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Wigner function and the successive measurement of position and momentum

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Wigner function is a "quasidistribution" that provides a representation of the state of a quantum mechanical system in the phase space of position and momentum. In this paper we find a relation between the Wigner function and appropriate measurements involving the system's position and momentum which generalize the von Neumann model of measurement. We introduce two *probes* coupled successively in time to projectors associated with the system's position and momentum. We show that one can relate the Wigner function to the Kirkwood joint quasidistribution of position and momentum, the latter, in turn, being a particular case of successive measurements. We first consider the case of a quantum mechanical system described in a continuous Hilbert space and then turn to the case of a discrete, finite-dimensional Hilbert space.

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I. INTRODUCTION

The Wigner function was originally introduced to provide a phase-space representation of the state of a quantum-mechanical system described in a continuous Hilbert space [1]. The Wigner function is termed a "quasidistribution," as it may become negative in some portions of phase space [2,3]. Indeed, as is well known, quantum mechanics (QM) precludes a proper joint probability distribution of position q and momentum p. However, in many respects the Wigner function plays a role similar to the phase-space distribution function in classical statistical mechanics [2–4]; therefore, we find it natural to inquire whether one can relate it to appropriate measurements involving the position and momentum of the system.

The idea we develop is to introduce the first stage of the measurement, or "pre-measurement," explicitly in the QM description, by coupling, successively in time, the system observables we wish to study to auxiliary degrees of freedom, or "probes," and detect the probes, not the system itself. This procedure represents a generalization of von Neumann's model of measurement [5–7]. Specifically, we couple projectors associated with the system's position and momentum to two independent probes, at times t_1 and t_2, respectively. It turns out that one can relate the Wigner function to correlations of observables—each belonging to one of the two probes—which, being distinct degrees of freedom and external to the system, *are compatible* and admit a joint probability distribution. These probe correlations are thus experimentally accessible.

We first relate the Wigner function to the so-called Kirkwood joint quasiprobability distribution of position and momentum [8,9] which is, in general, a complex quantity. It is then Kirkwood's distribution which can be expressed in terms of the abovementioned probe correlations, in the limit in which the coupling becomes very weak.

In a historical context, it is interesting to mention that Kirkwood introduced the joint quasiprobability distribution in phase space that bears his name a year later than Wigner introduced his, and with similar motivations related to statistical mechanics applications. In the next decade Dirac introduced essentially the same joint quasiprobability distribution for noncommuting observables, with the aim of "discussing trajectories for the motion of a particle in QM."

We should remark that in the field of quantum optics, the Wigner function has been related to a set of measurable quantities different from the ones considered in the present paper, namely, quadrature distributions which are experimentally available, a method that constitutes an application to QM of the computer-aided tomography scan [2–4].

Other quasidistributions have been proposed in the literature: e.g., Ref. [10] presents a family of quasidistribution functions, of which Wigner function—which is the distribution considered herewith—is a special case. As for the relation between Wigner's function and Kirkwood's quasidistribution, we also refer the reader to Refs. [11–13].

The paper is organized as follows. In the next section we develop the scheme we just outlined, for a quantum-mechanical system described in a continuous Hilbert space. In Sec. 3 we then turn to studying a discrete, finite-dimensional Hilbert space. The notion of Wigner function for a discrete Hilbert space is a topic which has been widely studied in the literature (a selection of these contributions is represented by Refs. [14–27]). Here, we adopt an alternative definition—which will be of interest in a geometrical context to be described elsewhere—as the starting point to develop the scheme presented above. We see that the discrete case is free from a number of divergences that are encountered in the continuous case. Various specific algebraic calculations have been carried out in the appendices in order not to interrupt the main presentation. We finally conclude in Sec. IV.

II. WIGNER FUNCTION AND KIRKWOOD QUASIDISTRIBUTION FOR A CONTINUOUS HILBERT SPACE

A. The Wigner transform of an operator defined in a continuous Hilbert space

The Wigner transform (WT) of an operator $\hat{A}$ is a mapping from Hilbert space to phase space [1]. It can be expressed as the inverse Fourier transform of the characteristic function of the operator. Using units in which q and p are dimensionless, and $\hbar = 1$, we have the definition [2–4]

$$W_{\hat{A}}(q,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{W}_{\hat{A}}(u,v)e^{i(uq+vp)}du\,dv, \quad (2.1a)$$

$$\tilde{W}_{\hat{A}}(u,v) = \mathrm{Tr}[\hat{A}e^{-i(u\hat{q}+v\hat{p})}]. \quad (2.1b)$$

When the operator $\hat{A}$ is the density operator $\hat{\rho}$, we speak of its WT as the Wigner function (WF) of the state. The definition (2.1) is equivalent to the standard one, presented, for convenience, in Eq. (A4).

The WT of an operator $\hat{A}$ can also be expressed as

$$W_{\hat{A}}(q,p) = \mathrm{Tr}[\hat{A}\hat{P}(q,p)], \quad (2.2)$$

$\hat{P}(q,p)$ being a Hermitian operator. Using the definition of WT given in Eq. (A4), $\hat{P}(q,p)$ can be written as

$$\hat{P}(q,p) = \int_{-\infty}^{\infty} e^{-ipy}\left|q-\frac{y}{2}\right\rangle\left\langle q+\frac{y}{2}\right|dy. \quad (2.3)$$

We can also use the *mutually unbiased bases* [4] (MUB) states $|x',\theta\rangle$, eigenstates of the operator $\hat{X}_\theta = \hat{q}\cos\theta + \hat{p}\sin\theta$—and hence eigenstates of the exponential operator appearing in Eq. (2.1b)—which satisfy the eigenvalue equation $\hat{X}_\theta|x',\theta\rangle = x'|x',\theta\rangle$, to express the operator $\hat{P}(q,p)$ as

$$\hat{P}(q,p) = \frac{1}{2\pi}\int_0^{\pi}d\theta\int_{-\infty}^{\infty}dx'\int_0^{\infty}dt \times |t|e^{-it(x'-q\cos\theta-p\sin\theta)}|x';\theta\rangle\langle x';\theta|, \quad (2.4a)$$

$$= -\frac{1}{\pi}\mathrm{P}\int_0^{\pi}d\theta\int_{-\infty}^{\infty}dx'\frac{\frac{\partial}{\partial x'}|x';\theta\rangle\langle x';\theta|}{x'-(q\cos\theta+p\sin\theta)} \quad (2.4b)$$

[cf. Ref. [4], Eq. (A6) (where $\rho(x,y)$ is to be identified with $W_{\hat{\rho}}(q,p)$ and $\rho_\theta(x')$ with $\langle x',\theta|\hat{\rho}|x',\theta\rangle$), and Eq. (23)].

The operator $\hat{P}(q,p)$ and the WT of an (arbitrary) operator $\hat{A}$ possess the following attributes.

(1) The matrix elements of the operator $\hat{P}(q,p)$ of Eq. (2.3) in the coordinate basis are given by

$$\langle q|\hat{P}(q',p')|\bar{q}\rangle = e^{ip'(q-\bar{q})}\delta(q+\bar{q}-2q'). \quad (2.5)$$

(2) The WT of a Hermitian operator $\hat{A}$ is real, which follows immediately from the hermiticity of $\hat{P}(q,p)$.

(3) The operators $\hat{P}(q,p)$ fulfill the following orthogonality and closure relations:

$$\frac{1}{2\pi}\mathrm{Tr}[\hat{P}(q,p)\hat{P}(q',p')] = \delta(q-q')\delta(p-p'), \quad (2.6a)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{P}(q,p)dq\,dp = \mathbb{I}, \quad (2.6b)$$

$\mathbb{I}$ being the unit operator.

(4) The WT of the operators $\hat{A}$ and $\hat{B}$ satisfy the "product formula," or "overlap formula" [see, e.g., Ref. [2], Eq. (3.20), and Ref. [3], Eq. (3.5)]:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_{\hat{A}}(q,p)W_{\hat{B}}(q,p)\frac{dq\,dp}{2\pi} = \mathrm{Tr}(\hat{A}\hat{B}). \quad (2.7)$$

(5) The WF for the state $\hat{\rho}$ satisfies the marginality relation

$$\mathrm{Tr}(\hat{\rho}\hat{\mathbb{P}}^{\theta}_{x'}) = \langle x',\theta|\hat{\rho}|x',\theta\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_{\hat{\rho}}(q,p)\delta(x'-(q\cos\theta+p\sin\theta))\frac{dq\,dp}{2\pi}, \quad (2.8)$$

[see Ref. [4], Eq. (22)] which states that, if the system is in state $\hat{\rho}$, the probability to find it in the pure state $|x',\theta\rangle$ is given by the integral of the WF along the line $q\cos\theta + p\sin\theta = x'$ in phase space. In particular, the marginal probability of q and that for p take the standard form. Expression (2.8) is referred to as the *Radon transform* [2–4] of the Wigner function $W_{\hat{\rho}}(q,p)$.

(6) The WF is normalized as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W_{\hat{\rho}}(q,p)\frac{dq\,dp}{2\pi} = 1. \quad (2.9)$$

B. Relation between the Wigner function and the Kirkwood quasidistribution for a continuous Hilbert space

As shown in Appendix A, one can express the WF in terms of Kirkwood's quasidistribution as

$$W_{\hat{\rho}}(q,p) = 2\iint dq'dp'e^{2i(q-q')(p-p')}K(p',q'). \quad (2.10)$$

Here, the quantity

$$K(p,q) = \mathrm{Tr}(\hat{\rho}\;\hat{\mathbb{P}}_p\hat{\mathbb{P}}_q), \quad (2.11a)$$

with the definition

$$\hat{\mathbb{P}}_q = |q\rangle\langle q|, \quad (2.11b)$$

$$\hat{\mathbb{P}}_p = |q\rangle\langle p|, \quad (2.11c)$$

is Kirkwood's joint quasidistribution [8,9] of q and p, which is, in general, complex. Similar results can be found in Refs. [11–13].

The operators $\hat{\mathbb{P}}_q$ and $\hat{\mathbb{P}}_p$ are not proper position and momentum projectors, since they are not idempotent. In order to use the formalism developed in Ref. [7] we use, instead, the operators $\hat{\mathbb{P}}_{q_n}$ and $\hat{\mathbb{P}}_{p_m}$ defined in Appendix B. For this purpose, we write Eq. (2.10) as

$$W_{\hat{\rho}}(q,p) = 2\sum_{n,m=-\infty}^{\infty}\int_{q_n-\delta/2}^{q_n+\delta/2}dq' \times \int_{p_m-\delta/2}^{p_m+\delta/2}dp'e^{2i(q-q')(p-p')}\mathrm{Tr}_s(\hat{\rho}_s\;\hat{\mathbb{P}}_{p'}\hat{\mathbb{P}}_{q'}) \quad (2.12a)$$

$$\approx 2 \sum_{n,m=-\infty}^{\infty} e^{2i(q-q_n)(p-p_m)} \mathrm{Tr}_s \times \left(\hat{\rho}_s \int_{p_m-\delta/2}^{p_m+\delta/2} dp' \, \hat{\mathbb{P}}_{p'} \int_{q_n-\delta/2}^{q_n+\delta/2} dq' \, \hat{\mathbb{P}}_{q'} \right) \tag{2.12b}$$

$$= 2 \sum_{n,m=-\infty}^{\infty} e^{2i(q-q_n)(p-p_m)} \mathrm{Tr}_s \left(\hat{\rho}_s \hat{\mathbb{P}}_{p_m} \hat{\mathbb{P}}_{q_n} \right) \tag{2.12c}$$

$$= 2 \sum_{n,m=-\infty}^{\infty} e^{2i(q-q_n)(p-p_m)} K(p_m,q_n), \tag{2.12d}$$

where $K(p_m,q_n)$ is Kirkwood's joint quasiprobability distribution of p_m and q_n defined in Eq. (B8). The discretization involved in going from Eq. (2.12a) to Eq. (2.12b) is an approximation. We expect that approximation to be justified if the interval δ is small enough that the factor $e^{2i(q-q')(p-p')}$ does not vary appreciably for q' and p' inside that interval. Alternatively, it could be justified using a "mean-value theorem" [28]. An argument where the approximation appears at the level of c-number functions can be found in Appendix B, right below Eq. (B3).

According to Eq. (B9), Kirkwood's distribution, in turn, can be expressed in terms of the position-position correlation of the two probes and their momentum-position correlation: these are compatible variables, detected in measurements described by von Neumman's model with very weak coupling; specifically, in this model the observables coupled in succession to the two probes are the projectors for the position and momentum of the system proper. Substituting the result of Eq. (B9) in Eq. (2.12) we thus find

$$W_{\hat{\rho}}(q,p) = 2 \sum_{n,m=-\infty}^{\infty} e^{2i(q-q_n)(p-p_m)} \times \left\{ \lim_{\epsilon_1 \to 0} \frac{1}{\epsilon_1 \epsilon_2} \left[\langle \hat{Q}_1 \hat{Q}_2 \rangle^{(\hat{\mathbb{P}}_{p_m} \leftarrow \hat{\mathbb{P}}_{q_n})} + \frac{i}{2\sigma_{P_1}^2} \langle \hat{P}_1 \hat{Q}_2 \rangle^{(\hat{\mathbb{P}}_{p_m} \leftarrow \hat{\mathbb{P}}_{q_n})} \right] \right\}. \tag{2.13}$$

This result states that the Wigner function, which is defined in the system phase space, can be related to a set of *measurable quantities*, consisting of the two-probe correlations detected in the experimental setup described above, and thereby reconstructed therefrom.

III. WIGNER FUNCTION AND KIRKWOOD QUASIDISTRIBUTION FOR A DISCRETE, FINITE-DIMENSIONAL HILBERT SPACE

The analysis performed in the previous section for a continuous Hilbert space is now extended with a similar philosophy to a discrete, finite-dimensional Hilbert space.

A. The Wigner transform for a discrete, finite-dimensional Hilbert space

The possibility of defining a WT for a Hilbert space of finite dimensionality has been studied extensively in the literature [14–27]. Here we propose, for the WT of an operator $\hat{A}$ defined in a Hilbert space of dimensionality N, the definition

$$W_{\hat{A}}(q,p) = \frac{1}{N} \left\{ \sum_{b=0}^{N-1} \sum_{k=1}^{N-1} \tilde{W}_{\hat{A}}(k,b) e^{i\frac{2\pi}{N}k(-p+bq)} + \sum_{l=0}^{N-1} \tilde{W}_{\hat{A}}(l) e^{i\frac{2\pi}{N}lq} \right\}, \tag{3.1a}$$

$$\tilde{W}_{\hat{A}}(k,b) = \mathrm{Tr}\{\hat{A}[(\hat{X}\hat{Z}^b)^k]^\dagger\}, \tag{3.1b}$$

$$\tilde{W}_{\hat{A}}(l) = \mathrm{Tr}[\hat{A}(\hat{Z}^l)^\dagger]. \tag{3.1c}$$

The variables $q,p = 0,1,\ldots,N-1$ denote the coordinate and momentum in our discrete phase space, which thus consists of an $N \times N$ set of points. The quantities $\hat{Z}$ and $\hat{X}$ appearing in Eqs. (3.1) are the Schwinger operators, defined, for convenience, in Appendix C. Definition (3.1) is, for the discrete case, analogous to that of Eqs. (2.1) for the continuous case. The $N(N-1)$ operators $(\hat{X}\hat{Z}^b)^k$ $(b = 0,1,\ldots,N-1;\ k = 1,\ldots,N-1)$ appearing in Eq. (3.1b), together with the N operators $\hat{Z}^l$ $(l = 0,\ldots,N-1)$ appearing in Eq. (3.1c), form a complete set of N^2 operators [see Eqs. (C7)].

We take the dimensionality N to be a prime number larger than 2, as in this case the integers $0,1,\ldots,N-1$ constitute a mathematical field, with addition, subtraction, multiplication, and division defined Mod$N$ (see, e.g., Refs. [17,27,30]). This field plays a role similar to that of the real numbers in the continuous case studied in the previous section. The quantity $\omega = \exp(2\pi i/N)$, one of the Nth roots of 1, appears frequently in our analysis; we agree that the numerical exponents of ω to be considered in what follows always belong to the Mod$N$ algebra. When the dimensionality N is a prime number, we also know that the problem admits exactly $N+1$ *mutually unbiased bases* (MUB) (see, e.g., Refs. [27,29]). The operators $\hat{X}\hat{Z}^b$, $b = 0,\ldots N-1$ define N of the $N+1$ MUB, [see Eq. (C8)], while the operator $\hat{Z}$ defines the so-called "reference basis," or "computational basis."

It is shown in Appendix D that the definition (3.1) can be written in terms of MUB as

$$W_{\hat{A}}(q,p) = \frac{1}{N} \sum_{b=\ddot{0}}^{N-1} \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}k[M_{q,p}(b)-m]} \times \langle m;b|\hat{A}|m;b\rangle - \mathrm{Tr}(\hat{A}), \tag{3.2a}$$

where the reference basis has been denoted, for convenience, as $\ddot{0}$. We have defined the quantity

$$M_{q,p}(b) = \begin{cases} (-p+bq)\ \mathrm{Mod}[N], & \text{for} \quad b = 0,\ldots,N-1, \\ q, & \text{for} \quad b = \ddot{0}. \end{cases} \tag{3.2b}$$

For a given pair of variables, q and p, Eq. (3.2b) states that, for $b = \ddot{0}$, $M_{q,p}(\ddot{0}) = q$; for $b = 0$, $M_{q,p}(0) = -p\ \mathrm{Mod}[N] = N - p$; and for subsequent values of b, $M_{q,p}(b) = (-p+bq)\ \mathrm{mod}[N]$. Thus, $M_{q,p}(b)$ may be viewed as specifying "points" in a b-m plane: b is along the x axis and takes the values $b = \ddot{0},0,1,\ldots,N-1$, which denote the $N+1$ bases; m is along the y axis and takes the

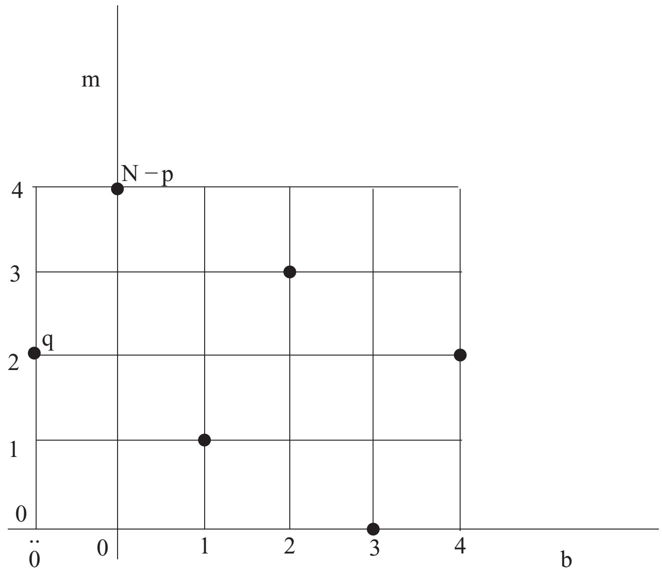

FIG. 1. Illustration of the function $m = M_{q,p}(b)$ in the b-m plane, for $N = 5$ and the particular pair of "phase-space" values $q = 2$ and $p = 1$.

values $m = 0,1,\ldots,N-1$, which denote the N states for each basis. This aggregate of points, for *fixed* q and p, may be described as a "line" in the b-m plane. This is illustrated, for a particular case, in Fig. 1.

Further study, based on such a view, is in progress. We thus refer to $M_{q,p}(b)$ as a line, and its corresponding operator, $\hat{P}_{q,p}$, Eq. (3.4c) below, as a line operator; it is similar to the "phase-point" operator introduced in Secs. V and VI of Ref. [17].

In Eq. (3.2a) we can do the sum over k, using the result

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}k[M_{q,p}(b)-m]} = \delta_{m,\, M_{q,p}(b)}, \tag{3.3}$$

where the arguments of the Kronecker δ are understood to be Mod[N]; in other words, for given q and p, the sum (3.3) vanishes unless m equals $(-p+bq)$ Mod[N] when $b \neq \ddot{0}$ or q when $b = \ddot{0}$. Equation (3.2a) can then be given the alternative forms

$$W_{\hat{A}}(q,p) = \sum_{b=\ddot{0}}^{N-1}\langle M_{q,p}(b);b|\hat{A}|M_{q,p}(b);b\rangle - \mathrm{Tr}(\hat{A}), \tag{3.4a}$$

$$= \mathrm{Tr}(\hat{A}\hat{P}_{q,p}), \tag{3.4b}$$

where we have defined the Hermitian operator

$$\hat{P}_{q,p} = \sum_{b=\ddot{0}}^{N-1}|M_{q,p}(b);b\rangle\langle M_{q,p}(b);b| - \hat{\mathbb{I}}, \tag{3.4c}$$

$\hat{\mathbb{I}}$ being the unit operator. From Eq. (3.2a), the line operator $\hat{P}_{q,p}$ can also be written more explicitly as

$$\hat{P}_{q,p} = \frac{1}{N}\sum_{b=0}^{N-1}\sum_{k=1}^{N-1}\sum_{m=0}^{N-1} e^{\frac{2\pi i}{N}k(-p+bq-m)}|m;b\rangle\langle m;b| + \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=0}^{N-1} e^{\frac{2\pi i}{N}k(q-n)}|n\rangle\langle n|. \tag{3.5}$$

Equations (3.5) and (3.4c) are analogous to Eqs. (2.4a) and (2.4b), which correspond to the continuous case. The integrals over θ, x', and $|t|$ of the continuous case correspond to the sums over b, m, and k of the discrete one.

The WT of Eq. (3.1) and the line operator of Eq. (3.4c) have the following properties, analogous to the ones for the continuous case.

(1) As shown in Appendix E, the matrix elements of the line operator with respect to the states of the reference basis are given by

$$\langle q|\hat{P}_{q',p'}|\bar{q}\rangle = \delta_{qq'}\delta_{\bar{q}q'} - \delta_{q\bar{q}}\ \delta_{2q,\,2q'+1} + \delta_{q+\bar{q},\,2q'+1}\ e^{\frac{2\pi i}{N}p'(q-\bar{q})}. \tag{3.6}$$

(2) The WT of a Hermitian operator $\hat{A}$ is real, i.e.,

$$W_{\hat{A}}(q,p) = W^{\star}_{\hat{A}}(q,p), \quad \text{for} \quad A^{\dagger} = A. \tag{3.7}$$

This follows immediately from the hermiticity of the line operators $\hat{P}_{q,p}$.

(3) The line operators $\hat{P}_{q,p}$, N^2 in number, form a complete orthonormal set of operators, in the following sense:

(i) It is shown in Appendix F that they fulfill the orthogonality relation

$$\frac{1}{N}\mathrm{Tr}[\hat{P}_{q,p}\ \hat{P}_{q',p'}] = \delta_{q,q'}\delta_{p,p'}, \tag{3.8}$$

which is the discrete version of Eq. (2.6a).

(ii) From expression (3.5), or from Eq. (3.4c), one finds, directly, that they satisfy the closure relation

$$\frac{1}{N}\sum_{q,p=0}^{N-1}\hat{P}_{q,p} = \mathbb{I}, \tag{3.9}$$

which is the discrete version of Eq. (2.6b).

(iii) An $N \times N$ matrix $\hat{A}$ can thus be written as a linear combination of the $\hat{P}_{q,p}$'s, i.e.,

$$\hat{A} = \frac{1}{N}\sum_{q,p=0}^{N-1}\mathrm{Tr}(\hat{A}\hat{P}_{q,p})\hat{P}_{q,p} \tag{3.10a}$$

$$= \frac{1}{N}\sum_{q,p=0}^{N-1}W_{\hat{A}}(q,p)\hat{P}_{q,p}. \tag{3.10b}$$

(4) The WTs of the operators $\hat{A}$ and $\hat{B}$ fulfill the so-called "product formula" [see also Ref. [17], Eq. (15)]

$$\frac{1}{N}\sum_{q,p=0}^{N-1}W_{\hat{A}}(q,p)W_{\hat{B}}(q,p) = \mathrm{Tr}(\hat{A}\hat{B}), \tag{3.11}$$

in analogy with Eq. (2.7) for the continuous case. This can be proved as follows. From Eq. (3.10b) applied to the operators $\hat{A}$ and $\hat{B}$, and using the orthogonality relation (3.8), we have

$$\mathrm{Tr}(\hat{A}\hat{B}) = \frac{1}{N^2}\sum_{q,p=0}^{N-1}\sum_{q',p'=0}^{N-1}W_{\hat{A}}(q,p)W_{\hat{B}}(q',p')\mathrm{Tr}[\hat{P}_{q,p}\ \hat{P}_{q',p'}] \tag{3.12a}$$

$$= \frac{1}{N}\sum_{q,p=0}^{N-1}W_{\hat{A}}(q,p)W_{\hat{B}}(q,p). \tag{3.12b}$$

(5) The WF $W_{\hat{\rho}}(q,p)$ satisfies the marginality property, written in terms of the projector $\hat{\mathbb{P}}_{mb} = |m,b\rangle\langle m,b|$,

$$\mathrm{Tr}(\hat{\rho}\ \hat{\mathbb{P}}_{mb}) = \langle m,b|\hat{\rho}|m,b\rangle = \frac{1}{N}\sum_{q,p=0}^{N-1} W_{\hat{\rho}}(q,p)\delta_{M_{q,p}(b),m}, \tag{3.13}$$

where we recall that $M_{q,p}(b)$ is defined in Eq. (3.2b). Equation (3.13), analogous to Eq. (2.8) for the continuous case, states that the probability to find the system in the state m of the basis b (of our set of $N+1$ MUBs) is $1/N$ times the sum of the WF over the points in the phase-space plane q,p that satisfy $M_{q,p}(b) = m$. The marginality relation, Eq. (3.13), is obtained from the product formula (3.12b) for $\hat{A} = \hat{\rho}$ and $B = \hat{\mathbb{P}}_{mb}$, the WT of the latter being, from Eq. (3.4b),

$$W_{\hat{\mathbb{P}}_{mb}}(q,p) = \mathrm{Tr}(\hat{\mathbb{P}}_{mb}\hat{P}_{q,p}) \tag{3.14a}$$

$$= \mathrm{Tr}\left\{|m;b\rangle\langle m;b|\left[\sum_{b'=\ddot{0}}^{N-1}|M_{q,p}(b');b'\rangle \times \langle M_{q,p}(b');b'| - \hat{\mathbb{I}}\right]\right\} \tag{3.14b}$$

$$= |\langle m;b|M_{q,p}(b);b\rangle|^2 + \sum_{b'=\ddot{0},(\neq b)}^{N-1}|\langle m;b|M_{q,p}(b');b'\rangle|^2 - 1 \tag{3.14c}$$

$$= \delta_{M_{q,p}(b),m} + N\frac{1}{N} - 1 \tag{3.14d}$$

$$= \delta_{M_{q,p}(b),m}. \tag{3.14e}$$

We comment in passing that the right-hand side of Eq. (3.13) can be considered as defining the *Radon transform* of the WF $W_{\hat{\rho}}(q,p)$ (see, e.g., Refs. [2–4,27]).
Two particular cases of the above marginality property are as follows: (i) for $b = \ddot{0}$, m is a coordinate, which we may call q_0, and the resulting summation in phase space [i.e., the right-hand side of Eq. (3.13); see Eq. (3.2b)] is over the *line in the* (q,p) *plane* containing all p's for that q_0; i.e.,

$$b = \ddot{0} : \mathrm{Tr}\big(\hat{\rho}\hat{\mathbb{P}}_{q_0,\ddot{0}}\big) = \langle q_0|\hat{\rho}|q_0\rangle = \frac{1}{N}\sum_{q,p} W_{\hat{\rho}}(q,p)\delta_{q,q_0} = \frac{1}{N}\sum_{p} W_{\hat{\rho}}(q_0,p); \tag{3.15}$$

(ii) for $b = 0$, we identify $m = N - p_0$ [see Eq. (C9c)], and the resulting summation in phase space [i.e., the right-hand side of Eq. (3.13); see Eq. (3.2b): $M_{q,p}(0) = N - p$] is over the *line in the* (q,p) *plane* containing all q's for that p_0; i.e.,

$$b = 0 : \mathrm{Tr}\big(\hat{\rho}\hat{\mathbb{P}}_{N-p_0,0}\big) = \frac{1}{N}\sum_{q,p} W_{\hat{\rho}}(q,p)\delta_{N-p,N-p_0}, \tag{3.16a}$$

$$\text{i.e.,}\quad \langle p_0|\hat{\rho}|p_0\rangle = \frac{1}{N}\sum_{q} W_{\hat{\rho}}(q,p_0). \tag{3.16b}$$

These are the standard marginality relations, which can also be obtained trivially from the form (3.4) for the WF, without using the product formula. For the case $b = 1,\ldots,N$, Eq. (3.13) states that

$$\langle m,b|\hat{\rho}|m,b\rangle = \frac{1}{N}\sum_{q,p} W_{\hat{\rho}}(q,p)\delta_{-p+bq,m}, \tag{3.17}$$

the sum on the right-hand side being over the points on the *line in phase space* (q,p) defined by $-p+bq = m\mathrm{Mod}[N]$, for fixed m,b.

(6) The WF is normalized as

$$\frac{1}{N}\sum_{p,q=0}^{N-1} W_{\hat{\rho}}(q,p) = 1, \tag{3.18}$$

just as in Eq. (2.9) for the continuous case.

Notice that the various properties mentioned in the previous section for the continuous case can be translated to the discrete case with the correspondence $1/2\pi \Rightarrow 1/N$.

B. Relation between the Wigner function and the Kirkwood quasidistribution for a discrete, finite-dimensional Hilbert space

Going back to our program of relating the WF to the Kirkwood quasidistribution, we show in Appendix G the relation

$$W_{\hat{\rho}}(q,p) = \sum_{q',p'=0}^{N-1} e^{\frac{2\pi i}{N}2(q-q'+\frac{N+1}{2})(p-p')}K_{p'q'} + \langle q|\hat{\rho}|q\rangle - \langle q+(N+1)/2|\hat{\rho}|q+(N+1)/2\rangle. \tag{3.19}$$

Notice that in this equation the labels occurring in bras and kets must be understood Mod$[N]$. It can be checked directly that the result (3.19) fulfills the normalization condition (3.18).

The result of Eq. (3.19) is analogous to that of Eqs. (2.10) and (2.12) for the continuous case. The Kirkwood distribution $K(p,q)$ is defined as in Eqs. (2.11) for the continuous case, except that the states $|q\rangle$ and $|p\rangle$ are to be defined as in Eqs. (C1a) and (C5a).

Just as in the previous section, we notice from Eq. (B9) that Kirkwood's distribution can be related to the correlations of two probes, in a very weak-coupling measurement designed to premeasure in succession the projectors for the position and momentum of the system. For the present discrete case, $(p,q = 0,\ldots,N-1)$, we write relation (B9) as

$$K(p,q) = \lim_{\epsilon_1\to 0}\frac{1}{\epsilon_1\epsilon_2}\left[\langle\hat{Q}_1\hat{Q}_2\rangle^{(\mathbb{P}_p\leftarrow\mathbb{P}_q)} + \frac{i}{2\sigma_{P_1}^2}\langle\hat{P}_1\hat{Q}_2\rangle^{(\mathbb{P}_p\leftarrow\mathbb{P}_q)}\right]. \tag{3.20}$$

Substituting this relation in Eq. (3.19) we thus find

$$W_{\hat{\rho}}(q,p) = \sum_{q',p'=0}^{N-1} e^{\frac{2\pi i}{N}2(q-q'+(N+1)/2)(p-p')}\left\{\lim_{\epsilon_1\to 0}\frac{1}{\epsilon_1\epsilon_2} \times\left[\langle\hat{Q}_1\hat{Q}_2\rangle^{(\mathbb{P}_{p'}\leftarrow\mathbb{P}_{q'})} + \frac{i}{2\sigma_{P_1}^2}\langle\hat{P}_1\hat{Q}_2\rangle^{(\mathbb{P}_{p'}\leftarrow\mathbb{P}_{q'})}\right]\right\} + \frac{1}{\epsilon}\langle\hat{Q}\rangle^{(\mathbb{P}_q)} - \frac{1}{\epsilon}\langle\hat{Q}\rangle^{(\mathbb{P}_{q+(N+1)/2})}. \tag{3.21}$$

The last two terms in Eq. (3.21) are the expectation values of the probe position in a single measurement designed to premeasure the projectors $\mathbb{P}_q$ and $\mathbb{P}_{q+(N+1)/2}$, respectively.

Result (3.21) is the discrete Hilbert-space counterpart of the duly discretized continuous case that was given in Eq. (2.13) of the previous section.

As a result, the Wigner function, which is defined in the system discrete phase space, can be related to a set of *measurable quantities*, consisting of the two-probe and single-probe expectation values obtained in the experimental setup described above, and reconstructed therefrom.

IV. CONCLUSIONS

In this paper we posed the question whether it is possible to find appropriate measurements involving the system's position and momentum that would allow the reconstruction of the Wigner function of the system state. We were able to give an affirmative answer to this question. The type of measurements needed are generalizations of the model envisaged by von Neumann in his model of QM measurement. They involve successive couplings of two probes with projectors associated with the system's position and momentum. In this model, what one detects are the correlation functions of the two probes, which are compatible dynamical variables, not the system itself.

We first considered the case in which the system is described in a continuous Hilbert space, and then we turned to the study of a description in a discrete, finite-dimensional Hilbert space.

The Wigner function for this latter case of a discrete, finite-dimensional Hilbert space, has been widely studied in the literature. Here we proposed an alternative version, formulated, in this paper, within a standard algebraic approach; however, as it turns out, this version can be reformulated entirely in terms of "finite-geometry" concepts, an approach that associates states and operators in Hilbert space with lines and points of the geometry [27]. This latter approach is conceptually very attractive, and its development will be postponed to a future publication.

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APPENDIX A: DERIVATION OF THE RELATION EQ. (2.10) BETWEEN WF AND KIRKWOOD QUASIDISTRIBUTION FOR THE CONTINUOUS CASE

The Wigner function is defined in Eq. (2.1). Using the Baker-Campbell-Hausdorff identity (Ref. [31], p. 323)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{A1}$$

valid when $\hat{A}$ and $\hat{B}$ commute with their commutator, and expressing the operators $e^{-iv\hat{p}}$ and $e^{-iu\hat{q}}$ in their spectral representation, we can write $\tilde{W}_{\hat{\rho}}(u,v)$ as

$$\tilde{W}_{\hat{\rho}}(u,v) = e^{-\frac{i}{2}uv} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dq'dp' e^{-i(uq'+vp')} K(p',q'), \tag{A2}$$

where $K(q,p)$ is Kirkwood's quasidistribution [8,9] of Eq. (2.11a). Introducing Eq. (A2) in Eq. (2.1a) and using the result

$$\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{i}{2}(\xi u+\eta v-uv)} du dv = e^{\frac{i}{2}\xi\eta}, \tag{A3}$$

we find Eq. (2.10).

An alternative derivation of this result is based on the standard definition of the WT of an operator $\hat{A}$ (see, e.g., Refs. [1–4])

$$W_{\hat{A}}(q,p) = \int_{-\infty}^{\infty} e^{-ipy} \left\langle q+\frac{y}{2}\right| \hat{A} \left| q-\frac{y}{2}\right\rangle dy. \tag{A4}$$

The Kirkwood distribution of Eq. (2.11a) can be written in terms of Wigner functions using the product formula of Eq. (2.7) as

$$K(p,q) = \iint W_{\hat{\rho}}(q',p') W_{\hat{\mathbb{P}}_p \hat{\mathbb{P}}_q}(q',p') \frac{dq'dp'}{2\pi}. \tag{A5}$$

We have

$$W_{\hat{\mathbb{P}}_p \hat{\mathbb{P}}_q}(q',p') = \int e^{-ip'y} \left\langle q'+\frac{y}{2}\right| \hat{\mathbb{P}}_p \hat{\mathbb{P}}_q \left| q'-\frac{y}{2}\right\rangle dy \tag{A6a}$$

$$= \frac{1}{\pi} e^{-2i(p-p')(q-q')}. \tag{A6b}$$

From Eq. (A5) we find

$$\iint K(\bar{p},\bar{q}) e^{2i(p-\bar{p})(q-\bar{q})} d\bar{q} d\bar{p}$$

$$= \iiiint W_{\rho}(q',p') W_{\mathbb{P}_{\bar{p}}\mathbb{P}_{\bar{q}}}(q',p') e^{2i(p-\bar{p})(q-\bar{q})} \times \frac{dq'dp'}{2\pi} d\bar{q} d\bar{p}, \tag{A7a}$$

$$= \iiiint W_{\rho}(q',p') \left[\frac{1}{\pi} e^{-2i(\bar{p}-p')(\bar{q}-q')}\right] e^{2i(\bar{p}-p)(\bar{q}-q)} \times \frac{dq'dp'}{2\pi} d\bar{q} d\bar{p}, \tag{A7b}$$

$$= \frac{1}{2} W_{\rho}(p,q), \tag{A7c}$$

which is the result (2.10). From Eq. (A7a) to Eq. (A7b) we have used the result (A6b). From Eq. (A7b) to Eq. (A7c) we have used the identity

$$\iint e^{-2i(\bar{p}-p')(\bar{q}-q')} e^{2i(\bar{p}-p)(\bar{q}-q)} d\bar{q} d\bar{p} = \pi^2 \delta(p'-p)\delta(q'-q). \tag{A8}$$

APPENDIX B: VON NEUMANN MODEL FOR POSITION AND MOMENTUM

The operator $\hat{\mathbb{P}}_q = |q\rangle\langle q|$ of Eq. (2.11b) is not a proper position projector, since it is not idempotent. In order to use the formalism developed in Ref. [7], we define the operators (with $q_n = n\delta$)

$$\mathbb{P}_{q_n} = \int_{q_n-\delta/2}^{q_n+\delta/2} |q\rangle dq \langle q|, \tag{B1}$$

which have the properties

$$\mathbb{P}_{q_n}\mathbb{P}_{q_{n'}} = \delta_{nn'}\mathbb{P}_{q_n}, \tag{B2a}$$

$$\sum_{n=-\infty}^{\infty} \mathbb{P}_{q_n} = 1. \tag{B2b}$$

Similarly, for the momentum we define the operators

$$\mathbb{P}_{p_m} = \int_{p_m-\delta/2}^{p_m+\delta/2} |p\rangle dp \langle p|, \tag{B3}$$

which have similar properties.

The approximation involved in going from Eq. (2.12a) to Eq. (2.12b) in the text can be justified in terms of c-number functions in the following way. For any two $\langle\phi|$ and $|\psi\rangle$, consider the following integral and the approximations to it given in the subsequent equations:

$$\langle\phi| \int_{-\delta/2}^{\delta/2} f(q')\mathbb{P}_{q'}dq'|\psi\rangle = \int_{-\delta/2}^{\delta/2} f(q')\phi^*(q')\psi(q')dq' \tag{B4a}$$

$$\approx f(q_1)\int_{-\delta/2}^{\delta/2} \phi^*(q')\psi(q')dq' \tag{B4b}$$

$$= \langle\phi|f(q_1)\mathbb{P}_{q_0}|\psi\rangle, \tag{B4c}$$

for some suitable $q_1 \in [-\delta/2,\delta/2]$. In Eq. (B4c) we have used the notation of Eq. (B1). With this argument we thus approximate the operator inside the first matrix element in Eq. (B4a) by the one in Eq. (B4c).

Reference [7] studies the extension to two probes of von Neumman's measurement model (vNM),

$$\hat{H}(t) = \epsilon_1\delta(t-t_1)\hat{\mathbb{P}}_{q_n}\hat{P}_1 + \epsilon_2\delta(t-t_2)\hat{\mathbb{P}}_{p_m}\hat{P}_2, \quad 0 < t_1 < t_2, \tag{B5}$$

in which $\hat{\mathbb{P}}_{q_n}$ plays the role of the observable to be premeasured first and $\mathbb{P}_{p_m}$ plays the role of the observable to be premeasured later.

The position-position and momentum-position correlations of the two probes are found to be

$$\frac{1}{\epsilon_1\epsilon_2}\langle\hat{Q}_1\hat{Q}_2\rangle^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})} = \mathrm{Re}W_{11}^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})}(\epsilon_1), \tag{B6a}$$

$$\frac{1}{\epsilon_1\epsilon_2}\langle\hat{P}_1\hat{Q}_2\rangle^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})} = 2\sigma_{P_1}^2\mathrm{Im}W_{11}^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})}(\epsilon_1), \tag{B6b}$$

where

$$W_{11}^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})}(\epsilon_1) = \sum_{n'} G_{n'n}(\epsilon_1)\mathrm{Tr}_s\left(\hat{\rho}_s\hat{\mathbb{P}}_{q_{n'}}\hat{\mathbb{P}}_{p_m}\hat{\mathbb{P}}_{q_n}\right), \tag{B7a}$$

$$G_{n'n}(\epsilon_1) = \delta_{nn'} + e^{-\frac{1}{2}\sigma_{P_1}^2\epsilon_1^2}(1-\delta_{nn'}). \tag{B7b}$$

A Gaussian distribution for the original state of the probes is assumed, and $\sigma_{P_1}^2$ denotes the momentum variance of probe 1. In the limit $\epsilon_1 \to 0$, the above expression (B7a) becomes

$$W_{11}^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})}(0) = \mathrm{Tr}_s\left(\hat{\rho}_s\hat{\mathbb{P}}_{p_m}\hat{\mathbb{P}}_{q_n}\right) \equiv K(p_m,q_n), \tag{B8}$$

which is Kirkwood's joint quasiprobability distribution [8,9] for the variables p_m and q_n, in the original state of the system $\hat{\rho}_s$.

Using Eqs. (B6), Kirkwood's joint quasidistribution can thus be expressed in terms of measurements performed on the probes as

$$K(p_m,q_n) = \lim_{\epsilon_1\to 0}\frac{1}{\epsilon_1\epsilon_2}\Bigg[\langle\hat{Q}_1\hat{Q}_2\rangle^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})} + \frac{i}{2\sigma_{P_1}^2}\langle\hat{P}_1\hat{Q}_2\rangle^{(\hat{\mathbb{P}}_{p_m}\leftarrow\hat{\mathbb{P}}_{q_n})}\Bigg]. \tag{B9}$$

APPENDIX C: SCHWINGER OPERATORS AND MUB

We consider our N-dimensional Hilbert space to be spanned by N distinct states $|q\rangle$, with $q = 0,1,\ldots,(N-1)$, which are subject to the periodic condition $|q+N\rangle = |q\rangle$. These states are designated as the "reference basis," or "computational basis," of the space. We shall follow Schwinger [32] and introduce the unitary operators $\hat{X}$ and $\hat{Z}$, defined by their action on the states of the reference basis by the equations

$$\hat{Z}|q\rangle = \omega^q|q\rangle, \quad \omega = e^{2\pi i/N}, \tag{C1a}$$

$$\hat{X}|q\rangle = |q+1\rangle. \tag{C1b}$$

The operators $\hat{X}$ and $\hat{Z}$ fulfill the periodic condition

$$\hat{X}^N = \hat{Z}^N = \hat{\mathbb{I}}, \tag{C2}$$

$\hat{\mathbb{I}}$ being the unit operator. These definitions lead to the commutation relation

$$\hat{Z}\hat{X} = \omega\hat{X}\hat{Z}. \tag{C3}$$

The two operators $\hat{Z}$ and $\hat{X}$ form a *complete algebraic set*, in that only a multiple of the identity commutes with both [32]. As a consequence, any operator defined in our N-dimensional Hilbert space can be written as a function of $\hat{Z}$ and $\hat{X}$.

We introduce the Hermitian operators $\hat{p}$ and $\hat{q}$, which play the role of "momentum" and "position," through the equations [29,33]

$$\hat{X} = \omega^{-\hat{p}} = e^{-\frac{2\pi i}{N}\hat{p}}, \tag{C4a}$$

$$\hat{Z} = \omega^{\hat{q}} = e^{\frac{2\pi i}{N}\hat{q}}. \tag{C4b}$$

What we defined as the reference basis can thus be considered as the "position basis." With Eq. (C3) and definitions (C4), the commutator of $\hat{q}$ and $\hat{p}$ in the continuous limit [29,33] is the standard one, $[\hat{q},\hat{p}] = i$.

The "momentum basis" consists of the eigenstates of $\hat{X}$, which can be expanded in terms of the position basis as

$$|p\rangle = \sum_{q=0}^{N-1}\frac{e^{\frac{2\pi i}{N}pq}}{\sqrt{N}}|q\rangle \tag{C5a}$$

and satisfy the eigenvalue equation [see Ref. [33], Eq. (12)]

$$\hat{X}|p\rangle = e^{-\frac{2\pi i}{N}p}|p\rangle. \tag{C5b}$$

The N^2-dimensional matrix space is spanned by the *complete orthonormal* N^2 *operators* $\hat{X}^m\hat{Z}^l$, with $m,l = 0,1,\ldots,(N-1)$, so that any $N \times N$ matrix can be written as a linear combination of these N^2 operators. A familiar example is a two-dimensional Hilbert space, where any 2×2 matrix can be written as a linear combination of the three Pauli matrices plus the unit matrix, which can also be written as σ_x, σ_z, $\sigma_x\sigma_z$, and I.

For N = prime > 2, we find the following identities:

$$(\hat{X}\hat{Z}^b)^k = \omega^{\frac{k(k-1)}{2}b}\hat{X}^k\hat{Z}^{kb} \tag{C6a}$$

$$= \omega^{-\frac{k(k+1)}{2}b}\hat{Z}^{kb}\hat{X}^k, \tag{C6b}$$

$$\hat{X}^k\hat{Z}^l = \omega^{-kl}\hat{Z}^l\hat{X}^k, \tag{C6c}$$

$$(\hat{X}\hat{Z}^b)^N = \hat{I}. \tag{C6d}$$

Our *complete orthonormal set of* N^2 *operators* can be taken as

$$(\hat{X}\hat{Z}^b)^k, \quad b = 0,1,\ldots,N-1, \quad k = 1,\ldots,N-1, \tag{C7a}$$

$$\hat{Z}^l, \quad l = 0,1,\ldots,N-1. \tag{C7b}$$

The operator $\hat{X}\hat{Z}^b$ possesses N eigenvectors, denoted by $|m,b\rangle$ (see Eqs. (10) and (11) of Ref. [27]):

$$\hat{X}\hat{Z}^b|m,b\rangle = \omega^m|m;b\rangle, \tag{C8a}$$

$$|m;b\rangle = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\omega^{\frac{b}{2}n(n-1)-nm}|n\rangle, \quad b,m = 0,1,\ldots,N-1. \tag{C8b}$$

Here, $|n\rangle$ $(n = 0,\ldots,N-1)$ denotes the N states of the reference basis. We have, altogether, $N+1$ MUB. The states with $b = 0$, i.e.,

$$|m;0\rangle = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}e^{-\frac{2\pi i}{N}mq}|q\rangle, \tag{C9a}$$

$$= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}e^{\frac{2\pi i}{N}(N-m)q}|q\rangle, \tag{C9b}$$

are eigenstates of $\hat{p}$ which, from Eq. (C5a), can be written as

$$|m;0\rangle = |p = -m = (N-m)\mathrm{Mod}[N]\rangle. \tag{C9c}$$

APPENDIX D: DERIVATION OF EQS. (3.2) FOR THE DISCRETE WIGNER FUNCTION

We can write the quantities $\tilde{W}_{\hat{A},\hat{B}}(k,b)$ and $\tilde{W}_{\hat{A},\hat{B}}(l)$ appearing in Eqs. (3.1) in terms of the MUB basis $|m,b\rangle$ defined in Eqs. (C8). The operator $\hat{X}\hat{Z}^b$ can be written in the spectral representation as

$$\hat{X}\hat{Z}^b = \sum_{m=0}^{N-1}|m,b\rangle\omega^m\langle m,b|, \tag{D1a}$$

$$[(\hat{X}\hat{Z}^b)^k]^\dagger = \sum_{m=0}^{N-1}|m,b\rangle\omega^{-mk}\langle m,b|, \tag{D1b}$$

so that

$$\tilde{W}_{\hat{A}}(k,b) = \sum_{m=0}^{N-1}\omega^{-mk}\langle m,b|\hat{A}|m,b\rangle. \tag{D2}$$

Similarly,

$$\hat{Z} = \sum_{m=0}^{N-1}|n\rangle\omega^n\langle n|, \tag{D3a}$$

$$(\hat{Z}^l)^\dagger = \sum_{n=0}^{N-1}|n\rangle\omega^{-nl}\langle n|, \tag{D3b}$$

so that

$$\tilde{W}_{\hat{A}}(l) = \sum_{n=0}^{N-1}\omega^{-nl}\langle n|\hat{A}|n\rangle. \tag{D4}$$

Substituting these results in Eqs. (3.1), we obtain Eqs. (3.2).

APPENDIX E: DERIVATION OF THE RELATION (3.6) FOR THE MATRIX ELEMENTS OF THE LINE OPERATOR

In the definition (3.4c) of the line operator we single out the first two terms to write

$$\hat{P}_{q',p'} = |q'\rangle\langle q'| + |N-p';0\rangle\langle N-p';0| + \sum_{b=1}^{N-1}|-p'+bq';b\rangle\langle -p'+bq';b| - \hat{\mathbb{I}}. \tag{E1}$$

Recall, from Eqs. (C9), that $|N-p';0\rangle = |p'\rangle$ and that $-p'+bq'$ is understood Mod$N$.

Using Eqs. (C8) for the states of the MUB, we write the matrix element $\langle q|\hat{P}_{q',p'}|\bar{q}\rangle$ as

$$\langle q|\hat{P}_{q',p'}|\bar{q}\rangle = \delta_{qq'}\delta_{\bar{q}q'} + \frac{1}{N}e^{\frac{2\pi i}{N}p'(q-\bar{q})} - \delta_{q,\bar{q}} + \frac{1}{N}\sum_{b=1}^{N-1}e^{\frac{2\pi i}{N}[\frac{b}{2}q(q-1)-q(-p'+bq')]} \times e^{-\frac{2\pi i}{N}[\frac{b}{2}\bar{q}(\bar{q}-1)-\bar{q}(-p'+bq')]} \tag{E2a}$$

$$= \delta_{qq'}\delta_{\bar{q}q'} + \frac{1}{N}e^{\frac{2\pi i}{N}p'(q-\bar{q})} - \delta_{q,\bar{q}} + \frac{1}{N}(\alpha-1)e^{\frac{2\pi i}{N}p'(q-\bar{q})}. \tag{E2b}$$

The quantity α is defined as

$$\alpha = \sum_{b=0}^{N-1}e^{\frac{2\pi i}{N}b[\frac{1}{2}(q-\bar{q})(q+\bar{q}-1-2q')]} \tag{E3}$$

and is nonzero only when

$$\tfrac{1}{2}(q-\bar{q})(q+\bar{q}-1-2q')=0\ \mathrm{Mod}[N]. \tag{E4}$$

That is,

$$\alpha = N[\delta_{q,\,\bar{q}}+\delta_{q+\bar{q},\,1+2q'}-\delta_{q,\,\bar{q}}\delta_{q+\bar{q},\,1+2q'}], \tag{E5}$$

where the arguments of the Kronecker δ's are understood, as always, Mod$[N]$. Substituting (E5) in Eq. (E2b), we find the result (3.6).

APPENDIX F: PROOF OF THE ORTHOGONALITY RELATION, EQ. (3.8)

The definition of a "line," Eq. (3.2b), implies that two distinct lines, i.e., such that their parameters q and/or p are *not* identical, have one, and only one point, i.e., $M(b)$, in common. We illustrate this in the case of two lines with common p but distinct q's: $q \neq q'$. We have then that $M(b)$ of the first equals $M'(b)$ of the second iff $bq = bq'$, which implies, for $q \neq q'$, that $b = 0$: i.e., the only common point is at $b = 0$, which is consistent with having a common p; there is no other common point for N a prime number.

Of course, two lines with the same q and p have all their points, $N + 1$ in number, in common.

From Eq. (3.4c), the trace appearing on the left-hand side of Eq. (3.8) can be written as

$$\begin{aligned}\mathrm{Tr}[\hat{P}_{q,p}\ \hat{P}_{q',p'}] &= \sum_b \mathrm{Tr}\,[|M_{q,p}(b),b\rangle\langle M_{q,p}(b),b|M_{q',p'}(b),b\rangle\langle M_{q',p'}(b),b|]\\ &\quad+\sum_{b\neq b'}\mathrm{Tr}[|M_{q,p}(b),b\rangle\langle M_{q,p}(b),b|M_{q',p'}(b'),b'\rangle\langle M_{q',p'}(b'),b'|]\\ &\quad-\sum_b \mathrm{Tr}[|M_{q,p}(b),b\rangle\langle M_{q,p}(b),b|]-\sum_{b'}\mathrm{Tr}[|M_{q',p'}(b'),b'\rangle\langle M_{q',p'}(b'),b'|]+\mathrm{Tr}\mathbf{I}\\ &\equiv A+B-C-C'+D.\end{aligned} \tag{F1}$$

That $D = N$ and $C = C' = N + 1$ is obvious. For two distinct lines, thus having one point in common, $A = 1$. For two identical lines, $A = N + 1$.

Now consider B. We have, for $b \neq b'$, $|\langle M_{q,p}(b),b|M_{q',p'}(b'),b'\rangle|^2 = 1/N$, since the bra and ket belong to two MUB. Since the summation in B contains $(N + 1)N$ terms, we find $B = N + 1$.

Thus

$$\mathrm{Tr}[\hat{P}_{q,p}\ \hat{P}_{q',p'}] = \begin{cases}1+(N+1)-2(N+1)+N=0, & \text{for } (q,p)\neq(q',p')\\ (N+1)+(N+1)-2(N+1)+N=N, & \text{for } (q,p)=(q',p').\end{cases} \tag{F2}$$

The result of Eq. (3.8) then follows.

APPENDIX G: DERIVATION OF THE RELATION EQ. (3.19) BETWEEN THE WF AND THE KIRKWOOD QUASIDISTRIBUTION FOR THE DISCRETE CASE

Here we proceed in analogy with the derivation given in Appendix A for the continuous case, starting from Eq. (A5).

Using the product formula, Eq. (3.11), the Kirkwood distribution can be written as

$$K_{p,q} = \mathrm{Tr}(\hat{\rho}\ \mathbb{P}_p\mathbb{P}_q) \tag{G1a}$$

$$= \frac{1}{N}\sum_{q',p'=0}^{N-1} W_\rho(q',p')W_{\mathbb{P}_p\mathbb{P}_q}(q',p'). \tag{G1b}$$

For the second WT we have

$$W_{\mathbb{P}_p\mathbb{P}_q}(q',p') = \mathrm{Tr}(\mathbb{P}_p\mathbb{P}_q\hat{P}_{q',p'}) \tag{G2a}$$

$$= \langle p|q\rangle\langle q|\hat{P}_{q',p'}|p\rangle \tag{G2b}$$

$$= \sum_{\bar{q}}\langle p|q\rangle\langle q|\hat{P}_{q',p'}|\bar{q}\rangle\langle\bar{q}|p\rangle, \tag{G2c}$$

and substituting the result (3.6) for the matrix element of the line operator, we find

$$W_{\mathbb{P}_p\mathbb{P}_q}(q',p') = \frac{1}{N}\{\delta_{qq'}-\delta_{2q,\,2q'+1}+e^{-\frac{2\pi i}{N}(p-p')[2(q-q')-1]}\}. \tag{G2d}$$

From Eq. (G1b) we construct the combination

$$\sum_{\bar{q}\bar{p}} K_{\bar{p}\bar{q}}e^{\frac{2\pi i}{N}2(q-\bar{q})(p-\bar{p})} = \frac{1}{N}\sum_{q'p'\bar{q}\bar{p}} W_{\hat{\rho}}(q',p')W_{\mathbb{P}_{\bar{p}}\mathbb{P}_{\bar{q}}}(q',p')e^{\frac{2\pi i}{N}2(q-\bar{q})(p-\bar{p})}. \tag{G3a}$$

Inserting the result (G2d), we find

$$= \frac{1}{N^2}\sum_{q'p'} W_{\hat{\rho}}(q',p') \sum_{\bar{q}\bar{p}}\{\delta_{\bar{q}q'}-\delta_{2\bar{q},\,2q'+1}+e^{-\frac{2\pi i}{N}(\bar{p}-p')[2(\bar{q}-q')-1]}\} \times e^{\frac{2\pi i}{N}2(q-\bar{q})(p-\bar{p})}. \tag{G3b}$$

Evaluating the various sums we obtain Kronecker δ's, thus giving

$$= \frac{1}{N^2} \sum_{q'p'} W_{\hat{\rho}}(q',p')[N\delta_{qq'} - N\delta_{2(q-q'),\,1} + N^2\delta_{pp'}\delta_{2(q-q'),\,1}]. \tag{G3c}$$

Recalling that $1/2 = (N+1)/2$ Mod$[N]$,

$$\sum_{\bar{q}\bar{p}} K_{\bar{p}\bar{q}} e^{\frac{2\pi i}{N}2(q-\bar{q})(p-\bar{p})} = \frac{1}{N}\sum_{p'} W_{\hat{\rho}}(q,p') - \frac{1}{N}\sum_{p'} W_{\hat{\rho}}(q-(N+1)/2,p') + W_{\hat{\rho}}(q-(N+1)/2,p) \tag{G4}$$

or

$$W_{\hat{\rho}}(q,p) = \langle q|\hat{\rho}|q\rangle - \langle q+(N+1)/2|\hat{\rho}|q+(N+1)/2\rangle + \sum_{\bar{q}\bar{p}} K_{\bar{p}\bar{q}} e^{\frac{2\pi i}{N}2(q-\bar{q}+(N+1)/2)(p-\bar{p})}, \tag{G5}$$

which is the desired relation (3.19).

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