From the quantum relative Tsallis entropy to its conditional form: Separability criterion beyond local and global spectra

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(Received 25 November 2013; published 27 January 2014)

The quantum relative Rényi entropy of two noncommuting density matrices was recently defined from which its conditional entropy is deduced. This framework is here extended to the corresponding Tsallis relative entropy and to its conditional form. This expression of Tsallis conditional entropy is shown to witness entanglement beyond the method based on global and local spectra of composite quantum states. When the reduced density matrix happens to be a maximally mixed state, this conditional entropy coincides with the expression in terms of Tsallis entropies derived earlier by Abe and Rajagopal [Physica A **289**, 157 (2001)]. The separability range in a noisy one-parameter family of W and Greenberger-Horne-Zeilinger states with three and four qubits is explored here and it is shown that the results inferred from negative Tsallis conditional entropy matches that obtained through Peres's partial transpose criterion for one-parameter family of W states, in one of its partitions. The criterion is shown to be nonspectral through its usefulness in identifying entanglement in isospectral density matrices.

DOI: 10.1103/PhysRevA.89.012331 PACS number(s): 03.67.−a, 03.65.Ud

I. INTRODUCTION

Entropic characterization of separability of composite quantum systems has attracted significant attention [1–9]. This is based on the observation that von Neumann conditional entropy of a pure bipartite entangled state is negative and this points towards a more general feature that entangled systems could be more disordered locally than globally, while it is not possible for separable states [10]. Positivity of generalized quantum conditional entropies such as Rényi and Tsallis conditional entropies is a more effective tool to investigate global vs local disorder in mixed states and leads to stricter constraints on separability than that obtained using conditional von Neumann entropy [3–8]. However, the entropic criterion is not sufficient for separability as it has been shown that any criterion based on global and local spectra is incapable of distinguishing a separable state from entangled ones [10].

Recently, a different quantum generalization of Rényi relative entropy was introduced [11,12] and there has been a surge of activity in establishing several properties of this recent version of Rényi entropy [11–15]. This generalized Rényi relative entropy of a pair of density operators reduces to the traditional one when the two density operators commute with each other. It is thus natural to anticipate that this quantity is more effective than its traditional version when noncommuting density matrices are involved. In Ref. [11] this generalization was termed "sandwiched" Rényi relative entropy in view of the sandwiching of noncommutative operators in a particular way and hence this nomenclature is an appropriate one for the generalized Rényi relative entropy.

In this paper we introduce a sandwiched version of Tsallis relative entropy and identify an alternative quantum version of conditional Tsallis entropy derived therefrom. We investigate entanglement based on nonpositive values of sandwiched Tsallis conditional entropy defined here. An earlier version of Tsallis conditional entropy, introduced by Abe and Rajagopal [3] using classical conditional Tsallis probabilities, has been found to be useful in exploring separability of several families of composite quantum systems [3–8]. The conditional version of the sandwiched Tsallis relative entropy (CSTRE) offers as a generalized version when the density matrix and its reductions do not commute with each other. The present version reduces to the Abe-Rajagopal (AR) q-conditional entropy when the reduced density matrix is maximally disordered. We show that the identification of inseparability based on nonpositive values of the CSTRE goes beyond the spectral criterion and is capable of distinguishing separable states from entangled states that are globally and locally isospectral.

The contents of the paper are organized into five sections, including the introduction and motivation behind this work detailed here in Sec. I. In Sec. II we introduce the sandwiched relative Tsallis entropy and its conditional version. We employ this conditional form to examine the separability range in one-parameter families of W and Greenberger-Horne-Zeilinger (GHZ) states in Sec. III. In Sec. IV we show that the CSTRE approach of identifying separability goes beyond spectral disorder criteria and can distinguish separable states from entangled ones when they share the same global and local spectra. Section V contains a summary and concluding remarks.

II. SANDWICHED RELATIVE TSALLIS ENTROPY AND ITS CONDITIONAL VERSION

The generalized entropies, the Rényi and Tsallis entropies, denoted, respectively, by $S_q^R(\rho)$ and $S_q^T(\rho)$, are given by [2,3,16]

$$S_q^R(\rho) = \frac{1}{1-q}\ln \mathrm{Tr}\,(\rho^q),$$
$$S_q^T(\rho) = \frac{\mathrm{Tr}\,(\rho^q) - 1}{1-q}, \tag{1}$$

where q is a real positive parameter. Both these reduce to von Neumann entropy in the limit $q \to 1$. The traditional quantum relative Rényi entropy for a pair of density operators ρ and σ is defined, by ignoring the ordering of the density matrices, as

$$D_q^R(\rho||\sigma) = \frac{\ln \mathrm{Tr}\,(\rho^q \sigma^{1-q})}{q-1} \quad \text{for} \quad q \in (0,1)\cup(1,\infty)$$
$$= \mathrm{Tr}\,[\rho(\ln\rho - \ln\sigma)] \quad \text{for} \quad q \to 1. \tag{2}$$

Recently, a generalized version of quantum relative Rényi entropy was introduced by Wilde *et al.* [11] and Müller-Lennert *et al.* [12] independently:

$$\tilde{D}_q^R(\rho||\sigma) = \frac{1}{q-1}\ln[\mathrm{Tr}\,(\sigma^{(1-q)/2q}\rho\sigma^{(1-q)/2q})^q]$$
$$\text{for} \quad q \in (0,1)\cup(1,\infty). \tag{3}$$

The quantum relative Rényi entropy (3) reduces to the traditional one given by (2) when the density matrices ρ and σ commute and hence the present version is an extension to the noncommutative case.

We consider the traditional form of Tsallis relative entropy

$$D_q^T(\rho||\sigma) = \frac{\mathrm{Tr}\,(\rho^q\sigma^{1-q}) - 1}{1-q} \tag{4}$$

and define the sandwiched Tsallis relative entropy in a similar manner,

$$\tilde{D}_q^T(\rho||\sigma) = \frac{\mathrm{Tr}\,[(\sigma^{(1-q)/2q}\rho\,\sigma^{(1-q)/2q})^q] - 1}{1-q}. \tag{5}$$

Note that when $\sigma = I$, the sandwiched Tsallis relative entropy (5) reduces to the Tsallis entropy $S_q^T(\rho) = \frac{\mathrm{Tr}\,(\rho^q)-1}{1-q}$ and in the limit $q \to 1$ it reduces to the von Neumann relative entropy $\lim_{q\to 1} D_q(\rho||\sigma) = \mathrm{Tr}\,[\rho(\ln\rho - \ln\sigma)]$ [17].

We now define the conditional version of $\tilde{D}_q^T(\rho_{AB}||\sigma)$ by taking $\sigma \equiv I_A \otimes \rho_B$ (or $\rho_A \otimes I_B$) where $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ [$\rho_A = \mathrm{Tr}_B(\rho_{AB})$] is the subsystem density matrix of the composite state ρ_{AB}. It is given by

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\tilde{Q}_q(\rho_{AB}||\rho_B) - 1}{1-q}, \tag{6}$$

where

$$\tilde{Q}_q(\rho_{AB}||\rho_B)$$
$$= \mathrm{Tr}\,\{[(I\otimes\rho_B)^{(1-q)/2q}\rho_{AB}(I\otimes\rho_B)^{(1-q)/2q}]^q\}. \tag{7}$$

The sandwiched conditional Tsallis entropy (6) reduces to AR q-conditional Tsallis entropy [3]

$$S_q^T(A|B) = \frac{1}{q-1}\left(1 - \frac{\mathrm{Tr}\,\rho_{AB}^q}{\mathrm{Tr}\,\rho_B^q}\right) \tag{8}$$

when the subsystem density matrix is a maximally mixed state. Negative values of AR q-conditional entropy indicate entanglement and it has been employed as a separability criterion for several classes of composite states [3–8]. Here we employ the form of sandwiched conditional Tsallis entropy derived from its relative entropy to infer about entanglement.

While the evaluation of the expression $\tilde{Q}_q(\rho_{AB}||\rho_B)$ does not seem trivial, construction of the unitary operator that diagonalizes the subsystem density matrix ρ_B makes the calculation a feasible one. If U_B is the unitary operator that diagonalizes ρ_B, we have

$$\sigma_D = U_\sigma\sigma^{(1-q)/2q}U_\sigma^\dagger$$
$$\text{for} \quad \sigma = I\otimes\rho_B, \quad U_\sigma = I\otimes U_B, \tag{9}$$
$$\sigma_D = \mathrm{diag}\big(\lambda_1^{(1-q)/2q}\cdots\lambda_n^{(1-q)/2q}\big).$$

Thus the expression for $\tilde{Q}_q(\rho_{AB}||\rho_B)$ in Eq. (7) simplifies to

$$\tilde{Q}_q(\rho_{AB}||\rho_B) = \mathrm{Tr}\,[(\sigma_D U_\sigma\rho U_\sigma^\dagger\sigma_D)^q]. \tag{10}$$

Defining $\Gamma = \sigma_D U_\sigma \rho U_\sigma^\dagger \sigma_D$, an evaluation of the eigenvalues γ_i of Γ immediately leads us to the quantity $\tilde{Q}_q(\rho_{AB}||\rho_B)$ as $\sum_{i=1}^d \gamma_i^q$, d being the dimension of ρ. Thus we finally obtain an expression for the CSTRE

$$\tilde{D}_q^T(\rho_{AB}||\rho_B) = \frac{\sum_{i=1}^d \gamma_i^q - 1}{1-q}. \tag{11}$$

We make use of Eq. (11) to determine the separability range in one-parameter families of W and GHZ states in the following section.

III. SEPARABILITY OF NOISY ONE-PARAMETER FAMILIES OF W AND GHZ STATES

The symmetric one-parameter families of N-qubit mixed states, involving a W or a GHZ state, are given, respectively, by

$$\rho_N^{(W)}(x) = \left(\frac{1-x}{N+1}\right)P_N + x|W\rangle_N\langle W| \tag{12}$$

and

$$\rho_N^{(\mathrm{GHZ})}(x) = \left(\frac{1-x}{N+1}\right)P_N + x|\mathrm{GHZ}\rangle_N\langle\mathrm{GHZ}|. \tag{13}$$

Here $0 \leqslant x \leqslant 1$ and $P_N = \sum_M |N/2,M\rangle\langle N/2,M|$ denotes the projector onto the symmetric subspace of N-qubits spanned by the $N+1$ angular momentum states $|N/2,M\rangle$, $M = N/2, N/2-1, \ldots, -N/2$, belonging to the maximum value $J = N/2$ of total angular momentum.

A systematic attempt to examine the separability range of the noisy one-parameter family of W and GHZ states using the AR q-conditional entropy has been carried out in Ref. [8]. While Prabhu *et al.* obtained a result matching that of the positive partial transpose (PPT) criterion [18] for two-qubit states of $\rho_{N=2}^{(W)}(x)$, the range of separability identified by them is weaker than that through the PPT criterion for both W and GHZ families when $N \geqslant 3$. Here we identify that noncommutativity of the density matrix ρ_{AB} with its subsystem state ρ_B does indeed play a role and the separability domain inferred through non-negative values of the CSTRE is stricter than that obtained from the AR q-conditional entropy, though it is weaker than the PPT criterion in some cases.

For $N = 3$, a direct evaluation of the subsystem density matrix ρ_{BC} of $\rho_3^W(x) \equiv \rho_{ABC}$ and the unitary matrix that diagonalizes it leads us to $\Gamma = \sigma_D U_\sigma \rho U_\sigma^\dagger \sigma_D$. Here we have

$$\sigma_D = I_2 \otimes \mathrm{diag}\left(\left(\frac{1}{3}\right)^{(1-q)/2q}, 0, \left(\frac{1-x}{3}\right)^{(1-q)/2q}, \times \left(\frac{1+x}{3}\right)^{(1-q)/2q} \right),$$

$$U_\sigma = I_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \rho = \rho_3^W(x).$$

The nonzero eigenvalues of Γ are found to be

$$\gamma_1 = \frac{3(1-x)3^{-1/q}}{4}, \quad \gamma_2 = \frac{3(1-x)^{1/q}3^{-1/q}}{4},$$
$$\gamma_3 = \frac{3^{-1/q}(1+3x)[1+x+2(1+x)^{1/q}]}{4(1+x)}, \tag{14}$$
$$\gamma_4 = \frac{3^{-1/q}[(1+x)(1-x)^{1/q} + 2(1-x)(1+x)^{1/q}]}{4(1+x)}.$$

One can now readily evaluate the expression for the CSTRE in Eq. (11) and for different values of q, we obtain $\tilde{D}_q^T(\rho_3^{(W)}(x)||\rho_{BC})$ as a function of x. The plots in Figs. 1 and 2 illustrate the stricter separability range for $\rho_3^{(W)}(x)$, in its A:BC partition, for increasing values of q. It can be readily seen from Figs. 1 and 2 that the separability range $0 \leqslant x \leqslant 0.1547$ in the A:BC partition of the one-parameter family of three-qubit W states obtained through the CSTRE approach is in complete agreement with that obtained from the partial transpose criterion. It should be noted (see Fig. 2) that AR q-conditional entropy yields a weaker separability range [8] $0 \leqslant x \leqslant 0.2$ for the A:BC partition of $\rho_3^{(W)}(x)$.

In a similar manner we evaluate the CSTRE $\tilde{D}_q^T(\rho_4^{(W)}(x)||\rho_{BCD})$ and arrive at the separability range in the A:BCD partition of the state $\rho_4^{(W)}(x)$. It can be seen that $\rho_4^{(W)}(x)$ is separable when $x \leqslant 0.1124$, in complete conformity with the separability range obtained through the PPT criterion. In Fig. 3 we illustrate our result for $\rho_4^{(W)}(x)$

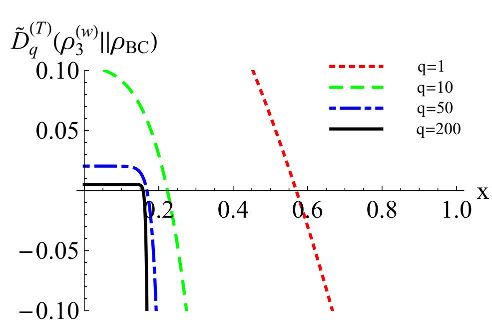

FIG. 1. (Color online) Conditional form of the sandwiched Tsallis relative entropy $\tilde{D}_q^T(\rho_3^{(W)}(x)||\rho_{BC})$ for a one-parameter family of three-qubit W states as a function of x for different values of q. It can be seen that the CSTRE is negative for $x \geqslant 0.57$ when $q = 1$, whereas it is negative as $x \to 0.1547$ (the value of x identified by the PPT criterion) for $q \gg 1$. All the quantities are dimensionless.

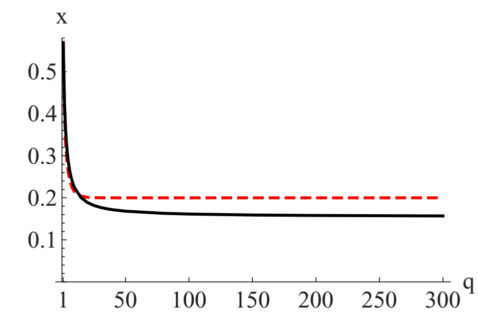

FIG. 2. (Color online) Implicit plot of $\tilde{D}_q^T(\rho_3^{(W)}||\rho_{BC}) = 0$ as a function of q (solid line) indicating that $x \to 0.1547$ as $q \to \infty$. In contrast, the implicit plot of the Abe-Rajagopal q-conditional entropy $S_q^T(A|BC) = 0$ (dashed line) leads to $x \to 0.2$ as $q \to \infty$. The quantities plotted are dimensionless.

through an implicit plot of $\tilde{D}_q^T(\rho_4^{(W)}||\rho_{BCD}) = 0$ and compare it with that of AR q-conditional entropy $S_q^T(A|BCD) = 0$. It is pertinent to point out that the state $\rho_N^{(W)}(x)$ and its reduced counterparts $I_2 \otimes \rho_{N-1}$ are noncommuting and thus are ideal examples to test the present CSTRE criterion for separability.

The separability range of the one-parameter family of N-qubit GHZ states has been examined using AR q-conditional entropy in Ref. [8], where the separability range matching that from the PPT criterion for $\rho_3^{(\mathrm{GHZ})}(x)$ only in the A:BC partition was obtained. It may be noted that the separability range in the A:BC partition of $\rho_3^{(\mathrm{GHZ})}(x)$ is $[0,\frac{1}{7}]$ through the PPT criterion as well as the AR q-conditional entropy criterion. An explicit evaluation using the CSTRE approach is seen to reproduce the same separability range, indicating that the CSTRE separability domain is bounded by the PPT range.

For $N = 4$ also, the separability ranges obtained through both the PPT criterion and the AR q-conditional entropy approach match each other only in the A:BCD partition of $\rho_4^{(\mathrm{GHZ})}(x)$. Here we identify the separability ranges through the present CSTRE approach and show that a separability range that is the same as that through the AR q-conditional approach is obtained in all possible partitions of the state

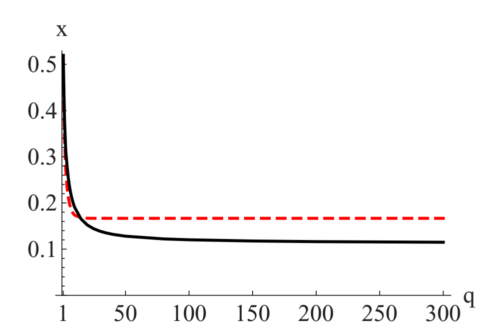

FIG. 3. (Color online) Implicit plot of $\tilde{D}_q^T(\rho_4^{(W)}||\rho_{BCD}) = 0$ (solid line) and $S_q^T(A|BCD) = 0$ (dashed line) as functions of q. Here $x \to 0.1124$ according to the CSTRE approach while $x \to 0.1666$ is inferred from the AR q-conditional entropy, both in the limit $q \to \infty$. The quantities plotted are dimensionless.

TABLE I. Comparison of separability range of one-parameter families of states through entropic criteria and a PPT.

Quantum state	von Neumann conditional entropy	AR q-conditional entropy	CSTRE	PPT
		$\rho_3^{(W)}$		
$A{:}BC$ partition	{0,0.5695}	{0,0.2}	{0,0.1547}	{0,0.1547}
$AB{:}C$ partition	{0,0.7645}	{0,0.4286}	{0,0.3509}	{0,0.1547}
		$\rho_3^{(\mathrm{GHZ})}$		
$A{:}BC$ partition	{0,0.5482}	{0,1/7}	{0,1/7}	{0,1/7}
$AB{:}C$ partition	{0,0.7476}	{0,1/3}	{0,1/3}	{0,1/7}
		$\rho_4^{(W)}$		
$A{:}BCD$ partition	{0,0.5193}	{0,0.1666}	{0,0.1123}	{0,0.1123}
$AB{:}CD$ partition	{0,0.6560}	{0,0.2105}	{0,0.2105}	{0,0.0808}
$ABC{:}D$ partition	{0,0.8222}	{0,0.5454}	{0,0.4174}	{0,0.1123}
		$\rho_4^{(\mathrm{GHZ})}$		
$A{:}BCD$ partition	{0,0.4676}	{0,0.0909}	{0,0.0909}	{0,0.0909}
$AB{:}CD$ partition	{0,0.6560}	{0,0.2105}	{0,0.2105}	{0,0.0625}
$ABC{:}D$ partition	{0,0.7868}	{0,0.375}	{0,0.375}	{0,0.0909}

$\rho_4^{(\mathrm{GHZ})}(x)$. Table I summarizes our results for the different partitions of the one-parameter family of W and GHZ states. It can be readily seen from Table I that the CSTRE approach yields a separability range that is either equal to or *stricter* than the range obtained through the AR q-conditional entropy and matches the PPT criterion in some of the noncommuting cases such as in $\rho^{(W)}_{N=3,4}$ in one of their $A{:}BC$ and $A{:}BCD$ partitions. We proceed further in Sec. IV to illustrate that the CSTRE separability criterion is nonspectral in nature and thus can distinguish separable and entangled states that share the same local and global eigenvalues.

IV. NONSPECTRAL NATURE OF THE CSTRE CRITERION

Generalized entropies serve as a measure of disorder in a given quantum state and negative values of traditional versions of generalized conditional entropies point towards more global order than local order in a composite system. Separable states are more locally ordered than globally as the eigenspectra of the whole composite separable state is majorized by that of its reduced systems [10]. Negative values of conditional entropies reflect the contrasting feature that local spectra *need not* majorize global spectra in entangled states. However, the separability criterion based purely on the spectra of composite and reduced states is shown to be insufficient [10]. This feature was illustrated through an example of an isospectral pair of two-qubit states ρ_{AB} and ϱ_{AB}, which share the same local and global spectra with ρ_{AB} being entangled while ϱ_{AB} is separable [10]:

$$\rho_{AB} = \frac{1}{3}\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{pmatrix}, \tag{15}$$

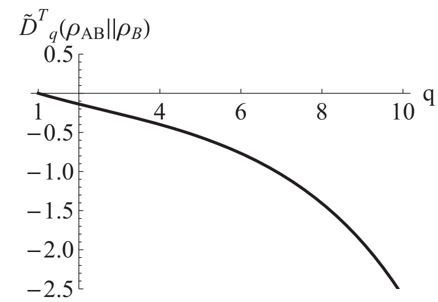

FIG. 4. Plot of the CSTRE $\tilde{D}^T_q(\rho_{AB}||\rho_B)$ of the two-qubit state (15) as a function of q. It can be readily seen that the CSTRE is negative for all values of $q > 1$, indicating that the state is entangled. Both quantities are dimensionless.

$$\varrho_{AB} = \frac{1}{3}\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&2\end{pmatrix}. \tag{16}$$

It is worth observing that the separable state ϱ_{AB} commutes with its reduced density matrices, whereas the entangled state ρ_{AB} and its subsystems are noncommutative. Interestingly, the present sandwiched conditional Tsallis entropy is capable of distinguishing the isospectral states and hence the approach proves to be superior to any spectral disorder criteria. We demonstrate the effectiveness of the CSTRE separability criterion in the above examples of two-qubit isospectral states.

For the state ρ_{AB} we obtain

$$\gamma_1 = 2^{(1-q)/q}3^{-1/q},$$
$$\gamma_2 = (1+2^{(1-q)/q})3^{-1/q}$$

and the CSTRE is given by

$$\tilde{D}^T_q(\rho_{AB}||\rho_B) = \frac{(1+2^{(1-q)/q})^q + 2^{1-q} - 3}{3(1-q)}. \tag{17}$$

A plot of $\tilde{D}^T_q(\rho_{AB}||\rho_B)$ as a function of q is given in Fig. 4.

It can be readily seen from Fig. 4 that $\tilde{D}^T_q(\rho_{AB}||\rho_B)$ is negative for all values of q, except for $q = 1$. In contrast, in the two-qubit separable state ϱ_{AB} we find that

$$\gamma_1 = \left(\frac{2}{3}\right)^{1/q}, \quad \gamma_2 = \left(\frac{1}{3}\right)^{1/q}, \tag{18}$$

leading to $\gamma_1^q + \gamma_2^q = 1$ and hence

$$\tilde{D}^T_q(\varrho_{AB}||\varrho_B) = 0 \tag{19}$$

for all values of q. The isospectral example of two-qubit states highlights that the CSTRE approach is essentially nonspectral in nature, unlike other entropic criteria [19].

V. CONCLUSION

Characterization of entanglement based on Rényi and Tsallis conditional entropies [1–9] is essentially based on the spectra of the composite state and its subsystems. Separable states are more disordered locally than globally [10] and this feature is reflected through their generalized conditional entropies being positive. Negative values of conditional entropies imply entanglement. However, the spectral criterion is only sufficient but not necessary to detect entanglement. There exist examples of separable and entangled states that share the same global and local spectra [10].

Motivated by the recently introduced sandwiched Rényi relative entropy [11,12], we defined a corresponding version of quantum Tsallis relative entropy for a pair of noncommuting density matrices. We have shown that conditional Tsallis entropy derived from sandwiched relative entropy of a composite quantum state and its subsystem is useful to characterize entanglement beyond the spectral disorder criteria. The present CSTRE reduces to the traditional form of Tsallis conditional entropy (AR q-conditional entropy) developed by Abe and Rajagopal [3] when the subsystem density matrix is a maximally mixed state. We have used the CSTRE to investigate separability of noisy one-parameter families of three- and four-qubit W and GHZ states. The results were compared with those obtained by AR q-conditional entropy and also Peres's PPT criterion. It was shown that the CSTRE is superior to AR q-conditional entropy, while the separability range is limited by that drawn from the PPT criterion. These results were collected in Table I. The CSTRE approach was shown here to be useful to distinguish between isospectral separable and entangled states. While the CSTRE approach was seen to be either identical to or weaker than the PPT criterion in the examples considered here, it is an open question whether this hierarchy is true for all states and whether this approach can identify bound entangled states.

ACKNOWLEDGMENT

A.S.N. acknowledges support from the Department of Science and Technology, Government of India through the INSPIRE fellowship.

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[19] Note that the AR q-conditional entropy of the isospectral states ρ_{AB} and ϱ_{AB} is zero for all values of q and so characterizing separability based on the AR q-conditional entropy is not useful.