12}| > 0$ cannot be prepared by general GLOCC starting from an (arbitrary) symmetric Gaussian state. In order to show that it is not in general the case that the sign of the determinants of the off-diagonal blocks cannot be changed via GLOCC, we present in the subsequent section a class of states with one positive and two negative determinants, from which states with three negative determinants can be obtained via GLOCC.

Let us first show that from a symmetric state with parameter λ and operating on mode 1 only, we obtain $(\lambda_1',\lambda_2',\lambda_2')$ with $\lambda \geqslant \lambda_2' \geqslant \lambda_1'$. Then we show that *any* ratio $\lambda_2'/\lambda_1' \geqslant 1$ can be obtained by a suitable TMS operation and suitable choice of the initial λ.

From Eq. (44) we see that λ_1' does not depend on O_1 and takes a global maximum for $x = 1$. Since a TMS operation yields

$$\lambda_1' = \frac{\lambda \mathrm{ch} r + 1}{\lambda + \mathrm{ch} r}, \tag{47}$$

which can take all values in $[1,\lambda]$, restricting to these operations does not constrain λ_1'. With Eq. (45), one readily checks that λ_2' is minimal[7] for $O_1 = \mathbb{1}$. Thus $\phi \neq 0$ only increases the ratio λ_2'/λ_1'. Since as we see below *all* such ratios $\geqslant 1$ can be obtained by TMS operations, we can set $\phi = 0$. Looking now at λ_2'/λ_1' for the case $\phi = 0$, we easily see that it is $\geqslant 1$.[8] Note that this is expected since the GLOCC (a partial measurement) is performed at mode 1 and thus our lack of knowledge about the local state there is less than at the unmeasured modes.

To complete the proof we have to show that all such ratios can be achieved by TMS operations. The parameter λ_2' after such a GLOCC is

$$\lambda_2' = \frac{\left[\lambda^2(\mathrm{ch}^2 r + 1) + \frac{3\lambda^4 + 6\lambda^2 - 1}{4\lambda} \mathrm{ch} r\right]^{1/2}}{\lambda + \mathrm{ch} r}. \tag{48}$$

That $\lambda_2' \geqslant \lambda_1'$ is easily seen using that $\lambda \geqslant 1$, which implies that the first term in the numerator is larger than or equal to $\lambda^2 \mathrm{ch} r^2 + 1$ and the second term is larger than or equal to $2\lambda \mathrm{ch} r$.

To see that *all* such pairs (λ_1',λ_2') can be achieved by suitable choice of the initial parameter λ and operation parameter r we can invert Eqs. (47) and (48) to find r,λ as functions of the target values λ_1' and $f \geqslant 0$, which determines the ratio λ_2'/λ_1' via

$$\left(\frac{\lambda_2'}{\lambda_1'}\right)^2 = 1 + f.$$

We find

$$\lambda = \left[\frac{(3+4f)\lambda_1'^2 + 1}{6\lambda_1'} + \sqrt{\left(\frac{(3+4f)\lambda_1'^2 + 1}{6\lambda_1'}\right)^2 + \frac{1}{3}}\right], \tag{49a}$$

$$\mathrm{ch} r = \frac{\lambda \lambda_1' - 1}{\lambda - \lambda_1'}. \tag{49b}$$

One readily checks that the values of λ,r are in the admissible range $(\lambda \geqslant 1, r \geqslant 0)$ for all valid target values $\lambda_1' \geqslant 1, f \geqslant 0$, which proves the statement.

However, it is not possible to obtain all pure three-mode Gaussian states from a symmetric initial state, not even by the most general GLOCC. This follows from the fact that the symmetric states all have the property that the three off-diagonal matrices D_{ij} all have nonpositive determinants $|D_{ij}| \leqslant 0$. As we show in the following lemma, this is a property that cannot be changed by GLOCC. However, we have already encountered states [such as the distributed two-mode squeezed states $\gamma(s,\theta)$; cf. Eq. (29)] which have one nonpositive determinant. These, therefore, cannot be reached by GLOCC from the symmetric states.

Lemma 5. It is impossible with GLOCC to transform a pure three-mode Gaussian state with three nonpositive determinants $|D_{ij}| \leqslant 0$ into a state with at least one (strictly) positive determinant.

Proof. We consider an arbitrary initial state with $|D_{ij}| \leqslant 0$ for all (ij), i.e. [cf. Eq. (23)], $\lambda_i^2 - \lambda_j^2 - \lambda_k^2 + 1 \leqslant 0 \ \forall\, (ijk)$ and apply an arbitrary measurement on the kth mode. Without loss of generality, we choose $(ijk) = (123)$. As before, we

[7] λ_2' is periodic with the angle ϕ with period π and monotonically increases in the interval $[0,\pi]$.

[8] We have (with $c_x = x^2 + x^{-2}, s_x = x^2 - x^{-2}$)

$$\frac{\lambda_2'^2}{\lambda_1'^2} = \frac{\lambda^2(1 + \mathrm{ch}^2 r) + \frac{3\lambda^4 + 6\lambda^2 - 1}{8\lambda} \mathrm{ch} r\, c_x + \frac{\lambda^2 - 1}{8\lambda}\sqrt{9\lambda^4 - 10\lambda^2 + 1}\, s_x}{\lambda^2 \mathrm{ch}^2 r + c_x \lambda \mathrm{ch} r}.$$

The difference between numerator and denominator is $\lambda^2 - 1 + \frac{(\lambda^2-1)(3\lambda^2+1)}{8\lambda} \mathrm{ch} r\, c_x + \frac{(\lambda^2-1)\sqrt{9\lambda^4 - 10\lambda^2 + 1}}{8\lambda} \mathrm{ch} r\, s_x$, which is positive since $c_x \geqslant |s_x|$ and $(3\lambda^2 + 1)^2 > 9\lambda^4 - 10\lambda^2 + 1$.

obtain for the matrices in the diagonal of γ

$$A_1' = \lambda_1 \mathbb{1} - T_1(\mathrm{ch}r\mathbb{1} + \lambda_3 X^2)^{-1} T_1^T, \tag{50}$$

$$A_2' = \lambda_2 \mathbb{1} - T_2(\mathrm{ch}r\mathbb{1} + \lambda_3 X^2)^{-1} T_2^T, \tag{51}$$

$$A_3' = \mathrm{ch}r\mathbb{1} - \mathrm{sh}r^2(\mathrm{ch}r\mathbb{1} + \lambda_3 X^2)^{-1}, \tag{52}$$

with $T_1 = D_{13}O_1X$ and $T_2 = D_{23}O_1X$. Again, we consider the term $C_3 \equiv \lambda_3'^2 - \lambda_2'^2 - \lambda_1'^2 + 1$, which now yields a more lengthy expression:

$$C_3 = \frac{[c_x A + B s_x \cos(2\phi)]\mathrm{ch}r + C}{4\lambda_3\left(\lambda_3^2 + \mathrm{ch}^2 r + c_x\lambda_3\mathrm{ch}r\right)}, \tag{53}$$

$$A = \lambda_1^4 - 2\left(\lambda_2^2 + \lambda_3^2 + 1\right)\lambda_1^2 + \lambda_3^4 + \left(\lambda_2^2 - 1\right)^2 - 2\left(\lambda_2^2 - 3\right)\lambda_3^2, \tag{54}$$

$$\begin{aligned} B = [&(\lambda_1 - \lambda_2 - \lambda_3 - 1)(\lambda_1 - \lambda_2 - \lambda_3 + 1)\\ &\times(\lambda_1 + \lambda_2 - \lambda_3 - 1)(\lambda_1 + \lambda_2 - \lambda_3 + 1)\\ &\times(\lambda_1 - \lambda_2 + \lambda_3 - 1)(\lambda_1 - \lambda_2 + \lambda_3 + 1)\\ &\times(\lambda_1 + \lambda_2 + \lambda_3 - 1)(\lambda_1 + \lambda_2 + \lambda_3 + 1)]^{1/2}, \end{aligned} \tag{55}$$

$$C = 4\lambda_3(\mathrm{ch}r^2 + 1)\left(\lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 1\right), \tag{56}$$

$$c_x = x^2 + x^{-2}, \tag{57}$$

$$s_x = x^2 - x^{-2}. \tag{58}$$

Since the denominator ch$r$ and B are positive,[9] to maximize this expression, $\cos 2\phi$ should have maximal modulus and the same sign as s_x; i.e., without loss of generality we can take $x > 1$ and $\phi = 0$. Note also that $C < 0$ by assumption since the state $(\lambda_1,\lambda_2,\lambda_3)$ has $|D_{12}| < 0$. To show finally that the whole numerator is always negative, we consider the expression for the other two determinants $|D_{ij}|$, namely, $C_2 = \lambda_2'^2 - \lambda_1'^2 - \lambda_3'^2 + 1$ and $C_1 = \lambda_1'^2 - \lambda_2'^2 - \lambda_3'^2 + 1$, which are

$$C_1 = \frac{\left(\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + 1\right)x^2\mathrm{sh}^2 r}{(\lambda_3 + x^2\mathrm{ch}r)(\mathrm{ch}r + \lambda_3 x^2)}, \tag{59}$$

$$C_2 = \frac{\left(\lambda_2^2 - \lambda_1^2 - \lambda_3^2 + 1\right)x^2\mathrm{sh}^2 r}{(\lambda_3 + x^2\mathrm{ch}r)(\mathrm{ch}r + \lambda_3 x^2)}, \tag{60}$$

i.e., up to a positive factor they are given by the *input* determinants $|D_{23}|$ and $|D_{13}|$, which thus do not change sign and remain nonpositive. Clearly $C_1 + C_2 + C_3 = 3 - \lambda_1'^2 - \lambda_2'^2 - \lambda_3'^2 \leqslant 0$. This relation must hold for all choices of x and r. Now consider the limit $x \to \infty$, for which both $C_1, C_2 \to 0$, and

$$C_3 \to \frac{A + B}{4\lambda_3^2}, \tag{61}$$

which must therefore be $\leqslant 0$. Hence $A + B \leqslant 0$ and therefore $(Ac_x + Bs_x)\mathrm{ch}r + C \leqslant (A + B)c_x\mathrm{ch}r + C \leqslant C < 0$, which shows that all three determinants remain nonpositive. This proves that a single-mode GLOCC does not allow us to transform a state with only nonpositive off-diagonal determinants $|D_{ij}| < 0$ into a final state with at least one positive determinant. Since any pure GLOCC operation is represented by a product CJ state and can therefore be decomposed into a sequence of three single-mode operations, we have shown that even the most general pure GLOCC cannot achieve this. ■

Thus, in particular, we have shown that from a symmetric state $\gamma_{\mathrm{sym}}(\lambda)$, which has $|D_{ij}| = -(\lambda^2 - 1)/2 \leqslant 0$, it is impossible to obtain via arbitrary GLOCC any state with $|D_{ij}| > 0$ for some (ij).

D. Initial states with positive determinant

To show that these signs are not GLOCC invariant, and that, in fact, a positive determinant $|D_{ij}| > 0$ can always be made negative by GLOCC we prove the following lemma.

Lemma 6. Given a pure three-mode Gaussian state with one positive determinant $|D_{ij}| > 0$, there exists a GLOCC to transform it into a state with three negative determinants.

Proof. Recall that for pure three-mode states there is at most one positive determinant; see, e.g., Eq. (21a). Assume, without loss of generality, that $|D_{12}| > 0$. From Eqs. (59) and (60) it is clear that to change the sign of $|D_{12}|$ we must perform a GLOCC at mode 3. The determinant after a general one-mode GLOCC is given by Eq. (53). Now consider the case $\phi = 0$ and the limits $x \to \infty$ and $x \to 0$. As before, the limit $x \to \infty$ proves that $A + B \leqslant 0$; cf. Eq. (61). For $x \to 0$, we obtain

$$C_3(x \to 0) = \frac{A - B}{4\lambda_3^2}. \tag{62}$$

Since $B > 0$ it follows that $C_3(x \to 0) < 0$, i.e., for sufficiently small x all three determinants C_i are negative. ■

Let us, as an example, consider the distributed two-mode squeezed states with CM $\gamma(s,\theta)$, discussed in Sec. IV. They are obtained by passing part of a two-mode squeezed state $\gamma(s)$ through a beam splitter with transmissivity $t = \cos^2\theta$; see Eq. (29). These states have one off-diagonal block proportional to the identity, say, $D_{23} = -\sin\theta\cos\theta(\mathrm{ch}s - 1)\mathbb{1}$, i.e., with positive determinant. The other two off-diagonal blocks are proportional to σ_z, i.e., $D_{12} = \cos\theta\,\mathrm{sh}s\,\sigma_z$ and $D_{13} = -\sin\theta\,\mathrm{sh}s\,\sigma_z$. When performing a GLOCC characterized by two-mode squeezing parameter r and local squeezing x, i.e., $\Gamma = [\mathbb{1} \oplus \mathrm{diag}(x,x^{-1})]\gamma(r)[\mathbb{1} \oplus \mathrm{diag}(x,x^{-1})]$ on mode 1 (the one with the largest local mixedness), we obtain from Eq. (42)

$$D_{23}' = D_{23} - D_{12}\left[\mathrm{ch}r\begin{pmatrix} x^2 & 0 \\ 0 & x^{-2}\end{pmatrix} + \mathrm{ch}s\mathbb{1}\right]^{-1} D_{13}; \tag{63}$$

therefore

$$d_{23}^{\pm'} = -\frac{\sin(2\theta)\mathrm{sh}^2(s/2)(x^{\pm2}\mathrm{ch}r - 1)}{\mathrm{ch}s + x^{\pm2}\mathrm{ch}r}, \tag{64}$$

i.e., for $x^2 < \mathrm{ch}r$ or $x^{-2} < \mathrm{ch}r$ one of the two coefficients is negative (while the other is positive), yielding $|D_{23}'| < 0$ for all $x \notin [\sqrt{1/\mathrm{ch}r}, \sqrt{\mathrm{ch}r}]$. Since the signs of $|D_{12}|$ and $|D_{13}|$ do not change, we have transformed the initial state with $\mathrm{sgn}(|D_{12}|) = -1$, $\mathrm{sgn}(|D_{13}|) = -1$, and $\mathrm{sgn}(|D_{23}|) = 1$ to a state with all signs negative.

In fact, we can even obtain all symmetric states starting from a distributed two-mode squeezed initial state. Let us consider the simple one-parameter family of degenerate states with CM

[9] Note that B is equivalent to the square root of w_1 given in Appendix C, which is necessarily positive.

$\gamma = \gamma(s,\theta = \pi/4)$. Clearly, in this case the two smaller of the three parameters are identical, i.e., our initial state is $\lambda_1 = \text{ch}s$ and $\lambda_2 = \lambda_3 = (\text{ch}s + 1)/2$. Then it suffices to perform a suitable measurement at mode 1 to obtain $\lambda_1' = \lambda_2' = \lambda_3'$. Moreover, by choosing s large enough, it is possible to obtain *all* symmetric states in this way. To see this, we use again Eqs. (44)–(46) for an operation characterized by $(r,x,\phi = 0)$. Then it is straightforward to see that by taking

$$x^2 + x^{-2} = \frac{\text{csch}^2\frac{s}{2}(3\text{ch}^2r\,\text{ch}s + \text{ch}^2r - \text{ch}s - 3)}{2\text{ch}r}, \qquad (65)$$

we can prepare the symmetric state $\gamma_{\text{sym}}(\lambda')$ with

$$\lambda' = \frac{4\text{ch}^2\frac{s}{2}(\text{ch}^2r\,\text{ch}s - 1)}{6(\text{ch}^2r - 1)\text{ch}s - 2\text{ch}^2r + \text{ch}(2s) + 1}. \qquad (66)$$

Note that the right-hand side of Eq. (65) is $\geqslant 2$ for all choices of s,r, and thus there is always $x \geqslant 1$ corresponding to the desired value. Considering the limits $s \to \infty$ and $s \to 0$, we see that $\lambda' \to \text{ch}^2r$ and $\lambda' \to 1$, respectively. Consequently, for any target state $\gamma_{\text{sym}}(\lambda')$ there exist $s,r \geqslant 0$ and $x > 0$ such that the symmetric state λ' can be prepared from the degenerate state $\gamma(s,\pi/4)$ by a the single-mode GLOCC with parameters $(r,x,0)$. Thus, the one-parameter family $\{\gamma(s,\pi/4) : s \geqslant 0\}$ is "more strongly entangled" than $\{\gamma_{\text{sym}}(\lambda) : \lambda \geqslant 1\}$ in the sense that the latter can be obtained from the former by deterministic GLOCC but the reverse is not possible. We leave as an open question whether *all* pure three-mode Gaussian states can be obtained from $\{\gamma(s,\pi/4) : s \geqslant 0\}$ by GLOCC. Since a TMS operation acting on the first mode allows an arbitrary reduction of the parameter $s \geqslant 0$ (without changing θ) [41], a positive answer would imply that there is a single (un-normalizable) pure three-mode state from which all others can be obtained by GLOCC.

Let us finally remark on the entanglement properties associated with the appearance of a positive determinant [say, $|D_{12}| = (\lambda_3^2 + 1 - \lambda_2^2 - \lambda_1^2)/2 > 0$]: First, it means that λ_1,λ_2 are too small (relative to λ_3), i.e., there is too little entanglement available between the modes (12): most (or in the case of the distributed two-mode squeezed states all) of the mixedness at these modes arises from the entanglement with mode 3. Since a two-mode Gaussian state is necessarily separable if the off-diagonal block of its CM has non-negative determinant [46], we see that in that case there is no residual entanglement between modes (23). As we have seen this strong concentration of entanglement into one mode cannot be generated by GLOCC. On the other hand, we have seen that a GLOCC on mode 1 allows residual entanglement to be induced between the modes (23) (e.g., by generating a symmetric state) even if their reduced state was separable initially.

For the special case $\theta = \pi/4$ and for a suitably chosen Gaussian operation (essentially, for sufficiently large x and r), one can readily check, using the partial-transpose separability criterion (e.g., in the simple form for Gaussian 1×1 states given in [6]), that all three reduced CMs are entangled.

VI. CONCLUSIONS

We presented an easily computable necessary and sufficient condition for Gaussian LU equivalence for an arbitrary number of modes and derived a standard from for pure three-mode Gaussian states. This showed, in particular, that the entanglement properties of an arbitrary pure three-mode Gaussian state are completely characterized by three bipartite entanglement measures, namely, the local purities. This also shows that for pure three-mode Gaussian states LU equivalence implies GLU equivalence.

In order to gain more insight into the relation among the GLU classes, we investigated the more general set of GLOCC operations. We provided simple expressions for GLOCC transformations between different GLU classes. For the pure three-mode states we showed that they can be divided into two classes (according to whether the sign of the largest determinant $|D_{ij}|$ is positive or not) such that no GLOCC can transform a state from the second class to the first. In particular, this shows that the set of symmetric states (GHZ/W states) does not suffice to generate an arbitrary state via GLOCC. Among the states unreachable from the symmetric states we identified a family which, in contrast, allows preparation of all symmetric states.

There are many questions concerning the GLOCC interconvertibility of pure multipartite Gaussian states that remain to be addressed: Is there a "maximally entangled" family [36] in the sense that *all* other states can be obtained from it by GLOCC? Is there a majorization relation governing which states can be GLOCC transformed into another? Are there mutually inaccessible subsets of GLU classes similar to the W and GHZ classes for three qubits? Can the observed restrictions on Gaussian-state transformations be lifted if several copies of the states are considered? Are there examples in which general (i.e., non-Gaussian) local unitaries allow the transformation between two pure Gaussian states that are not in the same GLU class or does LU equivalence of Gaussian states always imply their GLU equivalence? Answers to these questions might lead to a better understanding of the structure and qualitative features of pure Gaussian entanglement and be of practical use regarding which states are the most versatile in terms of state generation.

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APPENDIX A: PURE 1 × 1 × 1 STATES: STANDARD FORM

We show here that the condition $\gamma J \gamma = J$ implies that any three-mode CM γ is xp block diagonal (see Theorem 2). Let γ as given in Eq. (15) denote the standard form of the CM. In particular, $K = D_{12}$ is diagonal. However, instead of choosing O_3 such that $L = O D_{13}$, for some orthogonal matrix O and some diagonal matrix D_{13}, we chose here without loss of generality O_3 such that L has upper-triangular form.[10] The necessary condition for γ to correspond to a pure state, $\gamma J \gamma = J$, is equivalent to the following set of equations:

$$1 = \lambda_1^2 + |D_{12}| + |L|, \quad \text{(A1a)}$$
$$1 = \lambda_2^2 + |D_{12}| + |M|, \quad \text{(A1b)}$$
$$1 = \lambda_3^2 + |L| + |M|, \quad \text{(A1c)}$$
$$0 = \lambda_1 J D_{12} + \lambda_2 D_{12} J + L J M^T, \quad \text{(A1d)}$$
$$0 = \lambda_1 J L + \lambda_3 L J + D_{12} J M^T, \quad \text{(A1e)}$$
$$0 = \lambda_2 J M + \lambda_3 M J + D_{12}^T J L. \quad \text{(A1f)}$$

Note that $\lambda_i \geqslant 1$ (in particular $\lambda_i \neq 0$) for all i, must hold for any CM [see, e.g., condition Eq. (7b)]. Let us use the notation $x_{ij} = X_{ij}$ for $X \in \{K,L,M\}$. Writing Eqs. (A1d)–(A1f) elementwise we obtain

$$\left.\begin{aligned} 0 &= l_2 m_{21}, \\ 0 &= l_{12} m_1 - l_1 m_{12}, \\ 0 &= \lambda_1 k_2 + \lambda_2 k_1 - l_{12} m_{21} + l_1 m_2, \\ 0 &= \lambda_1 k_1 + \lambda_2 k_2 + l_2 m_1, \end{aligned}\right\} \quad \text{(A2a)}$$

$$\left.\begin{aligned} 0 &= \lambda_1 l_{12} + k_2 m_{12}, \\ 0 &= \lambda_3 l_{12} - k_1 m_{21}, \\ 0 &= \lambda_1 l_2 + \lambda_3 l_1 + k_1 m_2, \\ 0 &= \lambda_1 l_1 + \lambda_3 l_2 + k_2 m_1, \end{aligned}\right\} \quad \text{(A2b)}$$

$$\left.\begin{aligned} 0 &= \lambda_2 m_{21} - \lambda_3 m_{12}, \\ 0 &= k_1 l_2 + \lambda_2 m_2 + \lambda_3 m_1, \\ 0 &= k_2 l_1 + \lambda_2 m_1 + \lambda_3 m_2, \\ 0 &= k_2 l_{12} + \lambda_2 m_{12} - \lambda_3 m_{21}. \end{aligned}\right\} \quad \text{(A2c)}$$

We show that the above equations imply that $l_{12} = m_{12} = m_{21} = 0$, i.e., L and M are also diagonal. We first discuss the case $l_2 \neq 0$. Then the first of Eqs. (A2a) implies $m_{21} = 0$. Consequently, the second equation of Eqs. (A2b) implies $l_{12} = 0$ and the second of Eqs. (A2b) yields $m_{21} = 0$. If, instead, $l_2 = 0$, we have that $|L| = 0$ and we can without loss of generality[11] assume $l_{12} = 0$. Now consider first $L \neq 0$, i.e., $l_1 \neq 0$. Then the second of Eqs. (A2a) yields $m_{12} = 0$, and the first of Eqs. (A2c) implies $m_{21} = 0$. If, finally, $L = 0$, then $k_1 + k_2 = 0$ and $m_1 + m_2 = 0$ [Eqs. (A2a) and (A2c)], respectively, and then by the last two of Eqs. (A2b) $K = 0$ or $M = 0$, i.e., mode 1 or mode 3 factorizes. In either case, both L and M are diagonal and therefore γ is xp block diagonal.

APPENDIX B: PROOF OF LEMMA 2

Here we present the details of the proof of Lemma 2. In particular, we derive the conditions under which γ as given in Eq. (19) obeys the necessary condition $\gamma J \gamma = J$. In order to increase the readability of the appendix, we restate the equivalent conditions given in Eq. (21a):

$$\lambda_1^2 + |D_{12}| + |D_{13}| = 1, \quad \text{(B1a)}$$
$$\lambda_2^2 + |D_{12}| + |D_{23}| = 1, \quad \text{(B1b)}$$
$$\lambda_3^2 + |D_{13}| + |D_{23}| = 1, \quad \text{(B1c)}$$
$$\lambda_1 D_{12} + \lambda_2 \tilde{D}_{12} + \tilde{D}_{13} \odot D_{23} = 0, \quad \text{(B1d)}$$
$$\lambda_1 D_{13} + \lambda_3 \tilde{D}_{13} + \tilde{D}_{12} \odot D_{23} = 0, \quad \text{(B1e)}$$
$$\lambda_2 D_{23} + \lambda_3 \tilde{D}_{23} + \tilde{D}_{12} \odot D_{13} = 0. \quad \text{(B1f)}$$

As before, $\odot$ denotes the componentwise multiplication (Hadamard product). Here, we use the index-free notation $D_{12} = \mathrm{diag}(a,b)$, $D_{13} = \mathrm{diag}(c,d)$, and $D_{23} = \mathrm{diag}(e,f)$, and that $DJ = J\tilde{D}$ [i.e., $\tilde{D} = -JDJ$)] for any diagonal matrix D and therefore $DJD = |D|J$. Note that if $D = \mathrm{diag}(x,y)$, then $\tilde{D} = \mathrm{diag}(y,x)$. In order to solve those equations we distinguish between the following two cases:

(i) at least one of the diagonal matrices, D_{ij} is not invertible and

(ii) none of the determinants vanishes.

Let us first consider the case (i). Since we do not impose any order on the λ_i we assume without loss of generality that $e = 0$. It is then straightforward to verify that the solution to Eq. (B1) is given by

$$\lambda_1 = \sqrt{-1 + \lambda_2^2 + \lambda_3^2}, \quad \text{(B2a)}$$
$$a = (-1)^{k_1} \sqrt{\lambda_2 \left(-1 + \lambda_2^2\right)} \Big/ \sqrt{\lambda_1}, \quad \text{(B2b)}$$
$$b = -(-1)^{k_1} \sqrt{-1/\lambda_2 + \lambda_2} \sqrt{\lambda_1}, \quad \text{(B2c)}$$
$$c = (-1)^{k_2} \sqrt{\lambda_3 \left(-1 + \lambda_3^2\right) / \lambda_1}, \quad \text{(B2d)}$$
$$d = -(-1)^{k_2} \sqrt{-1/\lambda_3 + \lambda_3} \sqrt{\lambda_1}, \quad \text{(B2e)}$$
$$e = 0, \quad \text{(B2f)}$$
$$f = (-1)^{k_1+k_2} \sqrt{\left(\lambda_2^2 - 1\right)\left(\lambda_3^2 - 1\right)} \Big/ \sqrt{(\lambda_2 \lambda_3)}, \quad \text{(B2g)}$$

where $k_1, k_2 \in \{0,1\}$. Now, it is easy to see that the four solutions for the different values of k_1, k_2 are GLU equivalent by choosing $O = (-1)^{k_1} \mathbb{1} \oplus \mathbb{1} \oplus (-1)^{k_1+k_2} \mathbb{1}$. Thus, we chose without loss of generality $k_1 = k_2 = 0$. Given the expressions of the entries of the diagonal matrices D_{ij} [see Eq. (20)] it is straightforward to check that $a^2 = (d_{12}^-)^2$, $b^2 = (d_{12}^+)^2$, $c^2 = (d_{13}^-)^2$, $d^2 = (d_{13}^+)^2$, $e^2 = 0 = (d_{23}^-)^2$, and $f^2 = (d_{23}^+)^2$. Moreover, it is easy to see that $|D_{ij}| = d_{ij}^+ d_{ij}^-$, for all three matrices. Thus, the expressions coincide up to a (independent) global phase for the matrices D_1, D_2 (the sign of D_3 is thereby fixed). Let us denote these signs by k_1, k_2, k_3, respectively. Clearly, $d_{12}^+ \geqslant 0$, which implies, since $b \leqslant 0$, that $(-1)^{k_1} = -1$. Similarly, it is easy to see that $(-1)^{k_2} = -1$ and $(-1)^{k_3} = 1$ [which has to coincide with $(-1)^{k_1+k_2}$]. Thus, the orthogonal matrix $-\sigma_x \oplus -\sigma_x \oplus \sigma_x$ (corresponding to a GLU) sorts the entries in the diagonal matrices and applies the right signs to map γ into the form of Eq. (19), with the diagonal entries given in Eq. (20).

[10] Note that this is always possible for any L; as we will see in the case considered, the two conventions coincide.

[11] By choosing an appropriate orthogonal transformation at mode 3.

Let us now consider the more involved case (ii). First note that due to Eq. (B1) the following relations hold:

$$\begin{aligned} ab &= |D_{12}| = 1/2\big(1-\lambda_1^2-\lambda_2^2+\lambda_3^2\big),\\ cd &= |D_{13}| = 1/2\big(1-\lambda_1^2+\lambda_2^2-\lambda_3^2\big),\\ ef &= |D_{23}| = 1/2\big(1+\lambda_1^2-\lambda_2^2-\lambda_3^2\big). \end{aligned} \tag{B3}$$

Note that two of these determinants are nonpositive. More precisely, if $\lambda_i \geqslant \lambda_k,\lambda_l$ then $|D_{ik}| \leqslant 0$ and $|D_{il}| \leqslant 0$. Let us now define

$$\begin{aligned} x_1 &= \frac{1}{4\lambda_2\lambda_3|D_{13}|^2},\\ x_2 &= \lambda_2^6 + \left(-1+\lambda_1^2\right)^2\lambda_3^2 - 2\left(1+\lambda_1^2\right)\lambda_3^4\\ &\quad + \lambda_3^6 - \lambda_2^4\left(2+2\lambda_1^2\lambda_3^2\right)\\ &\quad + \lambda_2^2\big[\left(-1+\lambda_1^2\right)^2 + 4\left(1+\lambda_1^2\right)\lambda_3^2 - \lambda_3^4\big]. \end{aligned} \tag{B4}$$

The solution to Eq. (B1) is then given by

$$f = 2\frac{y_1 x}{y_2 + y_3 x^2}, \tag{B5a}$$

$$e = \frac{|D_{23}|}{f}, \tag{B5b}$$

$$d = \frac{\sqrt{\lambda_1}\sqrt{-|D_{12}|}}{\sqrt{ex+\lambda_2 x^2}}, \tag{B5c}$$

$$c = \frac{|D_{13}|}{d}, \tag{B5d}$$

$$b = xd, \tag{B5e}$$

$$a = \frac{|D_{12}|}{b}. \tag{B5f}$$

Here we have used

$$x = (-1)^k\left\{\frac{1}{\sqrt{2}}\sqrt{x_1[x_2+(-1)^l\sqrt{w}]}\right\}, \tag{B6}$$

$$\begin{aligned} w &= (-1+\lambda_1-\lambda_2-\lambda_3)(1+\lambda_1-\lambda_2-\lambda_3)\\ &\quad\times(-1+\lambda_1+\lambda_2-\lambda_3)(1+\lambda_1+\lambda_2-\lambda_3)\\ &\quad\times(-1+\lambda_1-\lambda_2+\lambda_3)(1+\lambda_1-\lambda_2+\lambda_3)\\ &\quad\times(-1+\lambda_1+\lambda_2+\lambda_3)(1+\lambda_1+\lambda_2+\lambda_3)\left(\lambda_2^2-\lambda_3^2\right)^2, \end{aligned} \tag{B7}$$

$$y_1 = \big(\lambda_3^2-\lambda_2^2\big)|D_{23}|, \tag{B8}$$

$$y_2 = -2\lambda_3|D_{12}|, \tag{B9}$$

$$y_3 = 2\lambda_2|D_{13}|, \tag{B10}$$

and $k,l \in \{0,1\}$.

Note that the denominator of x_1 is nonvanishing since D_{13} is invertible. Note further that the denominator of f is vanishing (for $\lambda_i \geqslant 1$) iff either (a) $\lambda_1 = \sqrt{1-\lambda_2^2+\lambda_3^2}$ or (b) $\lambda_j = \sqrt{1-\lambda_k^2+\lambda_1^2}$ for $j \neq k$, or (c) $\lambda_i = \sqrt{-1+\lambda_j^2+\lambda_k^2}$ or (d) $\lambda_2 = \lambda_3$. The cases (a)–(c) cannot occur here, since in those cases one of the determinants D_{ij} vanishes. Let us now first consider the case $\lambda_2 \neq \lambda_3$ [for case (d), $\lambda_2 = \lambda_3$, a similar argument applies].

Note that γ is real only if $w \geqslant 0$. Let $\lambda_i \geqslant \lambda_k,\lambda_j$ for mutually different values of $i,j,k \in \{1,2,3\}$ denote the largest value; then it can be easily seen that $w \geqslant 0$ iff either $\lambda_i \leqslant \lambda_j + \lambda_k - 1$ or $\lambda_i \geqslant \lambda_j + \lambda_k + 1$. As shown in the main text, the second choice is excluded due to the positivity of γ.

Note further that all values of k,l lead to a solution. Those equalities have been derived as follows. First we use the conditions $\lambda_1^2 + |D_{12}| + |D_{13}| = 1, \lambda_2^2 + |D_{12}| + |D_{23}| = 1, \lambda_3^2 + |D_{13}| + |D_{23}| = 1$ to compute a,c,e as functions of the other parameters. As can be easily seen, the conditions given in Eq. (B1) imply that $bc(f\lambda_2 + e\lambda_3) - ad(e\lambda_2 + f\lambda_3) = 0$, which implies that $b = xd$, where x is a function which depends only on f and λ_i. Using then that $-de - a\lambda_1 - b\lambda_2 = 0$ we compute d as a function of f, x, and λ_i. Next we compute c as a function of f, x, and λ_i by using that $-be - c\lambda_1 - d\lambda_3 = 0$. Thus, we have all variables as functions of f, x, and λ_i. Using then the condition $(\gamma J\gamma - J)_{2,1} = 0$ we derive $f = y_1x/(y_2 + y_3x^2)$. The equation $(\gamma J\gamma - J)_{1,4} = 0$ allows us then to compute x as given above. Note that we obtain two solutions for d, namely, $\pm d$ for d given in Eq. (B5c). However, changing the sign of d amounts to changing the signs of c,a,b [cf. Eqs. (B5d)–(B5f)] and corresponds therefore to the GLU $O_1 = -\mathbb{1}$, $O_2 = O_3 = \mathbb{1}$. It is tedious, but straightforward to show that all four solutions, $k,l \in \{0,1\}$ are GLU equivalent to the one with $k = l = 0$.

Similarly to the case (i) it can now be shown that the expressions we derived for the entries of the diagonal matrices coincide with the ones given in Eq. (20). However, here we have that $a = d_{12}^+$, etc. For $\lambda_2 = \lambda_3$ a similar argument can be used to arrive at the same conclusion, which completes the proof.

As shown in the following appendix, the necessary condition that $\gamma \geqslant 0$ is equivalent to the condition given in Eq. (22). Note that this implies that given the three local purities λ_i (or equivalently the bipartite entanglement shared in the three splittings $i|jk$), the state is uniquely determined. The reason for that is that $\lambda_1 = \sqrt{-1+\lambda_2^2+\lambda_3^2}$ iff $e = 0$ [also in case (ii)] and therefore knowing the parameters λ_i implies that we also know to which of the two cases the state belongs. Thus, an arbitrary state is uniquely determined (up to GLUs) by the bipartite entanglement.

APPENDIX C: POSITIVITY OF $\gamma(\lambda_1,\lambda_2,\lambda_3)$

To see that the conditions

$$\lambda_i + \lambda_j \geqslant \lambda_k + 1 \;\forall\; (ijk) \tag{C1}$$

[cf. Eq. (22)] imply positivity of the CM $\gamma = \gamma(\lambda_1,\lambda_2,\lambda_3)$ we proceed as follows: γ is by construction xp block diagonal and since it has been constructed to satisfy the purity condition $\gamma J\gamma = J$, it follows that $\gamma_p = \gamma_x^{-1}$, and hence positivity of γ_x implies positivity of γ. Using the Schur complement [57] positivity of γ_x is, as $\lambda_3 > 0$, equivalent to positivity of the

2×2 matrix Y,

$$Y = \begin{pmatrix} \lambda_1 & d_{12}^+ \\ d_{12}^+ & \lambda_2 \end{pmatrix} - \begin{pmatrix} d_{13}^+ \\ d_{23}^+ \end{pmatrix} \frac{1}{\lambda_3} (d_{13}^+ \quad d_{23}^+), \tag{C2}$$

which is equivalent to the two conditions

$$\mathrm{tr} Y \geqslant 0, \tag{C3}$$

$$\det Y \geqslant 0. \tag{C4}$$

The trace is found to be

$$\frac{\lambda_1 + \lambda_2}{8\lambda_1\lambda_2\lambda_3^2}(K_1 - 1 - \sqrt{w_1}),$$

where we have introduced

$$K_1 = -\sum_i \lambda_i^4 + 2\sum_i \left(\lambda_j^2\lambda_k^2 + \lambda_i^2\right), \tag{C5}$$

$$\begin{aligned} w_1 = {} & (-1 + \lambda_1 - \lambda_2 - \lambda_3)(1 + \lambda_1 - \lambda_2 - \lambda_3) \\ & \times (-1 + \lambda_1 + \lambda_2 - \lambda_3)(1 + \lambda_1 + \lambda_2 - \lambda_3) \\ & \times (-1 + \lambda_1 - \lambda_2 + \lambda_3)(1 + \lambda_1 - \lambda_2 + \lambda_3) \\ & \times (-1 + \lambda_1 + \lambda_2 + \lambda_3)(1 + \lambda_1 + \lambda_2 + \lambda_3). \end{aligned} \tag{C6}$$

It follows directly from Eq. (C1) that $w_1 \geqslant 0$. It is tedious but straightforward to show that

$$\det Y = \frac{\mathrm{tr} Y}{\lambda_1 + \lambda_2}. \tag{C7}$$

Thus we see that both conditions Eqs. (C3) and (C4) hold and therefore $\gamma \geqslant 0$ if

$$K_1 - 1 \geqslant \sqrt{w_1}. \tag{C8}$$

To see that $K_1 - 1 \geqslant 0$ we write it as a sum of positive terms [using that the conditions given in Eq. (22) are satisfied]:

$$\begin{aligned} K_1 - 1 = {} & \frac{1}{4}\{[(\lambda_3 - 1)^2 - (\lambda_2 - \lambda_1)^2][(\lambda_1 + \lambda_2)^2 -|,(\lambda_3 + 1)^2] \\ & + \left[\lambda_3^2 - (\lambda_1 - \lambda_2 - 1)^2\right]\left[(\lambda_1 + \lambda_2 - 1)^2 - \lambda_3^2\right] \\ & + \left[\lambda_3^2 - (\lambda_1 - \lambda_2 + 1)^2\right]\left[(\lambda_1 + \lambda_2 - 1)^2 - \lambda_3^2\right] \\ & + [(\lambda_3 - 1)^2 - (\lambda_1 - \lambda_2)^2][(\lambda_1 + \lambda_2)^2 - (\lambda_3 - 1)^2]\} \\ & + \sum_i \lambda_i^2(-\lambda_i + \lambda_j + \lambda_k) \\ & + \sum_i \lambda_i(2\lambda_i - 1) + 2\Pi_l \lambda_l, \end{aligned}$$

where (jk) in $\sum_i (\lambda_j + \lambda_k)^2$ refer in each term to the two indices distinct from i. Now the remaining condition $K_1 - 1 \geqslant \sqrt{w_1}$ can be checked for the squares of both sides and we find it trivially satisfied:

$$(K_1 - 1)^2 - w_1 = 64\lambda_1^2\lambda_2^2\lambda_3^2 \geqslant 0.$$

Therefore both $\det D \geqslant 0$ and $\mathrm{tr} D \geqslant 0$ and consequently $\gamma(\lambda_1,\lambda_2,\lambda_3) \geqslant 0$ whenever the λ's satisfy Eq. (22).

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