Classification scheme of pure multipartite states based on topological phases

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We investigate the connection between the concept of affine balancedness (a-balancedness) introduced by M. Johansson *et al.* [Phys. Rev. A **85**, 032112 (2012)] and polynomial local SU invariants and the appearance of topological phases, respectively. It is found that different types of a-balancedness correspond to different types of local SU invariants analogously to how different types of balancedness, as defined by A. Osterloh and J. Siewert, [New J. Phys. **12**, 075025 (2010)], correspond to different types of local special linear (SL) invariants. These different types of SU invariants distinguish between states exhibiting different topological phases. In the case of three qubits, the different kinds of topological phases are fully distinguished by the three-tangle together with one more invariant. Using this, we present a qualitative classification scheme based on balancedness of a state. While balancedness and local SL invariants of bidegree $(2n,0)$ classify the SL-semistable states [A. Osterloh and J. Siewert, New J. Phys. **12**, 075025 (2010); O. Viehmann *et al.*, Phys. Rev. A **83**, 052330 (2011)], a-balancedness and local SU invariants of bidegree $(2n-m,m)$ give a more fine-grained classification. In this scheme, the a-balanced states form a bridge from the genuine entanglement of balanced states, invariant under the SL group, towards the entanglement of unbalanced states characterized by U invariants of bidegree (n,n). As a byproduct, we obtain generalizations to the W state, i.e., states that are entangled, but contain only globally distributed entanglement of parts of the system.

I. INTRODUCTION

One element of quantum mechanics that appears counterintuitive is certainly entanglement. Though it is present everywhere where there is an interaction, its effects are most easily observed at low temperature and in carefully controlled environments. But even at ambient conditions, the extent to which entanglement plays a role in what we observe in nature is discussed (see, e.g., [1–5]). Local SU invariance is a feature of any entanglement property and thus enters as a minimal requirement for every entanglement measure [6]. Therefore, it has been used for the classification of states [7–10] and the definition of entanglement monotones as measures [11–16]. The group of local operations which leaves entanglement properties invariant was recognized rather early to be the special linear (SL) group [17–21], which is the group underlying the stochastic local operations and classical communication (SLOCC). However, the classification of multipartite entanglement is difficult and, due to the fact that almost all polynomial entanglement measures have a polynomial degree of at least four [61], their mixed state extension via the convex roof is also problematic, although solvable for certain cases [22–27].

The classification based on polynomial SL and unitary invariants is complete in the sense that it fully distinguishes inequivalently entangled states from each other. However, one can also consider other classification schemes that are less distinguishing, i.e., more coarse grained, but which focus on qualitative properties of the states. Such classification schemes can group sets of SLOCC classes or, alternatively, local SU(2) classes into families based on some common property [28–31].

The qualitative property we will consider here is balancedness. The concept of balancedness as defined in [32] is a property related to genuine entanglement, which is taken here to mean entanglement for which there are measures constructed from polynomial SLOCC invariants [32,33]. This relation between balancedness and polynomial invariants is described by the notion of semistability in geometric invariant theory [34,35]. Recently, it has also been shown in [36] that balancedness is a useful concept for describing the appearance of topological phases [36–43]. The analysis of the $SU(2)^{\otimes q}$ topological phases of pure q-qubit states [36] has generalized the balancedness condition [32] relevant for local SL(2) invariance [20,44–50] to an affine balancedness, or a-balancedness, relevant for local SU(2) invariance. In this analysis, a splitting of the three-qubit W SLOCC class was found, since the a-balanced state $|W'\rangle := \frac{1}{2}(|000\rangle + |100\rangle + |010\rangle + |001\rangle)$ is distinguished by a topological phase π from the unbalanced W state, $|W\rangle := \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, which has none. This raises the question of whether this qualitative difference between these two SLOCC equivalent states is captured by a local SU(2) invariant that distinguishes the states [36].

In this paper, we show that there indeed is a local SU(2) invariant that captures this difference between states in the SLOCC W class with different topological phases. Furthermore, we show that for an arbitrary number of qubits, different types of a-balanced states correspond to different local SU(2) invariant polynomials and these capture the qualitative properties related to topological phases. This is analogous to how the original balancedness condition is related to local SL(2) invariant polynomials and topological phases. In this light, we also discuss how multipartite entangled states can be classified based on their balancedness properties.

The paper is organized as follows. In the next section, we briefly discuss balancedness and its relation to topological phases. Section III is devoted to the three-qubit case and, in particular, the W' state, where we write down the SU(2)

invariant of bidegree (3,1) that detects it (i.e., assigns a nonzero value to it). It is then followed by Sec. IV, which is an investigation of the general case of SU(2) invariants of bidegrees $(2n - p, p)$ with $p \leqslant n$. In Sec. V, we discuss how a-balancedness can be used for a classification of multipartite entanglement. In the last section, we draw our conclusions and give an outlook.

II. BALANCED STATES

The first observation of the relation between three-partite entanglement and the property later termed balancedness in [32] was reported by Coffman *et al*. in [33]. This concept has been found to be far more general and has been extended and applied to genuinely multipartite entangled states in [32]. We briefly give the main definition of a *balanced state* here. Suppose we have a q-qubit state with a decomposition using L basis state vectors of a computational basis, called its length. Then, we construct a $q \times L$ matrix A_{kl} such that each column of A_{kl} corresponds to one of these basis state vectors in such a way that the kth element of the lth column is the kth entry of the lth basis state vector, but with every 0 replaced by -1. A state is balanced if and only if positive numbers in **N** exist such that

$$n_l A_{kl} = 0; \quad n_l \in \mathbf{N} \ \forall \ k = 1, \dots, q. \tag{1}$$

A state is called partly balanced if some of the numbers must be chosen to be zero. Furthermore, a balanced state is called *irreducible* if no part of the matrix is balanced by itself [32]. As an example, the q-qubit Greenberger-Horne-Zeilinger (GHZ) state $\alpha|0 \dots 0\rangle + \beta|1 \dots 1\rangle$ would be given by

$$A_{kl} = \begin{pmatrix} -1 & 1 \\ \vdots & \vdots \\ -1 & 1 \end{pmatrix}. \tag{2}$$

A state that has a balanced part in every computational product basis is genuinely entangled. The irreducible balanced states are particular in that they indicate a basis of entangled states detected by measures derived from SL invariants and are not separable over any bipartition. Therefore, they are genuinely multipartite entangled [62]. Furthermore, balancedness in every computational basis implies that the possible total phase factors that can be accumulated in a cyclic evolution generated by local SU(2) or SL(2) evolutions is a discrete set [36]. This discretization of the possible accumulated phases is related to the nontrivial topology of the local SU(2) or SL(2) orbits, and the phases are therefore called topological.

The study of topological phases generated by local SU(2) operations prompted the extension of the concept of balancedness. In this extension, the requirement that the numbers n_l in the definition of balancedness belong to the natural numbers **N** is replaced by the requirement that they are nonzero integers, i.e., they belong to $\mathbf{Z}\backslash 0$. The states balanced in **N** have been termed convexly balanced or c-balanced, while the states balanced in $\mathbf{Z}\backslash 0$ have been termed affinely balanced or a-balanced in [36].

Let us next focus on the a-balanced states. Those of the a-balanced states that are not also c-balanced do not have genuine entanglement, hence all local SL-invariant measures

vanish, but if they are a-balanced in every basis, then they have globally distributed entanglement. Therefore, consider a state which is irreducibly a-balanced in one (local) basis,

$$z_l A_{kl} = 0; \quad z_l \in \mathbf{Z} \setminus \{0\} \ \forall \ k = 1, \dots, q, \tag{3}$$

where q is the number of qubits and $l = 1, \dots, L$, where L is the length of the state. What changes in the definition, when compared to c-balancedness, is that negative numbers are also admitted, such that we are in **Z** and none of the z_l are zero. Therefore, some of the proofs of [32] on c-balanced states are also valid for a-balanced states. We therefore have the following:

Theorem II.1. Product states are never irreducibly a-balanced.

Theorem II.2. Every a-balanced q-qubit state with length larger than $q + 1$ is reducible.

Every c-balanced state is mapped into an a-balanced state if some of its components in the computational basis decomposition are being spin flipped ($\sigma_y^{\otimes q} C$), without producing a product state. As we will show, each such spin flip corresponds to a map from an invariant detecting the c-balanced state to an invariant detecting the a-balanced state. This map takes invariants into invariants of a different bidegree. As an example, the three-tangle of bidegree (4,0) can be mapped to an SU invariant of bidegree (3,1), and then further to a U invariant of bidegree (2,2). We have to consider, at most, half of the components being spin flipped (any multiplicity of a component in the balancedness relation is counted), since a state with $n \geqslant L/2$ could be obtained from the spin-flipped version of the c-balanced state with only $k = L - n \leqslant L/2$ components flipped.

A consequence of irreducible balancedness of a state, which we will make use of, is that the multiset of integers z_l is uniquely defined up to a common factor. Thus, there is a unique multiset $\{z_0, z_1, \dots, z_{L-1}\}$, up to a factor of -1, of integers without a common prime factor associated to an irreducible state. Furthermore, we will sometimes consider sets of states that are balanced in the same way, in the sense that there is a particular matrix A_{kl} such that each state in the set is local unitary equivalent to a form where its balancedness is described by A_{kl}. Such a set will be referred to as an A class of states.

III. THREE QUBITS

Before dealing with an arbitrary number of qubits, we consider the irreducible states, topological phases, and invariants of three qubits since this is a well-understood case [17,51–53]. Here the primary entangled states are the globally entangled W state, $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$, and the genuinely three-party entangled GHZ state, $|\text{GHZ}\rangle = (|111\rangle + |000\rangle)/\sqrt{2}$ [17]. The latter has an equivalent representation $|\text{GHZ}\rangle \overset{\text{SU}(2)}{\cong} |X\rangle := (|111\rangle + |100\rangle + |010\rangle + |001\rangle)/2$, which would be the generalization of the X state in [49]. Moreover, $\widetilde{|\text{GHZ}\rangle} := (\alpha|111\rangle + \beta|000\rangle)/\sqrt{2} \overset{\text{SL}(2)}{\cong} |\tilde{X}\rangle := (a|111\rangle + b|100\rangle + c|010\rangle + d|001\rangle)/2$ for general nonvanishing parameters, following [32]. However, it is easy to check that $\widetilde{|\text{GHZ}\rangle} \overset{\text{SU}(2)}{\ncong} |\tilde{X}\rangle$. The states $\widetilde{|\text{GHZ}\rangle}$ and $|\tilde{X}\rangle$ are distinguished by topological phases since $|\tilde{X}\rangle$ has topological

phases that are multiples of $\frac{\pi}{2}$, while $|\widetilde{\text{GHZ}}\rangle$ only has the phase π unless $|\alpha| = |\beta| = \frac{1}{\sqrt{2}}$.

For the W state, the situation is analogous. Here, $|W\rangle \overset{\text{SL(2)}}{\cong}$ $|W'\rangle := (|000\rangle + |100\rangle + |010\rangle + |001\rangle)/2$ [17], whereas $|W\rangle \overset{\text{SU(2)}}{\not\cong} |W'\rangle$. The latter is an a-balanced state [36] and has a topological phase of π, whereas the W state is completely unbalanced and has none. Now, if there was a nonvanishing SU(2) invariant of bidegree (d_1, d_2), where $d_1 \neq d_2$, it has been noted in [36] that a topological phase $\chi = \frac{2m\pi}{d_1 - d_2}$ for some integer m would appear. It would therefore be promising if an SU(2) invariant of a bidegree that corresponds to a π phase exists that is nonzero for $|W'\rangle$.

It has been described in [54] how to construct local SU(2) invariants. For three qubits, the algebra of local SU invariants has seven primary generators and three secondary generators [55]. The primary generators contain one invariant of bidegree $(4,0)$ as well as its complex conjugate of bidegree $(0,4)$, one of bidegree $(3,1)$ and its complex conjugate of bidegree $(1,3)$, one of bidegree $(1,1)$ (the squared modulus of the state), and two of bidegree $(2,2)$ (these correspond to the two independent reduced density matrices). For our purposes, we identify invariants with their complex conjugates. The $(4,0)$ invariant is the three-tangle $\tau_{(4,0)} = \tau_3$ [33]. The invariant of bidegree $(3,1)$ has been calculated from [54] to be

$$
\tau_{3,1} = \sum_{i_1, i_2 = 0}^{1} \left(|\psi_{0i_1 i_2}|^2 - |\psi_{1i_1 i_2}|^2 \right) (\psi_{000}\psi_{111} + \psi_{100}\psi_{011}
$$
$$
- \psi_{010}\psi_{101} + \psi_{001}\psi_{110}) + 2\big[\psi_{0i_1 i_2}(\psi_{100}\psi_{111}
$$
$$
- \psi_{110}\psi_{101})\psi_{1i_1 i_2}^* - \psi_{1i_1 i_2}(\psi_{000}\psi_{011} - \psi_{010}\psi_{001})\psi_{0i_1 i_2}^* \big],
$$
$$
\tag{4}
$$

where $\psi_{i_1 i_2 i_3}$ are the coefficients of the state vector in the computational basis. It, indeed, detects $|W'\rangle$ and not $|W\rangle$, and is manifestly SU(2) invariant. This explains the fact that the W' state has a topological phase of $\chi = \frac{2m\pi}{d_1 - d_2} = m\pi$ for $m \in \mathbf{N}$, whereas the W state has none. The invariant also detects $|\widetilde{\text{GHZ}}\rangle$ but not $|\tilde{X}\rangle$, which is only detected by the three-tangle, and this explains the different topological phases of these states.

From these observations, we can draw the full picture of polynomial SU invariants of bidegree (d_1, d_2), where $d_1 \neq d_2$, and topological phases for three qubits. As shown by Acín *et al.* [52] and Carteret *et al.* [53], each three-qubit state can be transformed by local unitary operations to a canonical form. In particular, we consider the canonical form where the set of basis vectors is invariant under permutation of the qubits [53],

$$
\kappa_0 e^{i\theta} |000\rangle + \kappa_1 |001\rangle + \kappa_2 |010\rangle + \kappa_3 |100\rangle + \kappa_4 |111\rangle, \tag{5}
$$

where κ_j for $j = 0,1,2,3,4$, and θ are real numbers. For a generic three-qubit state, all κ_j in the canonical form are nonzero. Thus, a generic entangled three-qubit state can be transformed by local unitaries to a balanced but not irreducibly balanced form. The states that can be transformed to an irreducible form are subsets characterized by fewer parameters. There are three different A classes of irreducible states. The first we will call the X class, with reference to [49], which is given by $\kappa_0 = 0$ and the other coefficients are nonzero;

the second is the $\widetilde{\text{GHZ}}$ class given by $\kappa_1 = \kappa_2 = \kappa_3 = 0$ while κ_4 and κ_0 are nonzero; and the W' class is given by $\kappa_4 = 0$. Thus, the set of local SU(2) orbits belonging to the X class and the set belonging to the W' class are both four-parameter subsets, while the SU(2) orbits of the $\widetilde{\text{GHZ}}$ class are a two-parameter set. The $\widetilde{\text{GHZ}}$ class intersects with the X class in the SU(2) orbit of the GHZ state.

The X class is detected by the three-tangle τ_3, but not by $\tau_{3,1}$. The W' class, on the other hand, is detected by $\tau_{3,1}$, but not by τ_3. While the modulus of τ_3 has its unique maximum on the local unitary orbit of the X state, the modulus of $\tau_{3,1}$ has its unique maximum on the local unitary orbit of W'. All genuinely entangled states except the X class are detected by both τ_3 and $\tau_{3,1}$. For three qubits, there is thus a clear relation between the irreducible balanced states, the polynomial SU invariants of bidegree (d_1, d_2), where $d_1 \neq d_2$, and topological phases. An experimental proposal to observe the topological phases in three-qubit systems was recently given in [43].

IV. THE GENERAL CASE: MORE THAN THREE QUBITS

In this section, we show that for every type of irreducible a-balancedness, there is at least one local SU-invariant polynomial that detects it—i.e., it gives a nonzero value to it—while not detecting other types of irreducible a-balancedness. Furthermore, if such an SU(2)$^{\otimes q}$-invariant polynomial of bidegree (d_1, d_2), where $d_1 \neq d_2$, detects a q-qubit state, then this state exhibits topological phases [36]. For three qubits, we have seen that the irreducible a-balanced state $|W'\rangle$ is detected by an SU(2)-invariant polynomial of this kind. Now we show that every irreducible a-balanced state for which the sum of the corresponding integers $\{z_0, z_1, \ldots, z_{L-1}\}$ is nonzero has topological phases, and that there always are SU(2) invariants of this type that detect them.

In Secs. IV A to IV D, we will go through the steps leading to this conclusion. First, in Sec. IV A, we introduce a mapping between irreducible c-balanced states and irreducible a-balanced states that are not c-balanced. We then show in Sec. IV B that this mapping between states induces a mapping between SL invariants that detect irreducible c-balanced states and SU invariants that detect the a-balanced but not c-balanced states. Although this induced mapping does not give the explicit form of the SU invariants, it allows us to deduce some of their properties in Sec. IV D.

A. The partial spin flip

We introduce a mapping between states. The mapping makes use of the universal spin-flip transformation $\sigma_y^{\otimes q} C$, where C is the complex conjugation that acts only on the state vector. This is the unique antiunitary transformation that flips arbitrary spins [56] since it is invariant under local SU and local SL transformations [48]. The spin-flip transformation can be applied to an arbitrary number of qubits and has been used, for example, to construct entanglement measures such as the concurrence [57]. Given a q-qubit state $|\psi\rangle$, we can express it as a sum of two components, $|\psi\rangle \equiv |\phi\rangle + |\theta\rangle$. We then apply a spin-flip transformation only to $|\theta\rangle$, which gives

$$
|\psi\rangle \rightarrow |\tilde{\psi}\rangle \equiv |\phi\rangle + \sigma_y^{\otimes q} C |\theta\rangle \equiv |\phi\rangle + |\tilde{\theta}\rangle. \tag{6}
$$

The spin-flip transformation $\sigma_y^{\otimes q} C$ is almost a conjugation, $\sigma_y^{\otimes q} C \sigma_y^{\otimes q} C = (-1)^q \mathbb{1}$, and therefore $\sigma_y^{\otimes q} C |\tilde{\theta}\rangle \equiv (-1)^q |\theta\rangle$. The kind of mapping in Eq. (6) will be referred to as a "partial spin-flip" map and it is only defined relative to a given decomposition of the state vector into two terms. In particular, for a computational basis $\{|ijk\dots\rangle\}$ where 0 is replaced by -1, we have

$$|\theta\rangle = \sum \theta_{ijk\dots}|ijk\dots\rangle \text{ and } |\tilde{\theta}\rangle = \sum \tilde{\theta}_{ijk\dots}|ijk\dots\rangle, \quad (7)$$

with

$$\tilde{\theta}_{ijk\dots} = (-i)^{(i+j+k+\cdots)}\theta^*_{-i-j-k\cdots}. \quad (8)$$

Assume that $|\psi\rangle$ is an irreducible c-balanced state of length L, where the terms are indexed by $\{0,1,\dots,L-1\}$. Choose a decomposition of $|\psi\rangle$ such that $|\theta\rangle$ corresponds to a subset s of the terms of $|\psi\rangle$ and apply the corresponding partial spin-flip map to produce a state $|\tilde{\psi}\rangle$. Then, $|\psi\rangle$ and $|\tilde{\psi}\rangle$ can be written as

$$
\begin{aligned}
|\psi\rangle &= \sum_{j=0}^{L-1} \psi_j \otimes_{k=1}^q |A_{kj}\rangle, \\
|\tilde{\psi}\rangle &= \sum_{j \notin s} \psi_j \otimes_{k=1}^q |A_{kj}\rangle \\
&\quad + \sum_{j \in s} (-i)^{\sum_k A_{kj}} \psi_j^* \otimes_{k=1}^q |-A_{kj}\rangle \\
&\equiv \sum_j \tilde{\psi}_j \otimes_{k=1}^q |\tilde{A}_{kj}\rangle,
\end{aligned}
\quad (9)
$$

where ψ_j and $\tilde{\psi}_j$ are the expansion coefficients of $|\psi\rangle$ and $|\tilde{\psi}\rangle$, respectively.

We remark that the state $|\tilde{\psi}\rangle$ is not a c-balanced state. Depending on the choice of s, it can be either an irreducible a-balanced state, a state where one or more qubits are in a tensor product with the remaining qubits, or, in some cases, an entangled state without topological phases. This will be further elaborated in Sec. IV E. Furthermore, every a-balanced state that is not already a c-balanced state can be mapped into a c-balanced state by some partial spin flip.

B. Induced mapping between invariants

Theorem IV.1. Consider a state $|\psi\rangle$ for which there is a nonvanishing polynomial SL-invariant P. For every state $|\tilde{\psi}\rangle$ obtained from $|\psi\rangle$ through a partial spin flip, there exists an invariant function $\tilde{P}$ on the SU(2) orbit of $|\tilde{\psi}\rangle$.

Proof. To see this, consider a c-balanced state $|\psi\rangle$ in an arbitrary local product basis,

$$|\psi\rangle = \sum \psi_{ijk\dots}|ijk\dots\rangle, \quad (10)$$

where $ijk\dots$ is a string of 1s and -1s. Assume that there is a SL(2)$^{\otimes q}$-invariant polynomial P that detects $|\psi\rangle$. Formally, P can be expressed as

$$P = \sum_\alpha b_\alpha \prod \psi_{ijk\dots}^{r(\alpha)_{ijk\dots}} \quad (11)$$

for some sets of exponents $\{r(\alpha)_{ijk\dots}\}$ and constants b_α.

Consider then a decomposition of $|\psi\rangle$ as $|\psi\rangle = |\phi\rangle + |\theta\rangle$. With use of the notation $|\phi\rangle = \sum \phi_{ijk\dots}|ijk\dots\rangle$, $|\theta\rangle = \sum \theta_{ijk\dots}|ijk\dots\rangle$, the polynomial P can be expressed, using (8), in the variables $\phi_{ijk\dots}$ and $\theta_{ijk\dots}$ as

$$P = \sum_\alpha b_\alpha \prod (\phi_{ijk\dots} + \theta_{ijk\dots})^{r(\alpha)_{ijk\dots}}. \quad (12)$$

Consider then the partially spin-flipped state $|\tilde{\psi}\rangle = |\phi\rangle + |\tilde{\theta}\rangle$ associated to the above decomposition, as given by Eq. (6). Using the notation $|\tilde{\theta}\rangle = \sum \tilde{\theta}_{ijk\dots}|ijk\dots\rangle$ where $\theta_{ijk\dots} = i^{(i+j+k+\dots)}\tilde{\theta}^*_{-i-j-k\dots}$, we can express P in $\phi_{ijk\dots}$ and $\tilde{\theta}_{-i-j-k\dots}$ as

$$P = \sum_\alpha b_\alpha \prod (\phi_{ijk\dots} + i^{(i+j+k+\dots)}\tilde{\theta}^*_{-i-j-k\dots})^{r(\alpha)_{ijk\dots}}. \quad (13)$$

This second expression for P in Eq. (13) defines a function $\tilde{P}$ on the SU(2)$^{\otimes q}$ orbit of $|\tilde{\psi}\rangle$ such that $P(|\psi\rangle) = \tilde{P}(|\tilde{\psi}\rangle)$. Furthermore, any SU(2)$^{\otimes q}$ operation commutes with the spin-flip operation $\sigma_y^{\otimes q} C$. Therefore, $P(U|\psi\rangle) = \tilde{P}(U|\tilde{\psi}\rangle)$ for $U \in$ SU(2)$^{\otimes q}$. Since P is, in particular, invariant on the SU(2)$^{\otimes q}$ orbit of $|\psi\rangle$, it follows that $\tilde{P}$ is invariant on the SU(2)$^{\otimes q}$ orbit of $|\tilde{\psi}\rangle$. ∎

Note that since only values of homogeneous invariants are concerned, $\tilde{P}$ itself does not need to be a homogeneous function. Note also that the above argument can be made in the other direction as well. That is, the existence of a polynomial SU invariant $\tilde{P}$ that detects states on the SU(2)$^{\otimes q}$ orbit of $|\tilde{\psi}\rangle$ implies the existence of an SU-invariant function P that evaluates to a nonzero value on the SU(2)$^{\otimes q}$ orbit of $|\psi\rangle$ related to $\tilde{P}$ by the induced mapping.

C. Irreducible c-balanced states and SL-invariant polynomials

The irreducibility of a state places a constraint on the homogeneous degrees of any polynomial SL invariant that detects the state. Consider an irreducible q-qubit c-balanced state $|\psi\rangle$ of length L,

$$|\psi\rangle = \sum_{j=0}^{L-1} \psi_j \otimes_{k=1}^q |A_{kj}\rangle, \quad (14)$$

where $A_{kj} = 1, -1$. We assume that the terms are indexed such that the multiset of integers $\{z_0, z_1, \dots, z_{L-1}\}$ associated with the state, where each z_j is associated to the term with coefficient ψ_j and satisfies $|z_0| \geqslant |z_1| \geqslant |z_2| \geqslant \cdots \geqslant |z_{L-1}|$. Furthermore, we assume that the z_j have no common divisor. Since the irreducible c-balanced states are genuinely multipartite entangled, there is a homogeneous SL(2)$^{\otimes q}$-invariant polynomial P that evaluates to a nonzero value for this state.

In the particular basis that we have chosen for the state, the monomials of P that detect $|\psi\rangle$ are of the form $\sum_\alpha \prod_j \psi_j^{r_{\alpha j}}$, up to constant factors, for some sets of exponents $\{r_{\alpha j}\}$. We call the polynomial in the ψ_j's made up of these monomial terms P_A. P and P_A evaluate to the same value for every state related to $|\psi\rangle$ by local filtering operations.

Consider a particular local filtering operation F on the kth qubit,

$$F = \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}. \quad (15)$$

This operation multiplies the αth monomial of P_A by $t^{\sum_j A_{kj} r_{\alpha j}}$. Since the polynomial P is SL(2)$^{\otimes q}$ invariant, the sum of all the monomials of P_A after the filtering must equal the sum before the filtering. Moreover, this must be true for any t. This is possible only if $t^{\sum_j A_{kj} r_{\alpha j}} = 1$ for each α. Since this must be true for local filterings on any qubit, this gives us a system of q linear equations. Using the convention that the term of $|\psi\rangle$ with coefficient ψ_0 is $|11\ldots1\rangle$, this system of equations can be formulated as a matrix equation:

$$\begin{pmatrix} A_{21} & A_{31} & \cdots & A_{(L-1)1} \\ A_{22} & A_{32} & \cdots & A_{(L-1)2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{2q} & A_{3q} & \cdots & A_{(L-1)q} \end{pmatrix} \begin{pmatrix} r_{\alpha1} \\ r_{\alpha2} \\ \vdots \\ r_{\alpha L-1} \end{pmatrix} = -r_{\alpha0} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$
(16)

for each α. Since the state is irreducible, and thus the columns of the matrix on the left-hand side are linearly independent, the matrix on the left-hand side uniquely determines the solution up to a common multiplicative factor h. Equation (16) is the same equation that appeared in [36] to determine the set $\{z_0, z_1, \ldots, z_{L-1}\}$. The solutions are given by $\{z_0, z_1, \ldots, z_{L-1}\}$ such that $r_{\alpha j} = h_\alpha z_j$ for some h_α. Since the polynomial is homogeneous, we have only one $h_\alpha \equiv h$. Thus, the polynomial P_A is a single monomial P_A given, up to a constant, by

$$P_A = \left(\prod_{j=0}^{L-1} \psi_j^{z_j} \right)^h.$$
(17)

We will briefly comment on the nature of h. As the $\{z_0, \ldots, z_L\}$ are assumed to have no common divisor, any solution to Eq. (16) that occurs is $h \in \mathbb{N}$, where an $h \geqslant 2$ means that the length $l = \sum_j z_j$ of the state fits h times in the polynomial degree of the corresponding invariant [32,50]. We can, therefore, conclude that the homogeneous degree of any nonzero SL-invariant polynomial is $h \sum_{j=0}^{L-1} z_j$ for some integer h and that the polynomial contains the monomial $(\prod_{j=0}^{L-1} \psi_j^{z_j})^h$.

D. Irreducible a-balanced states and SU invariants

We have established that a partial spin-flip operation mapping a c-balanced state $|\psi\rangle$ to an a-balanced state $|\tilde{\psi}\rangle$ induces a mapping between a homogeneous polynomial SL invariant that detects the c-balanced state and an invariant function that detects the a-balanced state. Furthermore, we have reviewed the multiplicative scaling behavior of SL invariants on irreducible c-balanced states.

Now we address the question of the multiplicative scaling behavior of the invariant $\tilde{P}$ for the case where the map $P \rightarrow \tilde{P}$ is induced by a partial spin flip of some terms of an irreducible state. In other words, we assume that $|\tilde{\psi}\rangle$ is irreducible and investigate how $\tilde{P}$ scales when $|\tilde{\psi}\rangle$ is multiplied by a factor $\lambda \in \mathbb{C}\backslash\{0\}$. Let us assume that we have constructed the irreducible a-balanced state $|\tilde{\psi}\rangle$ from an irreducible c-balanced state by applying the partial spin-flip operation to a subset s of the L terms that it has [see Eq. (9)].

Theorem IV.1. If $|\psi\rangle$ is an irreducible c-balanced state detected by a polynomial SL invariant P of homogeneous

bidegree $(h \sum z_j, 0)$, and $|\tilde{\psi}\rangle$ is the irreducible a-balanced state constructed from $|\psi\rangle$ by spin flipping a subset s of the terms, then the SU invariant $\tilde{P}$, constructed from P by the induced mapping, has homogeneous bidegree $(h \sum_{j \notin s} z_j, h \sum_{j \in s} z_j)$.

Proof. Consider an irreducible a-balanced state $|\tilde{\psi}\rangle = |\phi\rangle + |\tilde{\theta}\rangle$ and multiply it with a constant λ: $|\tilde{\psi}\rangle_\lambda = \lambda|\phi\rangle + \lambda|\tilde{\theta}\rangle$. A spin flip on the part s (the $\tilde{\theta}$ part) of the rescaled state gives $|\tilde{\psi}\rangle_\lambda \rightarrow |\psi\rangle_\lambda = \lambda|\phi\rangle + \lambda^*|\theta\rangle$. We immediately extract from (17) that the SU invariant $\tilde{P}$ satisfies $\tilde{P}(\lambda|\tilde{\psi}\rangle) = \lambda^{h \sum_{j \notin s} z_j} \lambda^{*h \sum_{j \in s} z_j} P(|\psi\rangle) = \lambda^{h \sum_{j \notin s} z_j} \lambda^{*h \sum_{j \in s} z_j} \tilde{P}(|\tilde{\psi}\rangle)$. We therefore have that $\tilde{P}$ is an *SU* invariant of bidegree $(h \sum_{j \notin s} z_j, h \sum_{j \in s} z_j)$. ∎

Corollary IV.1. Let $|\psi\rangle$ be an irreducible a-balanced state with associated integers $\{z_0, z_1, \ldots, z_{L-1}\}$ that satisfy $\sum_{j=0}^{L-1} z_j \neq 0$. Then, $|\psi\rangle$ is detected by an SU(2) invariant of bidegree (d_1, d_2), where $d_1 \neq d_2$, and this implies that $|\psi\rangle$ is a-balanced in every basis.

Consider again the irreducible c-balanced state $|\psi\rangle$ and the irreducible a-balanced state $|\tilde{\psi}\rangle$ produced by a spin-flip operation on a subset s of the terms as given by Eq. (9). From Eq. (17), we can easily find that the form of the part of $\tilde{P}$ that evaluates to a nonzero value, $\tilde{P}_A$, is

$$\tilde{P}_A = i^{\left(\sum_{j \in s} \sum_{k=1}^{q} A_{kj}\right)} \prod_{j \notin s} \tilde{\psi}_j^{hz_j} \prod_{j \in s} \tilde{\psi}_j^{*hz_j}.$$
(18)

Every local SU invariant that detects states in the A class of $|\tilde{\psi}\rangle$ contains a monomial of this form.

In [32], it was shown that every irreducible c-balanced state is detected by a local SL(2)-invariant polynomial. In a similar way, every irreducible a-balanced state is detected by a local SU(2)-invariant polynomial. Furthermore, in [36], it was pointed out that if a state is detected by polynomial local SU(2) invariants of bidegree (d_1, d_2) such that $d_1 \neq d_2$, then it exhibits topological phases. As found here, every state that is irreducibly a-balanced and satisfies $\sum_j z_j \neq 0$ is of this kind.

The states which are irreducibly a-balanced but not c-balanced belong to SLOCC-zero classes, that is, they are not detected by any SL invariant. In other words, they constitute the SL-null cone. Therefore, the above observation implies that SLOCC-zero classes can be split into states that exhibit topological phases and those that do not.

E. Derived irreducible states, invariants, and topological phases

We now elaborate on how the irreducible a-balanced states can be constructed from a given irreducible c-balanced state. Methods to construct irreducible c-balanced states were discussed in [32,36]. Consider that we have an irreducible c-balanced state with an associated multiset of integers $\{z_0, z_1, \ldots, z_{L-1}\}$ that is detected by an invariant polynomial P of bidegree $(h \sum_{j=0}^{L-1} z_j, 0)$. By spin flipping different submultisets s such that the associated submultisets of the integers $\{z_0, z_1, \ldots, z_{L-1}\}$ satisfy $\sum_{j \in s} z_j \neq \sum_{j \notin s} z_j$, we can produce different irreducible a-balanced states that feature topological phases. Two different partial spin flips corresponding to different submultisets s_1 and s_2 may map the state to the same A class, but if $\sum_{j \in s_1} z_j \neq \sum_{j \in s_2} z_j$ (or if one identifies with the complex conjugate classes: $\sum_{j \in s_1} z_j \neq \sum_{j \notin s_2} z_j$), then the A classes are certainly distinct.

If we collect all states derived from $|\psi\rangle$, such that the set of spin-flipped terms s satisfies $\sum_{j \in s} z_j < \sum_{j \notin s} z_j$, these are representatives of each A class that can be constructed in this way from $|\psi\rangle$. To each A class, there is a corresponding invariant of bidegree $(h \sum_{j \notin s} z_j, h \sum_{j \in s} z_j)$. Moreover, since the topological phases for $|\psi\rangle$ are multiples of $\frac{2\pi}{\sum_j z_j}$, we deduce that the topological phases for a state $|\tilde{\psi}\rangle$ constructed by spin flipping a set s are multiples of χ where

$$\chi = \frac{2\pi}{\left(\left| \sum_{j \notin s} z_j \right| - \left| \sum_{j \in s} z_j \right| \right)}. \tag{19}$$

Let us consider the case when the submultiset s satisfies $\sum_{j \in s} z_j = \sum_{j \notin s} z_j$, that is, when it defines an equal partition of $\{z_0, z_1, \ldots, z_{L-1}\}$. In each row of the matrix A_{jk}, the 1s and -1s also define an equal partition of $\{z_0, z_1, \ldots, z_{L-1}\}$. If s defines an equal partition that also corresponds to a row of A_{jk}, the partial spin flip produces a row of only 1s or only -1s. In this case, the qubit corresponding to this row is in a tensor product with the remaining qubits. However, if there are more equal partitions of the $\{z_0, z_1, \ldots, z_{L-1}\}$ than A_{jk} has rows, it is possible to select an s such that a nonproduct state results. In this case, one has produced an entangled state that can only be detected by polynomials that are invariant under the full $U(2)^{\otimes q}$ group and that does not have topological phases. These states are thus natural extensions of the W state.

Returning to the case of three qubits, we can see that a partial spin flip of a single term of an irreducible state in the X class will produces a state in the W' class. A spin flip of two terms produces a state where one qubit is in a tensor product with a possibly entangled two-qubit state.

As a further example, consider the five-qubit irreducible c-balanced state from [32],

$$|11111\rangle + |11000\rangle + |10110\rangle + |01000\rangle$$
$$+ |00101\rangle + |00011\rangle, \tag{20}$$

which, by a spin flip on the basis state $s = \{1\}$, is transformed into the irreducible a-balanced state

$$|00000\rangle + |11000\rangle + |10110\rangle + |01000\rangle$$
$$+ |00101\rangle + |00011\rangle, \tag{21}$$

with nonzero invariant of bidegree $(5, 1)$, then by spin flip on the basis state $s = \{2\}$ into an irreducible a-balanced state,

$$|00000\rangle + |00111\rangle + |10110\rangle + |01000\rangle$$
$$+ |00101\rangle + |00011\rangle, \tag{22}$$

with nonzero invariant of bidegree $(4, 2)$, and finally by a spin flip on the basis state $s = \{5\}$ into a state with a nonzero invariant of bidegree $(3, 3)$,

$$|00000\rangle + |00111\rangle + |10110\rangle + |01000\rangle$$
$$+ |11010\rangle + |00011\rangle. \tag{23}$$

This final state, which comes out of the initial irreducible c-balanced state (20) by a spin flip on the part $s = \{1, 2, 5\}$, is a state that is only detected by $U(2)^{\otimes 5}$ invariants, similarly to the $|W\rangle$ states, which, however, are completely unbalanced.

We can, in fact, go even further than spin flipping submultisets s, and instead split a basis product state vector into two

parts, followed by a spin flip on only one of the parts. However, the resulting state is then a GHZ state plus an unbalanced state. In order to give an example with the splitting of a basis product state vector, we take the X state from [49],

$$\sqrt{2}|1111\rangle + W \leftrightarrow A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{24}$$

This state is mapped by a partial spin flip on $s = \{1, 3, 4\}$ into

$$|1111\rangle + |0000\rangle, \tag{25}$$

$$|0111\rangle + |1011\rangle + |0010\rangle + |0001\rangle. \tag{26}$$

The splitting into a GHZ (c-balanced) state and the W-like (unbalanced) state,

$$|W-\text{like}\rangle = |0111\rangle + |1011\rangle + |0010\rangle + |0001\rangle, \tag{27}$$

is easily seen.

The number of different types of irreducible c-balancedness for q qubits increases rapidly with q. The problem of finding these different types can be rephrased as a combinatorial problem which can be solved algorithmically. However, the algorithmic search constructed in Ref. [36] already becomes computationally expensive for $q > 7$. Constructing the invariants corresponding to the irreducible a-balanced or c-balanced states is also a demanding task and, in general, the invariants of qubit systems have only been studied for up to five qubits [47,50].

V. INDUCED CLASSIFICATION

Any qualitative entanglement property can be used to classify entangled states. Local SU(2) interconvertibility already groups the states into classes, since belonging to the same orbit of some group is an equivalence relation. A complete generating set of invariants distinguishes all the orbits of the underlying group. In this way, a finite number of invariants produces an infinite number of classes. However, for some purposes, it may be useful to classify states based on some property of interest that yields a finite number of classes. Here, we reflect upon how balancedness can be used to construct such a classification.

One approach is to classify states based on the balancedness of their minimal form. However, states may have several minimal forms with different balancedness, thus making the assignment nonunique. Moreover, the minimal forms do not always give us the full picture. One such example is the intersection of the GHZ class and the X class in Sec. III. While the states in the local SU orbit of the GHZ state have the GHZ state itself as a minimal form, they can also be put on the form of the X state.

Any SL- or SU-invariant polynomial that is not invariant under U(1) transformations detects only balanced parts of states. However, the choice of homogeneous generators for the polynomial algebra of invariants is typically not unique. Given a generator, products of other generators can often be added without changing the bidegree of the generator. As we saw in Sec. IV, for every irreducible state, there is at least

one invariant polynomial containing a monomial term that precisely captures the balancedness of the state. This allows us to choose generators of the polynomial algebra such that a generator detects only a given type of irreducibly balanced states while not detecting any other types. This connection between the different kinds of c- or a-balancedness and invariant polynomials is a direct result of SL or SU invariance, respectively. We therefore use the invariants chosen in this way as a starting point for a classification.

Because our interest is in a qualitative classification, we consider only which invariants out of a generating set take nonzero values, rather than what their precise values are. Assume that a complete generating set for the invariants is known. Then we put two states in the same class, if they are detected by the same set of invariants. This is an equivalence relation (self-similarity and transitivity) and groups the states in finitely many classes for finitely many qubits. Given that we have chosen the generators, as outlined above, this also captures the balancedness of the states.

Here, it is worth mentioning that some carefully chosen elements of the zero class of the underlying symmetry group can be added without changing the class. That is, if a state vector from the zero class can be added to a given vector without creating new balancedness in the resulting state, this careful adding does not change the class. This can be elements out of the (n,n) class plus the unbalanced class for classifications with respect to SU(2), or the $(2n - p, p)$ class, $p \notin \{0, 2n\}$, classifying with respect to the group SL(2).

Beyond the connection to topological phases, it is unclear to us what physical sense a classification of the kind given here may carry, but it is a very natural one that relies on the SU invariance (or SL invariance) of entanglement properties, and we consider further analysis worthwhile.

A. Examples of how the classification works

As an example, we can first consider the connection between different types of c-balancedness and polynomial local SL invariants in the case of four qubits, which was previously investigated in [31,32]. The generating set of polynomial local SL invariants contains four generators of polynomial degree 2, 4, 4, and 6, respectively [44]. In Ref. [49], irreducible c-balanced states that represent up to seven different types (invariant under qubit permutations) of genuine four-qubit entanglement were identified [58]. One is the four-qubit GHZ state,

$$|0000\rangle + |1111\rangle, \quad (28)$$

which is a representative of the only type of genuinely multipartite entanglement that is detected by the generator of degree 2. The second is the cluster state

$$|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle, \quad (29)$$

which is local unitary equivalent to $|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle$, and the states related to the state in Eq. (29) by permutation of the qubits. These are only detected by the two degree-4 generators after functional dependencies of the generators have been removed. The last state is the four-qubit X state,

$$|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle, \quad (30)$$

which is only detected by the degree-6 generator. As found in [36], these states are also distinguished by their respective topological phases. The four-qubit GHZ state has only the topological phase π, while the four-qubit cluster state has a topological phase $\frac{\pi}{2}$ and the four-qubit X state has a phase $\frac{\pi}{3}$.

In [31], a classification scheme was suggested for four-qubit entangled states where SLOCC-equivalence classes were grouped into families in terms of "tangle patterns." The tangle patterns are defined in terms of a generating set of polynomial invariants, and the highest degree of a generator that detects a state determines which family it belongs to. In this way, the states are sorted into a hierarchy of four families. The four families are precisely the c-unbalanced states, and three families corresponding to the four generators of degree 2, 4, and 6, respectively. The entanglement types of the three genuine entangled families have been named after representative states of the respective family that is detected only by its particular highest-degree generator. These are the states identified in [49] and therefore the types are called X type, cluster type, and GHZ type [31]. In addition to this, there is the W-type family which is not detected by SL-invariant polynomials.

We note that the families are distinguished by topological phases under cyclic local SL evolution, just as are their representative states. Only the X-type family contains states with topological phase $\frac{\pi}{3}$ under cyclic local SL evolution. Only the cluster-type family contains states with topological phase $\frac{\pi}{2}$. All states in the GHZ-type family have the phase π and the states of the W type have no topological phases under cyclic local SL evolution.

In contrast to this, our classification scheme gives a slightly different answer and is more fine grained. From the chosen invariants $(\mathcal{A}, \mathcal{B}^I, \mathcal{B}^{II}, \mathcal{C})$ [31] of polynomial degrees $(2,4,4,6)$, we can construct the invariants $(\mathcal{A}, \mathcal{B}^I - \mathcal{A}^2, \mathcal{B}^{II} - \mathcal{A}^2, \mathcal{C} + A^3)$. These satisfy the property that each invariant detect only one kind of irreducible balancedness.

Using these invariants, we give a 4-tuple (a, b_1, b_2, c) in $\mathbb{Z}_2^4$. The class $(1,0,0,0)$ corresponds to the GHZ-type entanglement, while $(0,1,0,0)$ and $(0,0,1,0)$ correspond to the two different kinds of irreducibly balanced states of length 4, i.e., the two different kinds of cluster-type states related by permutations of the second and third qubit. The irreducibly balanced states of length 6, i.e., states with X-type entanglement, belong to the class $(0,0,0,1)$. The two different kinds of biseparable states of the type $|\phi_1\rangle \otimes |\phi_2\rangle$, where each $|\phi_i\rangle$ is an entangled state of two qubits, are found in the two classes $(1,1,0,1)$ and $(1,0,1,1)$. These two classes also contain the reducibly balanced states of length 4 that are not biseparable. Almost every state except a set of zero measure belongs to the class $(1,1,1,1)$ containing all the different kinds of c-balancedness. $(0,0,0,0)$ is the class of states that are not c-balanced. They will be treated in what follows. This classification gives us, at most, 15 classes of genuine four-party entanglement.

When looking at a classification of states based on their a-balancedness and their topological phases under local SU(2) evolution, we make the more fine-grained division of local SU(2) orbits into families based on which local SU(2) invariants of bidegree (d_1, d_2), where $d_1 \neq d_2$, detect them. We

note that such a classification scheme captures the structure on the set of entangled three-qubit states that was described in Sec. III. Here, the GHZ SLOCC class is subdivided into the X family detected only by the three-tangle and the rest of the GHZ SLOCC class detected by both the three-tangle and $\tau_{3,1}$. The W family is divided into the W' family detected by $\tau_{3,1}$ with the W' state as a representative and the unbalanced states with the W state as a representative. In particular, we see that such a scheme divides the W SLOCC class of states, i.e., the states that are not c-balanced, into subfamilies based on a-balancedness through the associated local SU(2)-invariant polynomials.

As a further example of this, we can consider the four-qubit states that are not c-balanced. For four qubits, the irreducible a-balanced states derived from the X state are detected by invariants of bidegree (5,1) or of bidegree (4,2) and display topological phases $\frac{\pi}{2}$ or π, respectively. The irreducible a-balanced states that can be derived from the cluster state in Eq. (29) are detected by polynomials of bidegree (3,1) and display the topological phase π. By considering the different combinations of these invariants that can detect a state, and treating states related by qubit permutations as equivalent, we get, at most, seven subfamilies. In addition to this, we have the family of a-unbalanced states that are only detected by invariants of the full local unitary group.

As an example of the classification scheme involving c-balanced states, we can consider the local SU orbits of the four-qubit X-type family from [31]. These are all detected by the generator of bidegree (6,0), but can be further subdivided into subfamilies based on which of the invariants of bidegree (4,0), (3,1), (5,1), (4,2), and (2,0) detects the states. This gives, at most, a total of 32 subfamilies. Notably, only the subfamily detected by none of the polynomials other than that of bidegree (6,0) contains states with topological phase $\frac{\pi}{3}$, while the other subfamilies only contain states with phase π.

This classification scheme gives us a hierarchy of entanglement families based on the concept of balancedness. Furthermore, it is closely connected to the qualitative feature of topological phases displayed by states in the respective family. Further investigation of this concept would be highly desirable.

VI. CONCLUSIONS

Some examples of irreducible a-balanced states with topological phases are known from [36], but in this work we have demonstrated that topological phases are a feature of all irreducibly a-balanced states for which the associated set of integers has a nonzero sum. We have also shown that every such state is detected by a local SU-invariant polynomial of bidegree (d_1, d_2), where $d_1 \neq d_2$. It distinguishes these states from the unbalanced states that do not have any topological phase and which are only detected by invariants of bidegree (d_1, d_1).

We have shown this by introducing a partial spin flip, which maps between states that are c-balanced and those which are only a-balanced. The partial spin-flip map also induces a map between invariants, such that from an initial SL(2) invariant the existence of SU(2) invariants that detect irreducible a-balanced states follows. For three qubits, an invariant $\tau_{3,1}$

of bidegree (3,1) detects the irreducible a-balanced state $|\tilde{W}\rangle$ with topological phase π, but not the unbalanced W state that has no topological phases. The invariant $\tau_{3,1}$ also distinguishes between the genuinely three-partite entangled states with topological phase π and those with topological phases $\frac{m\pi}{2}$ for integer m. The remaining states are detected only by invariants of the group of local U(2), and thus are in the zero class of both SL(2) and SU(2) invariants. The set of these states contains bipartite product states besides globally entangled states, which are W-like states. Which class a state belongs to can be clearly foreseen from the original (irreducible) c-balanced form of an SL(2)-invariant state.

Furthermore, we have discussed how balancedness and, in particular, a-balancedness can be used for the classification of entangled states. The connection to polynomial invariants as well as topological phases suggests that c-balancedness as well as a-balancedness are useful concepts for the description of multipartite entangled states.

APPENDIX

Let us now illustrate what a-balancedness means by showing that irreducible a-balanced states for which $\sum_j z_j \neq 0$ always have an a-balanced part after local SU transformations on a single qubit. To see this, let an SU transformation U act on a single qubit of an a-balanced state $|\psi\rangle$ whose balancedness is represented by a matrix A. We can express the resulting state as a sum of two parts $U_1|\psi\rangle$ and $U_2|\psi\rangle$ corresponding to the action of the diagonal part $U_1 = \frac{1}{2}[\mathbb{1}\mathrm{tr}\, U + \mathrm{tr}\,(U\sigma_z)\sigma_z]$ and the off-diagonal part $U_2 = \frac{1}{2}[\mathrm{tr}\,(U\sigma_x)\sigma_x + \mathrm{tr}\,(U\sigma_y)\sigma_y]$ of U, respectively. Both parts $U_1|\psi\rangle$ and $U_2|\psi\rangle$ are a-balanced with the same set of integer numbers $z_1, \ldots, z_L$ and their respective matrices are $A_1 = A$, and A_2 related to A by multiplication of a row by -1. If no pair of columns in A_1 is also found in A_2, the matrix of the full state contains the columns of both A_1 and A_2 and is then clearly balanced. However, if A_1 and A_2 has one or more pairs of columns in common, it is possible to choose the SU transformation such that one or possibly several of the terms corresponding to these columns cancel out. In this case, these columns do not appear in the matrix of the full state and therefore a-balancedness may be lost.

However, if the original state is irreducibly a-balanced, i.e., if no proper subset of columns in A are linearly dependent, and if $\sum_j z_j \neq 0$, it follows that at most one pair of columns is common to A_1 and A_2. Even if a term corresponding to such a column is canceled out, the full state is still balanced. To see this, let z_l and $z_{l'}$ be the integers that correspond to the columns belonging to the canceled term. By choosing $m_1 z_l + m_2 z_{l'} = 0$, with m_1 and m_2 relatively prime, the integer corresponding to the canceled term is zero. The corresponding integers for the remaining

state are $m_1(z_1, \ldots, z_{l-1}, z_{l+1}, \ldots, z_L)_{A_1}$, $m_2(z_1, \ldots, z_{l'-1}, z_{l'+1}, \ldots, z_L)_{A_2}$ and thus the state is balanced also in the absence of this column, corresponding to the irreducible balancedness of A_1 and A_2, respectively.

As an example, consider the irreducible a-balanced three-qubit state

$$|W'\rangle = |000\rangle + |100\rangle + |010\rangle + |001\rangle, \qquad (A1)$$

with integer numbers $(-1,1,1,1)$ describing its a-balancedness, and apply the Hadamard transformation on the first qubit. Here, the term $|100\rangle$ is canceled out by the transformation and the resulting state is

$$H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad (A2)$$

$$H_1|W'\rangle = |000\rangle + |010\rangle + |110\rangle + |001\rangle + |101\rangle. \quad (A3)$$

Therefore, we find the integers corresponding to the matrices A_1 and A_2 by choosing $m_1 = m_2 = 1$,

$$A_1 = \begin{matrix} -1 & 1 & 1 & 1 \\ \begin{pmatrix} -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \end{matrix};$$

$$A_2 = \begin{matrix} -1 & 1 & 1 & 1 \\ \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \end{matrix}. \qquad (A4)$$

The matrix and integers of the full state are

$$A_{H_1|\psi\rangle} = \begin{matrix} 0 & 1 & 1 & 1 & 1 \\ \begin{pmatrix} -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & 1 \end{pmatrix} \end{matrix}. \qquad (A5)$$

As in the example, one or several of the integer numbers attributed to product basis states can become zero, but the remaining components are (a- or c-)balanced. In particular, the resulting state cannot be a product state.

A second example is the four-qubit state

$$|W'\rangle = |0000\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle, \quad (A6)$$

with corresponding integers $(-2,1,1,1,1)$. Using the same transformation on the first qubit, we get the matrices

$$A_1 = \begin{matrix} -2 & 1 & 1 & 1 & 1 \\ \begin{pmatrix} -1 & 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \end{matrix};$$

$$A_2 = \begin{matrix} -2 & 1 & 1 & 1 & 1 \\ \begin{pmatrix} 1 & -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix} \end{matrix}. \qquad (A7)$$

Here, $|1000\rangle$ is canceled ($m_1 = 2$, $m_2 = 1$) and we have

$$H_1|W'\rangle = |0000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$$
$$+ |1100\rangle + |1010\rangle + |1001\rangle, \qquad (A8)$$

with corresponding integers $(-3,2,2,2,1,1,1)$.

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[61] For measures quadratic in ψ, see Refs. [57,59].

[62] We mention here that the term "genuinely multipartite entangled" is widely used in the literature to indicate that a state is not bipartite; see, e.g., [17,60]. We use the more strict terminology from [48] and call such states "globally entangled."