Certifying nonlocality from separable marginals

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Imagine three parties, Alice, Bob, and Charlie, who share a state of three qubits such that all two-party reduced states A-B, A-C, and B-C are separable. Suppose that they have information only about these marginals but not about the global state. According to recent results, there exists an example for a set of three separable two-party reduced states that is only compatible with an entangled global state. In this paper, we show a stronger result by exhibiting separable two-party reduced states A-B, A-C, and B-C, such that any global state compatible with these marginals is nonlocal. Hence, we obtain that nonlocality of multipartite states can be certified from information only about separable marginals.

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I. INTRODUCTION

Entanglement [1] and nonlocality [2] are two defining aspects of quantum mechanics providing powerful resources for numerous applications in quantum information science. Although long ago they were thought to be two facets of the same phenomenon, these two notions of inseparability have turned out to be quite different [3]. Crucially, due to Bell's theorem [4], distant parties sharing an entangled quantum state can generate nonlocal correlations, witnessed by violation of a Bell inequality, which rules out any local realistic model. However, it is difficult to fully identify the set of entangled states which are nonlocal, i.e., give rise to Bell violation.

In the simplest bipartite scenario, for instance, there exists a family of quantum states, the so-called Werner states [5], which are entangled but nevertheless are local (i.e., admit a local realistic model for any single-shot measurement). Similarly, in the tripartite case, there exists a family of entangled three-qubit states having a local realistic model for any single-shot von Neumann measurement [6]. Interestingly, some of these three-qubit states are genuinely multipartite entangled, representing a very strong form of multipartite entanglement. Conversely, it has been recently shown that a three-qubit bound entangled state (where entanglement in the system presents in a very weak, almost invisible form) exhibits nonlocality; that is, it violates a tripartite Bell inequality [7].

This selection of works already suggests that the relation between entanglement and nonlocality is very subtle. In our present work, we wish to give a further example linking the two concepts to each other in an intriguing way. The question we pose is the following. Does there exist a three-party system with a set of two-party separable reduced states for which any global state compatible with these reduced states is nonlocal? Note that two related questions have already been addressed: (i) Can one deduce that a global state is entangled from the observation of separable reduced states [8,9]? (ii) Can one deduce that a global state is nonlocal from the observation of local marginal correlations [9,10]?

To both questions the answer turns out to be yes. However, as posed above, our goal in this paper is to answer a question which is strictly stronger than both questions above. In particular, we wonder if there exist reduced states which are nonentangled where, however, any three-qubit state compatible with these marginals is nonlocal. In this case, subcorrelation Bell inequalities [9–12] come to our aid. These types of Bell inequalities do not involve full-correlation terms (that is, correlation terms which consist of all parties), and in the special case of three parties they contain only two- and one-body expectation values.

As a starting point for our study, we exhibit in Sec. II a tripartite quantum state which has separable two-party reduced states. Then, we introduce in Sec. III a subcorrelation Bell inequality (involving only one- and two-body mean values) which is violated by the above quantum state provided well-chosen measurements are performed on it. Note that due to the special form of our Bell inequality, the quantum expectation values and, consequently, the quantum violation of the Bell inequality depend only on the two-party reduced states and not on the global state itself. Then, we obtain that any extension of the above set of separable two-party reduced states to a global state results in a nonlocal global state. This already implies our main result stated in the abstract. However, it turns out that the Bell violation with the above two-party reduced states is very small (in the range of 10^{-2}) and therefore very sensitive to noise, which arises inevitably in any experimental setup.

In order to propose a scheme which is more robust to noise, we give a simple method in Sec. IV based on semidefinite programming (SDP), which allows us to decide whether a global state is fully determined by its reduced states. By applying this method to our set of three two-party reduced states introduced in Sec. II, we find out that these marginals in fact fully determine the global three-party state. This implies that the violation of an arbitrary three-party Bell inequality (possibly consisting of all-correlation terms as well) signals the nonlocality of any global state compatible with the two-party reductions of the global state. Therefore, in the following we do not have to restrict ourselves to the study of two-body Bell inequalities. Indeed, we provide in Sec. V a three-party Bell inequality which is violated by a large amount using our unique state with separable marginals. The relatively big violation suggests that our example is promising from the viewpoint of possible experimental implementation as well.

II. A FAMILY OF THREE-QUBIT STATES

Our starting point is the following family of states:

$$\varrho = p_0|0\rangle\langle 0| \otimes |\psi_0\rangle\langle\psi_0| + |1\rangle\langle 1| \otimes (p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|), \tag{1}$$

where Alice holds the first qubit and the pure two-qubit states $|\psi_i\rangle, i = 0,1,2$, possessed by Bob and Charlie have the special parametric form

$$|\psi_0\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle,$$
$$|\psi_1\rangle = (\cos\beta|0\rangle + \sin\beta|1\rangle) \otimes (\cos\gamma|0\rangle + \sin\gamma|1\rangle), \tag{2}$$
$$|\psi_2\rangle Y = \frac{1}{\sqrt{2}}(\sin\delta|00\rangle + \cos\delta|01\rangle + \cos\delta|10\rangle - \sin\delta|11\rangle).$$

Note that $|\psi_1\rangle$ is a product state, whereas $|\psi_0\rangle$ is a partially entangled state for generic angle α and $|\psi_2\rangle$ is a maximally entangled state. Also note that, due to construction, the state is biseparable with respect to cut $A|BC$, which implies that both ϱ_{AB} and ϱ_{AC} two-party reduced states are separable (for a review of different notions of separability, we refer the reader to Ref. [13]). On the other hand, tracing out Alice's qubit, we get the reduced state $\rho_{BC} = \sum_{i=0,1,2} p_i|\psi_i\rangle\langle\psi_i|$. Let us now fix weights p_i,

$$p_0 = 0.759101,$$
$$p_1 = 0.015596, \tag{3}$$
$$p_2 = 0.225303,$$

and the angles

$$\alpha = 0.093586,$$
$$\beta = 1.228106,$$
$$\gamma = 1.063034, \tag{4}$$
$$\delta = 0.050725$$

entering the three-party state (1) along with two-qubit pure states (2) held by Bob and Charlie.

In the following, let us denote by ϱ^* the state (1) with the specially chosen parameters (2), (3), and (4). Using the Peres transposition map [14], we find that the two-party reduced state $\rho_{BC} = \sum_{i=0,1,2} p_i|\psi_i\rangle\langle\psi_i|$ of the global state ϱ^* is separable as well. Hence, we can conclude that all three two-party reduced states of the state ϱ^* are separable.

III. TWO-BODY BELL INEQUALITY

We now present a three-party Bell inequality, where each party has a maximum of two possible binary measurements $A_i, B_i, C_i, i = 1,2$. The Bell expression consists of only single-party marginals and two-body correlation terms defined by the following sum of expectation values:

$$B = -A_1 + (B_1 - B_2 - C_2)(1 + A_1) + Q_{CHSH,BC} \leqslant 3, \tag{5}$$

where the last term on the left-hand side defines the Clauser-Horne-Shimony-Holt (CHSH) quantity [15],

$$Q_{CHSH,BC} = B_1C_1 + B_1C_2 + B_2C_1 - B_2C_2. \tag{6}$$

Let us briefly mention that the above Bell inequality (5) defines a facet of the polytope of classical correlations which now lives in the reduced space of single- and two-party correlators (i.e., neglecting correlators of order 3). One may arrive at the above Bell inequality, for instance, by means of a geometric approach similar to the one used in [9].

Let us remark that Alice in the above Bell inequality (5) performs only a single measurement A_1. In the classical case, (that is, in case of local realistic models) the Bell expression (5) is bounded by the value of 3. However, by performing suitable measurements on the state ϱ^*, it becomes possible to beat this bound. Here we show it by giving the actual measurements. All of them are of equatorial von Neumann type, which can be written in the form $A_i = \cos\theta_i^a\sigma_z + \sin\theta_i^a\sigma_x$, where σ_x and σ_z are Pauli matrices. The measurements B_i, C_i for Bob and Charlie are denoted analogously. The corresponding measurement angles are defined by

$$\theta_1^a = 0,$$
$$\theta_1^b = 0.320997,$$
$$\theta_1^c = 1.442524, \tag{7}$$
$$\theta_2^b = 2.707329,$$
$$\theta_2^c = -3.108820.$$

Indeed, the measurements defined by the angles (7) acting on the state ϱ^* lead to the value of $Q = 3.017583$ in the Bell expression (5), giving rise to a small (but nonzero) violation of the inequality.

In order to arrive at the above Bell violation, we applied the simplex uphill method [16] to find the best measurement operators and the state with the given form (1) fulfilling the condition that the two-qubit marginal ρ_{BC} is separable. This latter condition was imposed by the simple two-qubit separability condition [17], requiring that a two-qubit state ρ_{BC} is separable if and only if $\det(\rho_{BC}^{T_B}) \geqslant 0$, where the operation T_B denotes partial transposition [14].

Note that in the case of optimality, the value of the angle δ in Eq. (4) is close to zero; hence, the maximally entangled state $|\psi_2\rangle$ in (2) is close to the Bell state $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. By fixing the form of state $|\psi_2\rangle$ to be state $|\Psi^+\rangle$, one gets a Bell violation of 3.017454, which is only slightly lower than the optimal one presented above with the parametric form of $|\psi_2\rangle$.

On the other hand, one may wonder what the largest quantum violation is if one does not stick to the form of the family of states (1) but one allows the most general form of a three-qubit state with separable two-qubit marginals. In that case, using a seesaw-type iteration technique [18], the best state found gives the slightly higher quantum violation of 3.017924.

Let us stress again the unusual feature of the Bell inequality (5), namely, that Alice performs only one measurement on her share of the quantum state. In fact, this measurement acts as a filter, heralding the desired entangled state for the remaining two parties. Let us next analyze the Bell violation from this perspective by giving an alternative way to arrive at the quantum value of $Q = 3.017583$ obtained above with the state ϱ^* and particular measurement angles (7).

Alice, by measuring in the standard basis, which corresponds to the observable $A_1 = \sigma_z$, will collapse ϱ^* into another state. In particular, whenever the result is $A_1 = +1$, which occurs with a probability of p_0, the projected state becomes

$$\varrho_{BC}^{+} = |\psi_0\rangle\langle\psi_0|, \tag{8}$$

whereas for the outcome $A_1 = -1$, which occurs with a probability of $1 - p_0$, the projected state becomes

$$\varrho_{BC}^{-} = \frac{p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2|}{p_1 + p_2}, \tag{9}$$

where we have written both states in a normalized form. Similarly, the three-party Bell inequality (5) traces back to two different two-party Bell inequalities depending on the outcomes $A_1 = \pm 1$,

$$\begin{aligned} B_+ &= B(A_1 = +1) \\ &= 2(B_1 - B_2 - C_2) - 1 + Q_{CHSH,BC} \leqslant 3, \\ B_- &= B(A_1 = -1) = Q_{CHSH,BC} + 1 \leqslant 3, \end{aligned} \tag{10}$$

where $Q_{CHSH,BC}$ is defined by Eq. (6). Above, the $B_\pm$ expressions are obtained by substituting $A_1 = \pm 1$ into the Bell expression (5). Our task now is to compute the overall quantum Bell value Q by weighting the probability of occurrences of the two distinct cases, $Q_+ = \mathrm{Tr}\,(\rho_{BC}^{+} B_+) = 2.898134$ and $Q_- = \mathrm{Tr}\,(\rho_{BC}^{-} B_-) = 3.393981$,

$$Q = p_0 Q_+ + (1 - p_0) Q_- = 3.017583 > 3. \tag{11}$$

Despite the fact that only the second inequality B_- is violated, due to the non-negligible probability $(1 - p_0) = 0.240899$ of the $A_1 = -1$ outcome occurring, we get the net violation of $Q = 3.017583$ reported above.

Let us next analyze how economic the above-devised Bell test is regarding the number of settings and the state used. First, let us look at the number of settings in Eq. (5). Alice has one setting, whereas the other two parties can choose between two alternative settings. By removing one setting from any of the parties we get either a trivial or, effectively, a two-party Bell inequality. In neither case can we arrive at the conclusion that separable two-party marginals imply nonlocal quantum correlations. So, regarding the number of settings, the Bell inequality (5) defines a minimal construction.

Regarding the state, suppose that we set to zero the small p_1 weight defined by (3) in the state ϱ^*. Then, there will be at most two terms in the eigendecomposition of the two-party marginal $\varrho_{BC} = \sum_{i=0,1,2} p_i |\phi_i\rangle\langle\phi_i|$. Then, it is known that for any natural measure the set of (2×2)-dimensional separable states occupies a nonzero volume [19]. However, due to a recent work [20], the respective volume is zero for rank-2 states, such as in the case of the above ϱ_{BC} with $p_1 = 0$. This implies that for $p_1 = 0$ the reduced state ϱ_{BC} almost certainly becomes entangled. Hence, we found that the small nonzero component p_1 takes care of the separability of the reduced state ϱ_{BC} of ϱ^*. This means that the rank-3 biseparable state ϱ^* with nonzero weight p_1 is also a minimal construction in terms of the number of pure-state decompositions. However, regarding the global state, we assumed the special form of (1), and it remains an open question whether rank-2 (or even rank-1) genuinely tripartite entangled states [13] would suffice to prove our result.

As stated before, the example analyzed in this section already implies the existence of nonentangled two-party reduced states which are only compatible with nonlocal global states. However, by looking more closely at the state ϱ^*, it will turn out that its three separable bipartite reduced states define a unique extension to a global state (which is the state ϱ^* itself). We find this result via a compatibility test which will be described in Sec. IV. Then, based on the uniqueness property of the global three-qubit state ϱ^*, violation of a generic three-party Bell inequality with the state ϱ^* demonstrates the existence of separable two-party reduced states that are compatible with nonlocal global states. Indeed, in Sec. V we find a large violation of a three-party Bell inequality with the particular state ϱ^*, which implies our stated result.

IV. COMPATIBILITY TEST OF THE STATE

In this section, we show a simple method based on SDP which allows us to decide whether a given three-qubit state is fully determined by its two-qubit reduced states. The method below is related in spirit to the SDP used in Refs. [9,21] and can be easily generalized to higher-dimensional states and more particles as well. However, we conjecture that the complexity of the problem will increase rapidly with the dimension of the state and the number of parties involved.

First, note that any three-qubit density matrix ρ can be expressed in a tensor form,

$$\rho = \frac{1}{8} \sum_{i_1,i_2,i_3=0}^{3} T_{i_1,i_2,i_3} \sigma_{i_1} \otimes \sigma_{i_2} \otimes \sigma_{i_3}, \tag{12}$$

where $\sigma_{i_k} \in \{\mathbb{1}\sigma_1,\sigma_2,\sigma_3\}$ are the Pauli matrices of the kth observer. On the other hand, the tensor components of a three-qubit state can be readily obtained by the expectation values,

$$T_{i_1,i_2,i_3} = \mathrm{Tr}\,\rho\sigma_{i_1} \otimes \sigma_{i_2} \otimes \sigma_{i_3}. \tag{13}$$

In particular, let us denote by $T^*_{i_1,i_2,i_3}$ the tensor components of our particular state ϱ^* defined by Eqs. (1), (2), (3), and (4). Also, note that the B-C two-party reduced state of a general three-qubit state ρ can be expressed as

$$\mathrm{Tr}_A\,\rho = \frac{1}{4} \sum_{i_2,i_3=0}^{3} T_{0,i_2,i_3} \sigma_{i_2} \otimes \sigma_{i_3}, \tag{14}$$

and analogous expressions hold for the other bipartite states, A-B and B-C, as well. Let us now solve separately the following two SDP problems for all $i_1,i_2,i_3 = 1,2,3$:

$$\begin{aligned} T^U_{i_1,i_2,i_3} = \underset{\rho,T}{\text{maximize}} \quad & T_{i_1,i_2,i_3} \\ \text{subject to} \quad & \rho \succeq 0, \\ & T_{0,j_2,j_3} = T^*_{0,j_2,j_3}, \\ & T_{j_1,0,j_3} = T^*_{j_1,0,j_3}, \\ & T_{0,j_2,j_3} = T^*_{0,j_2,j_3}, \\ & \forall\ j_1,j_2,j_3 = 0,1,2,3 \end{aligned} \tag{15}$$

and

$$\begin{aligned} T^L_{i_1,i_2,i_3} = \underset{\rho,T}{\text{minimize}} \quad & T_{i_1,i_2,i_3} \\ \text{subject to} \quad & \rho \succeq 0, \\ & T_{0,j_2,j_3} = T^*_{0,j_2,j_3}, \\ & T_{j_1,0,j_3} = T^*_{j_1,0,j_3}, \\ & T_{0,j_2,j_3} = T^*_{0,j_2,j_3}, \\ & \forall\ j_1,j_2,j_3 = 0,1,2,3. \end{aligned} \tag{16}$$

The above SDP optimization problems (note that ρ is a linear function of the tensor components T_{i_1,i_2,i_3}) are actually the same problems with the only difference being that in the first case a maximization is carried and in the second case a minimization is carried out.

Having solved the SDP problem with the particular T^* components coming from the state ϱ^* and making use of formula (14), we find that $T^L_{i_1,i_2,i_3} = T^U_{i_1,i_2,i_3}$ for all $i_1,i_2,i_3 = 1,2,3$ up to a precision of $\sim 10^{-10}$, which is roughly the numerical accuracy of our SDP solver SEDUMI [22]. Hence, we conclude that state ϱ^* is completely determined by its two-party reduced states up to high numerical precision. In other words, all the information of state ϱ^* is stored within its two-party reduced states. It is interesting to note that states with such a property are generic among multipartite pure states [23,24]. In particular, it was shown by Jones and Linden [25] that generic N-party pure quantum states (with equidimensional subsystems) are uniquely determined by the reduced states of just over half the parties. For a special set of multipartite states, the so-called n-qubit ring cluster states, an even stronger result has been obtained by Tóth *et al.* [8], who proved that for $n \geqslant 6$ all neighboring three-party reduced states are separable and determine uniquely the global state. However, we are not aware of such results from the literature in the case of mixed three-qubit states.

V. GENERIC THREE-PARTY BELL INEQUALITIES

In the previous section, we have seen that the two-party marginals of the state ϱ^* defined by Eqs. (1), (2), (3), and (4) determine the state completely; hence it is legitimate to use generic Bell inequalities to test the nonlocal nature of the state ϱ^*. Namely, if we find a violation of a three-party Bell inequality (possibly consisting of three-body terms as well) with our state ϱ^*, we can be certain that the only global state compatible with the two-party marginals of state ϱ^* is nonlocal. Our goal now is to find a three-party Bell inequality which gives the biggest Q/L ratio, where L defines the local bound on the Bell inequality in question and Q is the maximum quantum value attainable by using state ϱ^*. The magnitude of the Q/L ratio indicates how useful the Bell inequality is for our purposes.

Let us first pick the Mermin inequality [26], which consists of three-party correlation terms,

$$M = -A_1B_1C_1 + A_1B_2C_2 + A_2B_1C_2 + A_2B_2C_1 \leqslant 2. \tag{17}$$

This is equivalent to number 2 in the complete list of two-setting three-party Bell inequalities collected by Sliwa [27]. Let us now choose the following settings:

$$\begin{aligned} A_1 &= A_2 = \sigma_z, \\ B_1 &= \sigma_z, \quad B_2 = \sigma_y, \\ C_1 &= \cos\theta_1\sigma_z + \sin\theta_1\cos\theta_2\sigma_x + \sin\theta_1\sin\theta_2\sigma_y, \\ C_2 &= -\cos\theta_1\sigma_z - \sin\theta_1\cos\theta_2\sigma_x + \sin\theta_1\sin\theta_2\sigma_y, \end{aligned} \tag{18}$$

with $\theta_1 = 3.500760$ and $\theta_2 = 1.605042$. Note the optimal settings are not on the XY plane as usual for a Greenberger-Horne-Zeilinger state equal to $(1/\sqrt{2})(|000\rangle + |111\rangle)$ [28]. With our settings (18), we get a quantum violation of $Q = 2.086929$. However, the optimal quantum violation with the same state ϱ^* but allowing completely general settings is marginally bigger, given by $Q = 2.087190$.

By listing all the inequalities in Sliwa's set, number 4 happens to give the biggest Q/L ratio. The inequality looks as follows:

$$S_4 = (1 - A_1)Q_{CHSH,BC} + 2A_1 \leqslant 2, \tag{19}$$

where $Q_{CHSH,BC}$ is defined by Eq. (6). Compared to inequality (5), Alice still performs a single measurement, but the inequality now contains three-body terms as well. The quantum maximum of (19) using state ϱ^* turns out to be $Q = 2.334184$, which easily follows from our previous analysis in Sec. III.

Namely, let Alice measure in the standard basis. With probability $p_0 = 0.759101$ she projects the state on $|\psi_0\rangle$ [defined by (8)] and with probability $1 - p_0$ on state (9). Then, if Alice gets outcome $A_1 = +1$, we obtain the trivial two-party Bell inequality $B_+ = 2$ with local bound $Q_+ = 2$, independent of the performed measurements of Bob and Charlie. On the other hand, in the case of outcome $A_2 = -1$, the resulting two-party Bell inequality is $B_- = 2Q_{CHSH,BC} - 2$. Because B_+ does not depend on the actual form of Bob's and Charlie's measurements, we can apply the Horodecki formula [29] for the calculation of the CHSH value corresponding to state (9), which turns out to be $Q_{CHSH,BC} = 2.693620$. This way, we obtain the quantum value $Q_- = 2Q_{CHSH,BC} - 2 = 3.387240$. Hence, the overall maximum is $Q = p_0Q_+ + (1 - p_0)Q_- = 2.334184$, entailing the ratio $Q/L = 1.167092$. Interestingly, Bob's and Charlie's settings now require complex numbers, whereas in the case of the two-body Bell inequality of Sec. III it was enough to consider real-valued measurements.

We wish to note that using the software developed in [30] based on the geometric method [31], we could not find a better Q/L ratio with our state ϱ^* up to five measurement settings per party. Hence, we conjecture that the ratio $Q/L = 1.167092$ found for ϱ^* is optimal or at least very close to optimality for any number of measurement settings.

VI. CONCLUSION

We have provided an affirmative answer to the following open question: Is there an example of a set of separable two-party marginals, such that any global state compatible with these marginals is nonlocal, witnessed by violation of a Bell inequality? We found such a state, which is, in fact, uniquely determined by its two-party reduced states. Among two-setting Bell inequalities, this state is maximally violated by Sliwa's inequality 4, giving a ratio of 1.167092 for the quantum per classical bound. Interestingly, the same state also violates a Bell inequality built up from only two-body correlation terms. An intriguing open question is whether our result could be strengthened by considering the stronger notion of the genuine nonlocality scenario [32] instead of the standard nonlocality scenario considered in the present study. That is, we inquire whether there exist two-party separable marginals such that any global three-party state compatible with these marginals is genuinely tripartite nonlocal. The state we considered here was not genuinely multipartite entangled and hence cannot lead to genuine nonlocality. Therefore new insight is very likely needed to tackle this interesting open problem.

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