Multipolar hierarchy of efficient quantum polarization measures

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We advocate a simple multipole expansion of the polarization density matrix. The resulting multipoles appear as successive moments of the Stokes variables and can be obtained from feasible measurements. In terms of these multipoles we construct a whole hierarchy of measures that accurately assess higher-order polarization fluctuations.

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I. INTRODUCTION

The standard notion of polarization comes from the treatment of light as a beam. This hints at a well-defined direction of propagation, and thus at a specific transverse plane, wherein the tip of the electric field describes an ellipse. This polarization ellipse can be elegantly visualized by using the Poincaré sphere and is determined by the Stokes parameters, the degree of polarization being simply the length of the Stokes vector [1].

This geometric representation not only provides remarkable insight but also greatly simplifies otherwise complex problems and, as a result, has become an indisputable tool to deal with polarization phenomena. However, the necessity of addressing new issues, such as highly nonparaxial fields [2], narrow-band imaging systems [3], and the recognition of associated propagation questions [4], has brought about significant modifications of this simple classical picture [5–13].

In the quantum domain, the classical setting can be immediately mimicked in terms of the Stokes operators, which can be obtained from the Stokes parameters by quantizing the field amplitudes [14]. However, the appearance of hurdles such as hidden polarization [15], the fact that the Poincaré sphere cannot accommodate photon-number fluctuations [16], and the difficulties in defining polarization properties of two-photon entangled fields [17], to cite only a few examples, show that the resulting theory is insufficient.

The root of these difficulties can be traced to the fact that classical polarization is chiefly built on first-order moments of the Stokes variables, whereas higher-order moments can play a major role for quantum fields. Polarization squeezing [18], a nonclassical effect that is actually defined only by the variances of the Stokes operators, illustrates that point in the most clear way.

Nowadays, there is a general consensus in that a full understanding of the subtle polarization effects arising in the realm of the quantum world would require a characterization of higher-order polarization fluctuations, as it happens in coherence theory, where one needs, in general, a hierarchy of correlation functions. Some results along these lines have already been reported, but either they use magnitudes difficult to determine in practice, such as distances [19], generalized visibilities [20–23], and central moments [24], or they go only up to second order [25,26], and the pertinent extensions are difficult to discern.

In this paper, we propose a systematic and feasible solution to such a fundamental and longstanding problem. To that end, we resort to a multipole expansion of the density matrix that naturally sorts successive moments of the Stokes variables. The dipole term, being just the first-order moment, can be identified with the classical picture, while the other multipoles account for higher-order moments. The probability distribution for these multipoles provides thus a complete information about the polarization properties of any state; in terms of it we propose a suitable measure for the quantitative assessment of those fluctuations.

II. SETTING THE SCENARIO

Throughout, we assume a monochromatic quantum field specified by two operators, $\hat{a}_H$ and $\hat{a}_V$, representing the complex amplitudes in two linearly polarized orthogonal modes, which we denote as horizontal (H) and vertical (V), respectively. The Stokes operators can be concisely defined as

$$\hat{S}_\mu = \frac{1}{2}(\hat{a}_H^\dagger \ \hat{a}_V^\dagger)\,\sigma_\mu \begin{pmatrix} \hat{a}_H \\ \hat{a}_V \end{pmatrix}, \tag{2.1}$$

the subscript $\dagger$ denoting the Hermitian adjoint. The Greek index μ runs from 0 to 3, where $\sigma_0 = \mathbb{1}$ and $\{\sigma_k\}$ ($k = 1,2,3$) are the Pauli matrices.

Note carefully that $\hat{S}_0 = \hat{N}/2$, where $\hat{N} = \hat{a}_H^\dagger \hat{a}_H + \hat{a}_V^\dagger \hat{a}_V$ is the operator for the total number of photons. On the other hand, with our definition the average of $\hat{\mathbf{S}} = (\hat{S}_1, \hat{S}_2, \hat{S}_3)$ differs by a factor of 1/2 from the classical Stokes vector [14]. However, in this way $\{\hat{S}_k\}$ satisfy the commutation relations of the SU(2) algebra

$$[\hat{S}_k, \hat{S}_\ell] = i\epsilon_{k\ell m}\,\hat{S}_m, \tag{2.2}$$

where $\epsilon_{k\ell m}$ is the Levi-Civita fully antisymmetric tensor. This noncommutability precludes the simultaneous exact measurement of the physical quantities they represent, which can be formulated quantitatively by the uncertainty relation

$$\Delta^2 \hat{\mathbf{S}} = \Delta^2 \hat{S}_1 + \Delta^2 \hat{S}_2 + \Delta^2 \hat{S}_3 \geqslant \tfrac{1}{2}\langle \hat{N} \rangle, \tag{2.3}$$

where the variances are given by $\Delta^2 \hat{S}_i = \langle \hat{S}_i^2 \rangle - \langle \hat{S}_i \rangle^2$. In other words, the electric vector of a monochromatic quantum field never traces a definite ellipse.

In classical optics, the states of definite polarization are specified by $\langle \hat{\mathbf{S}} \rangle^2 = \langle \hat{S}_0 \rangle^2$ and the average intensity is a well-defined quantity. In the three-dimensional space of the Stokes parameters this defines a sphere with radius equal to the intensity: the Poincaré sphere. In contradistinction, in quantum optics we have that $\hat{\mathbf{S}}^2 = \hat{S}_0(\hat{S}_0 + \hat{\mathbb{1}})$. As fluctuations in the number of photons are, in general, unavoidable, we are forced to work with a full three-dimensional Poincaré space that can be regarded as a set of nested spheres with radii proportional to the different photon numbers that contribute to the state.

The Hilbert space $\mathscr{H}$ of these fields is spanned by the Fock states $\{|n_H, n_V\rangle\}$ for both polarization modes. However, since $[\hat{N}, \hat{\mathbf{S}}] = 0$, each subspace with a fixed number of photons N (i.e., fixed spin $S \equiv S_0 = N/2$) must be handled separately. In other words, in the previous onionlike picture of the Poincaré space, each shell has to be addressed independently. This can be underlined if we employ the relabeling

$$|S, m\rangle \equiv |n_H = S + m, n_V = S - m\rangle. \tag{2.4}$$

In this angular momentum basis, $S = N/2, m = (n_H - n_V)/2$, and, for each S, m runs from $-S$ to S. This can be seen as the basis of common eigenstates of $\{\hat{\mathbf{S}}^2, \hat{S}_3\}$, and these states span a $(2S+1)$-dimensional subspace wherein $\hat{\mathbf{S}}$ acts in the standard way.

III. THE POLARIZATION SECTOR AND THE MULTIPOLE EXPANSION

From the previous discussion, it is clear that the moments of any energy-preserving observable (such as $\hat{\mathbf{S}}$) do not depend on the coherences between different subspaces. The only accessible information from any state described by the density matrix $\hat{\varrho}$ is thus its polarization sector [27], which is given by the block-diagonal form

$$\hat{\varrho}_{\text{pol}} = \bigoplus_S P_S \, \hat{\varrho}^{(S)}, \tag{3.1}$$

where P_S is the photon-number distribution (S takes on the values $0, 1/2, 1, \ldots$) and $P_S \, \hat{\varrho}^{(S)}$ is the reduced density matrix in the subspace with spin S. Any $\hat{\varrho}$ and its associated block-diagonal form $\hat{\varrho}_{\text{pol}}$ cannot be distinguished in polarization measurements, and so, accordingly, we drop henceforth the subscript *pol*. This is consistent with the fact that polarization and intensity are, in principle, separate concepts: in classical optics the form of the ellipse described by the electric field (polarization) does not depend on its size (intensity).

To proceed further we need to represent every component $\hat{\varrho}^{(S)}$ in a polarization basis. Instead of using directly the states $\{|S, m\rangle\}$, it is more convenient to write such an expansion as

$$\hat{\varrho}^{(S)} = \sum_{K=0}^{2S} \sum_{q=-K}^{K} \varrho_{Kq}^{(S)} \, \hat{T}_{Kq}^{(S)}, \tag{3.2}$$

TABLE I. Values of $\mathscr{W}_K$ and the degree $\mathbb{P}_K$ for three different quantum polarization states. $|S; \theta, \phi\rangle$ stands for an SU(2) coherent state in the S subspace, and $|\alpha_H, \alpha_V\rangle$ is a two-mode quadrature coherent state with $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$ the average number of photons.

State	$\mathscr{W}_K$	$\mathbb{P}_K$
$\|S, m\rangle$	$\frac{2K+1}{2S+1} (C_{Sm,K0}^{Sm})^2$	$\left[\frac{\sum_{\ell=1}^{K} \frac{2\ell+1}{2S+1} (C_{Sm,\ell 0}^{Sm})^2}{\sum_{\ell=1}^{K} \frac{2\ell+1}{2S+1} (C_{SS,\ell 0}^{SS})^2} \right]^{1/2}$
$\|S; \theta, \phi\rangle$	$\frac{2K+1}{2S+1} (C_{SS,K0}^{SS})^2$	1
$\|\alpha_H, \alpha_V\rangle$	$\sum_{S=K/2}^{\infty} \frac{\bar{N}^{2S} e^{-\bar{N}}}{(2S)!} \frac{2K+1}{2S+1} (C_{SS,K0}^{SS})^2$	$\sum_{S=K/2}^{\infty} \frac{\bar{N}^{2S} e^{-\bar{N}}}{(2S)!}$

where the irreducible tensor operators $\hat{T}_{Kq}^{(S)}$ are [28]

$$\hat{T}_{Kq}^{(S)} = \sqrt{\frac{2K+1}{2S+1}} \sum_{m,m'=-S}^{S} C_{Sm,Kq}^{Sm'} \, |S, m'\rangle\langle S, m|, \tag{3.3}$$

with $C_{Sm,Kq}^{Sm'}$ being the Clebsch-Gordan coefficients that couple a spin S and a spin K ($0 \leqslant K \leqslant 2S$) to a total spin S.

Although at first sight Eq. (3.3) might look a bit intricate, $\hat{T}_{Kq}^{(S)}$ is related to the Kth power of the Stokes operators, a simple observation that will turn out crucial in the following. In particular, the monopole $\hat{T}_{00}^{(S)}$, being proportional to the identity, is always trivial, while the dipole $\hat{T}_{1q}^{(S)}$ is proportional to $\hat{S}_q$ and thus renders the classical picture, in which the state is depicted by its average value. Therefore, higher-order multipoles embody the polarization fluctuations we wish to appraise [29].

The expansion coefficients $\varrho_{Kq}^{(S)} = \text{Tr}[\hat{\varrho}^{(S)} \, T_{Kq}^{(S)\dagger}]$ are known as state multipoles, and they contain complete information, but sorted in a manifestly SU(2)-invariant form.

Alternatively, one can look at

$$\mathscr{W}_K^{(S)} = \sum_{q=-K}^{K} |\varrho_{Kq}^{(S)}|^2, \tag{3.4}$$

which is just the square of the state overlapping with the Kth multipole patterns in the Sth subspace. When there is a distribution of photon numbers, we sum over all of them: $\mathscr{W}_K = \sum_S P_S \, \mathscr{W}_K^{(S)}$. One can easily find out that

$$\sum_K \mathscr{W}_K = \text{Tr}(\hat{\varrho}^2), \tag{3.5}$$

so it is just the purity. Actually, as shown in the Appendix, $\mathscr{W}_K$ can be interpreted as a measure of the localization of the state in phase space.

In Table I we have worked out the values of $\mathscr{W}_K$ for three outstanding examples of quantum states that will serve as a guide: the state $|S, 0\rangle$ (which reads $|N, N\rangle$, with $N = S$, in the basis $|n_H, n_V\rangle$), the SU(2) coherent state $|S; \theta, \phi\rangle$ (defined in the Appendix), and a two-mode quadrature coherent state $|\alpha_H, \alpha_V\rangle$, summing up over the Poissonian photon-number distribution (with $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$). In Fig. 1 we also plot these cases in point; as we can see, for the classical quadrature

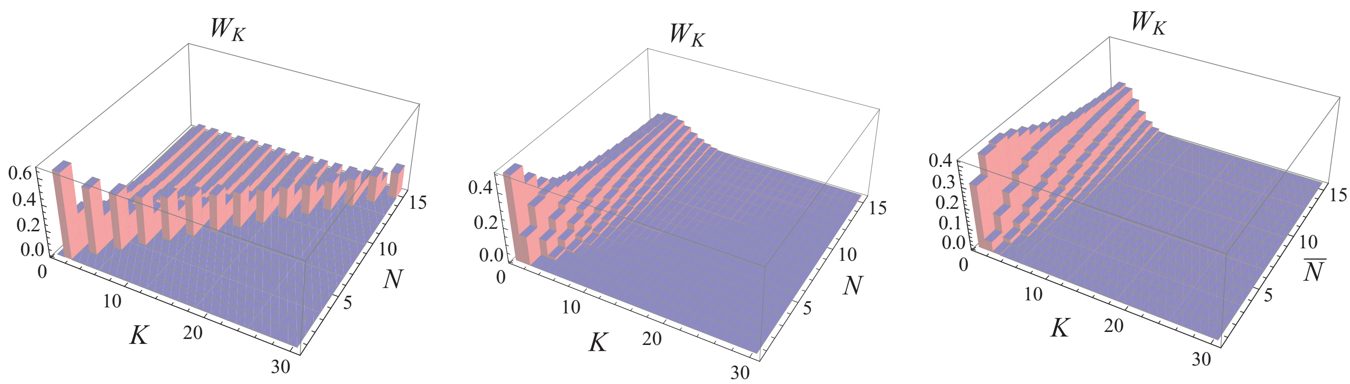

FIG. 1. (Color online) Distribution $\mathscr{W}_K$ as a function of the multipole order K for the examples in Table I. From left to right, the state $|S,0\rangle$ ($|N,N\rangle$, with $S=N$, in the basis $|n_H,n_V\rangle$), the SU(2) coherent state $|S;\theta,\phi\rangle$, and a two-mode quadrature coherent state with average number of photons $\bar{N}=|\alpha_H|^2+|\alpha_V|^2$.

coherent state the first multipoles contribute the most, whereas for the nonclassical $|S,0\rangle$ state the converse holds.

IV. RECONSTRUCTING THE MULTIPOLES

The analysis thus far confirms that multipoles constitute a natural tool to deal with polarization properties. We will show next that, in addition, they can be experimentally determined.

The polarization state is customarily analyzed with a Stokes measurement setup (see Fig. 2), consisting of a quater-wave plate (QWP) with the axis at angle ϕ, followed by a half-wave plate (HWP) at angle θ and a polarizing beam splitter (PBS) that separates the H and V modes. The wave plates effectively perform a displacement of the state that can be described by the operator $\hat{D}(\theta,\phi)=e^{i\theta\hat{S}_2}e^{i\phi\hat{S}_3}$, and (θ,ϕ) are angular coordinates on the sphere. Each of the two outputs of the PBS are measured by photon detectors: the photocurrent sum gives directly the eigenvalue of $\hat{N}$, while the difference gives the observable $\hat{S}_{\mathbf{n}}=\mathbf{n}\cdot\hat{\mathbf{S}}$, where $\mathbf{n}$ is the unit vector in the direction (θ,ϕ) [30].

Altogether, this indicates that the scheme yields the probability distribution for $\hat{S}_{\mathbf{n}}$, from which we can equivalently infer the moments

$$\mu_\ell^{(S)}(\theta,\phi)=\mathrm{Tr}\left[\hat{S}_{\mathbf{n}}^\ell\,\hat{\varrho}^{(S)}\right]. \tag{4.1}$$

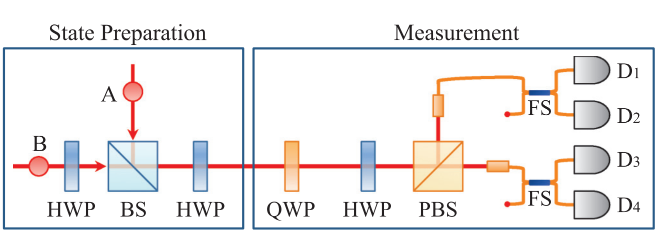

FIG. 2. (Color online) Experimental setup. Single photons A and B, both horizontally polarized, are prepared by spontaneous parametric down-conversion. (P)BS denotes a (polarization) beam splitter. HWP and QWP denote half-wave and quarter-wave plates, respectively. FS denotes a 50:50 fiber splitter, and D1–D4 denote single-photon avalanche photodiodes.

For simplicity, we restrict ourselves to a subspace with a fixed number of photons S, but everything can be smoothly extended to the whole polarization sector.

We start by noticing that the measurable moments can be expressed in terms of the state multipoles as

$$\begin{aligned}\mu_\ell^{(S)}(\theta,\phi)&=\mathrm{Tr}\left[\hat{S}_3^\ell\,\hat{D}(\theta,\phi)\,\hat{\varrho}^{(S)}\,\hat{D}^\dagger(\theta,\phi)\right]\\&=\mathrm{Tr}\left[\hat{S}_3^\ell\sum_{K=0}^{2S}\sum_{q,q'=-K}^{K}\varrho_{Kq}^{(S)}\,D_{qq'}^{K}(\theta,\phi)\,\hat{T}_{Kq}^{(S)}\right],\end{aligned} \tag{4.2}$$

where $D_{mm'}^S(\theta,\phi)=\langle S,m|\hat{D}(\theta,\phi)|S,m'\rangle$ is the Wigner D function [28]. To proceed further we need to compute

$$\begin{aligned}\mathrm{Tr}\left[\hat{S}_3^\ell\,\hat{T}_{Kq}^{(S)}\right]&=\delta_{q0}\left[\frac{S(S+1)(2S+1)}{3}\right]^{\ell/2}\frac{3^{\ell/2}\sqrt{2K+1}}{(2S+1)^{(\ell+1)/2}}\\&\quad\times\sum_{m=-S}^{S}\left(C_{Sm,10}^{Sm}\right)^\ell C_{Sm,K0}^{Sm}.\end{aligned} \tag{4.3}$$

Interestingly, we have that $C_{Sm,10}^{Sm}=m/\sqrt{S(S+1)}$ and

$$\sum_{m=-S}^{S}m^\ell C_{Sm,K0}^{Sm}=i^{\ell-K}\partial_\omega^\ell\chi_K^S(\omega)\big|_{\omega=0}\equiv f_{K\ell}^{(S)}\qquad K\leqslant\ell, \tag{4.4}$$

with $\chi_S^m(\omega)$ the generalized SU(2) character [28]. Collecting all those results together, the moments come out connected with the multipoles in quite an elegant way:

$$\mu_\ell^{(S)}(\theta,\phi)=\sqrt{\frac{4\pi}{2S+1}}\sum_{K=0}^{\ell}\sum_{q=-K}^{K}\varrho_{Kq}^{(S)}\,f_{K\ell}^{(S)}\,Y_{Kq}(\theta,\phi), \tag{4.5}$$

$Y_{Kq}(\theta,\phi)$ being the spherical harmonics.

We can benefit from the orthonormality of $Y_{Kq}(\theta,\phi)$ to integrate Eq. (4.5) so as to obtain

$$\varrho_{Kq}^{(S)}=\frac{1}{f_{K\ell}^{(S)}}\sqrt{\frac{2S+1}{4\pi}}\int_{\mathscr{S}^2}d\Omega\,\mu_\ell^{(S)}(\theta,\phi)\,Y_{Kq}^*(\theta,\phi), \tag{4.6}$$

where $K\leqslant\ell$ and the integral extends over the whole unit sphere $\mathscr{S}^2$ with $d\Omega=\sin\theta d\theta d\phi$ being the solid angle. The reconstruction of the state requires the knowledge of *all* the multipoles: this implies measuring *all* the moments in *all* the directions, which proves to be very demanding [16].

Nonetheless, we can attack the problem in a much more economic way. The central idea is that to determine the Kth multipole it is enough to perform a Stokes measurement in $2K+1$ independent directions. As a matter of fact, the proposal proceeds in a recurrent way: first, we measure the first-order moments in the three coordinate axis (or other equivalent ones) and reconstruct $\varrho_{1q}^{(S)}$. That is, from the values of $\mu_1^{(S)}(\theta,\phi)$, which can write down as

$$\mu_1^{(S)}(\theta,\phi) = f_{11}^{(S)} \sqrt{\frac{4\pi}{2S+1}} \sum_{q=-1}^{1} \varrho_{1q}^{(S)} Y_{1q}(\theta,\phi), \tag{4.7}$$

we need to know $\varrho_{1q}^{(S)}$. By taking into account that $f_{11}^{(S)} = (2S+1)\sqrt{S(S+1)}/3$, we can solve the resulting linear system, getting

$$\begin{pmatrix} \varrho_{11}^{(S)} \\ \varrho_{10}^{(S)} \\ \varrho_{1-1}^{(S)} \end{pmatrix} = \sqrt{\frac{3}{2S(S+1)(2S+1)}} \begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \begin{pmatrix} \mu_{1,1}^{(S)} \\ \mu_{1,2}^{(S)} \\ \mu_{1,3}^{(S)} \end{pmatrix}, \tag{4.8}$$

from which we infer all the first-order properties. Here $\mu_{1,k}^{(S)}$ indicate the first-order moment in the kth direction.

The measurement of the second moments gives us

$$\mu_2^{(S)}(\theta,\phi) = \frac{1}{2S+1} f_{02}^{(S)} + f_{22}^{(S)} \sqrt{\frac{4\pi}{2S+1}} \sum_{q=-2}^{2} \varrho_{2q}^{(S)} Y_{2q}(\theta,\phi), \tag{4.9}$$

with

$$\begin{aligned} f_{02}^{(S)} &= \tfrac{1}{3} S(S+1)(2S+1), \\ f_{22}^{(S)} &= \frac{4(2S+1)}{5!} \sqrt{S(2S-1)(S+1)(2S+3)}, \end{aligned} \tag{4.10}$$

while $f_{12}^{(S)} = 0$. We need to fix five optimal directions to invert that system. For example, thinking of the measurements as lines, we can choose the directions as

$$\mathbf{n}_{1,2} \propto \begin{pmatrix} 0 \\ \pm 2 \\ 1+\sqrt{5} \end{pmatrix}, \quad \mathbf{n}_{3,4} \propto \begin{pmatrix} \pm 2 \\ 1+\sqrt{5} \\ 0 \end{pmatrix}, \quad \mathbf{n}_{5} \propto \begin{pmatrix} 1+\sqrt{5} \\ 0 \\ 2 \end{pmatrix}, \tag{4.11}$$

which maximizes the minimum angle between the lines and thus in some sense spreads out the measurements over the Poincaré sphere as much as possible [31]. The system can be then solved, and all we need to characterize the process at second order is known.

For the Lth moment, we have

$$\boldsymbol{\mu}_L^{(S)} = \sqrt{\frac{4\pi}{2S+1}} f_{KL}^{(S)} \mathbf{Y}_K \, \boldsymbol{\varrho}_K^{(S)}, \tag{4.12}$$

where $\boldsymbol{\mu}_L^{(S)} = (\mu_L^{(S)}(\theta_1,\phi_1), \ldots, \mu_L^{(S)}(\theta_{2L+1},\phi_{2L+1}))$ and similarly for $\boldsymbol{\varrho}_K^{(S)}$ and $[\mathbf{Y}_L]_{ij} = Y_{Lj}(\theta_i,\phi_i)$. Observe that, in general, the right-hand side hinges on the results of lowest-order measurements. The linear inversion of that equation can be formally written down as

$$\boldsymbol{\varrho}_K^{(S)} = \frac{1}{f_{KL}^{(S)}} \sqrt{\frac{2S+1}{4\pi}} \frac{4\pi}{2L+1} \mathbf{P}_L^{-1} \mathbf{Y}_L^{\dagger} \boldsymbol{\mu}_L^{(S)}, \tag{4.13}$$

where $\mathbf{P}_L = 4\pi/(2L+1)\mathbf{Y}_L\mathbf{Y}_L^{\dagger}$, with $[\mathbf{P}_L]_{ij} = P_L(\omega_{ij})$, $\cos\omega_{ij} = \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \sin(\phi_i - \phi_j)$, and $P_L(\omega_{ij})$ is the Legendre polynomial. The choosing of the appropriate directions is, in general, a tricky question if one wants to be sure about the linear independence, but it has been thoroughly studied [32]. In practice, methods such as maximum likelihood are much more efficient in handling that inversion [33].

To check the proposed strategy, we have performed an experiment using spontaneous parametric down-conversion. The photon pairs centered at 780 nm were generated in a 2-mm-thick type-I β-barium-borate (BBO) crystal pumped by a femtosecond laser pulse centered at 390 nm and subsequently filtered by an interference filter with a 4-nm bandwidth and brought to the inputs of a Hong-Ou-Mandel interferometer. After the interferometer, either the state $|1_H,1_V\rangle$ or the state $|2_H,0_V\rangle$ can be postselected, depending on the relative polarizations of the incident photons.

The setup is sketched in Fig. 2. At each output of the PBS, a two-photon detector is simulated by a 50:50 fiber beam splitter (FS) and two single-photon detectors (PerkinElmer, SPCM-AQRH). The photon detection efficiency of each single-photon detector channel is used to calibrate the measurement of the Stokes parameters. To achieve full information about the first- and second-order moments, we have measured these coincidences in five distinct measurement bases and then reconstructed the multipoles via linear inversion. Each measurement is done for 3 s and repeated three times to improve the precision.

In Table II we summarize the results obtained for the state $|2_H,0_V\rangle$ (which is $|1,1\rangle$ in the angular momentum basis). The agreement with the theory is pretty good. Although this instance might look a bit naive, it constitutes quite a conclusive proof of principle of our method.

V. ASSESSING HIGHER-ORDER POLARIZATION CORRELATIONS

Even though the polarization information is encoded in the set $\{\mathscr{W}_K^{(S)}\}$, for most of the states only a limited number of multipoles play a substantive role and the rest of them have an exceedingly small contribution, so that gaining a good feeling of the corresponding behavior may be tricky.

A possible way to bypass this disadvantage is to look at the cumulative distribution

$$\mathscr{A}_K^{(S)} = \sum_{\ell=1}^{K} \mathscr{W}_\ell^{(S)}, \tag{5.1}$$

TABLE II. Experimental and theoretical results obtained for the state $|2_H,0_V\rangle$ (which is the $|1,1\rangle$ state in the angular momentum basis). The number in parentheses indicates the error in the last figure. The directions of measurement are the three coordinate axes for μ_1 and (4.11) for μ_2.

	Experiment		Theory			Experiment		Theory	
Direction	μ_1	μ_2	μ_1	μ_2	Multipole	$K=1$	$K=2$	$K=1$	$K=2$
1	−0.10 (3)	0.84 (7)	0	0.8618	ϱ_{K-2}		−0.01 (7) −0.01 (1) i		0
2	0.06 (2)	0.87 (1)	0	0.8618	ϱ_{K-1}	−0.05 (2) + 0.03 (1)i	0.07 (6) −0.02 (2) i	0	0
3	0.99 (3)	0.50 (2)	1	0.5000	ϱ_{K0}	0.70 (1)	0.39 (4)	0.7071	0.4082
4		0.52 (1)		0.5000	ϱ_{K1}	0.05 (2) + 0.03 (1) i	−0.07 (7) −0.02 (1) i	0	0
5		0.70 (2)		0.6382	ϱ_{K2}		−0.01 (1) +0.01 (1) i		0

which conveys the whole information *up* to order K. We know from probability that it has remarkable properties [34]. Moreover, our previous reconstruction puts in clear evidence that to obtain the Kth multipole one needs to determine all the previous moments.

As with any cumulative distribution, $\mathscr{A}_K^{(S)}$ is a monotone nondecreasing function of the multipole order, with $\mathscr{A}_{2S}^{(S)}$ being proportional to the state purity [except by the monopole contribution, $K=0$, which is not included in Eq. (5.1)]. One might be interested in dealing instead with magnitudes satisfying $0 \leqslant \mathbb{P}_K \leqslant 1$ for every K, as any sensible degree of polarization [35]. To that end, we note that for SU(2) coherent states we have

$$\mathscr{A}_{K,\mathrm{SU}(2)}^{(S)} = \frac{2S}{2S+1} - \frac{[\Gamma(2S+1)]^2}{\Gamma(2S-K)\Gamma(2S+K+2)}. \tag{5.2}$$

We conjecture that $\mathscr{A}_{K,\mathrm{SU}(2)}^{(S)}$ is indeed maximal for any K in each subspace S. This seems to suggest a degree of polarization up to the Kth order as

$$\mathbb{P}_K = \sum_S P_S \sqrt{\frac{\mathscr{A}_K^{(S)}}{\mathscr{A}_{K,\mathrm{SU}(2)}^{(S)}}}. \tag{5.3}$$

According to the definition Eq. (5.3), $\mathbb{P}_K = 1$ (for every K) for SU(2) coherent states, which is compatible with the idea that they are the most localized states over the sphere. On the other hand, for quadrature coherent states, which constitute an acid test for any new proposal in polarization, the result, as indicated in Table I, reads

$$\mathbb{P}_K = \sum_{S=K/2}^{\infty} \frac{e^{-\bar{N}}\bar{N}^{2S}}{(2S)!} \simeq \frac{1}{2}\mathrm{erfc}\left(\frac{K-\bar{N}}{\sqrt{2\bar{N}}}\right). \tag{5.4}$$

Here, $\bar{N}$ is the average number of photons, and the second equality, in terms of the complementary error function, holds true for $\bar{N} \gg 1$. From the properties of this function, we can estimate that the multipoles that contribute effectively are, roughly speaking, from 1 to $\bar{N}$. In Fig. 3 we plot $\mathbb{P}_K$ for the states $|S,0\rangle$ and $|\alpha_H,\alpha_V\rangle$.

To round off our understanding of $\mathbb{P}_K$, in Fig. 4 we have depicted $\mathbb{P}_K$ for two other relevant quantum states routinely treated in this context: NOON and two-mode squeezed vacuum states, defined as

$$\begin{aligned} |\mathrm{NOON}\rangle &= \frac{1}{\sqrt{2}}(|N,0\rangle + |0,N\rangle), \\ |\mathrm{TMSV}\rangle &= \sqrt{1-\lambda^2}\sum_N \lambda^N |N,N\rangle. \end{aligned} \tag{5.5}$$

To follow the standard notation, in both cases we have employed the $\{|n_H,n_V\rangle\}$ basis and $\lambda = \tanh r$, with r the squeezing parameter.

For the particular yet significant case of the dipole ($K=1$), Eq. (5.3) reduces to

$$\mathbb{P}_1 = \sum_S^{\infty} P_S \frac{\sqrt{\langle \hat{S}_1\rangle^2 + \langle \hat{S}_2\rangle^2 + \langle \hat{S}_3\rangle^2}}{\langle \hat{S}_0\rangle}, \tag{5.6}$$

and the average values are calculated in every subspace S. Interestingly, this definition has been recently proposed as a way to circumvent the shortcomings of the standard degree of

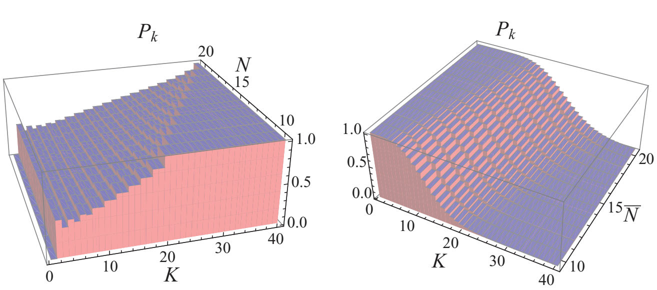

FIG. 3. (Color online) Degree of polarization $\mathbb{P}_K$ as a function of the multipole order K for the state $|S,0\rangle$ (left panel) and a quadrature coherent state $|\alpha_H,\alpha_V\rangle$ with average number of photons $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$ (right panel).

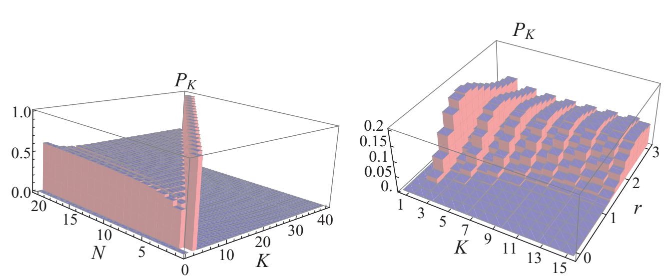

FIG. 4. (Color online) Degree of polarization $\mathbb{P}_K$ for the NOON (left panel) and two-mode squeezed vacuum (right panel) using the squeezing parameter r as a measure of the average number of photons.

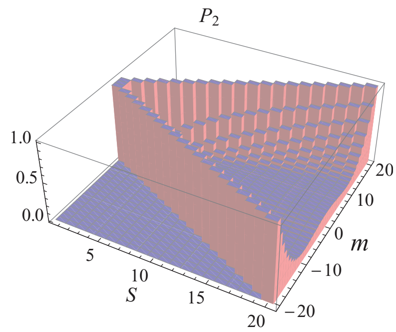

FIG. 5. (Color online) Second-order degree of polarization $\mathbb{P}_2$ for the state $|S,m\rangle$.

polarization [36]; in our approach, it emerges quite in a natural way.

To close our paper, we briefly consider the instance of $\mathbb{P}_2$. For two-mode quadrature coherent states $|\alpha_H,\alpha_V\rangle$ we immediately get

$$\mathbb{P}_2(|\alpha_H,\alpha_V\rangle) = 1 - (1+\bar{N})\exp(-\bar{N}), \tag{5.7}$$

which tends to the unity when the average number of photons $\bar{N}$ becomes large enough, in agreement with previous second-order approaches [25]. For the states $|S,m\rangle$, we have

$$\mathbb{P}_2(|S,m\rangle) = \sqrt{\frac{45m^4 + 5S^2(S+1)^2 - 9m^2[2S(S+1)+1]}{4S^2(2S-1)(4S+1)}}. \tag{5.8}$$

This expression is exactly unity whenever $m = \pm S$ or $\pm\sqrt{1+2S-3S^2}/\sqrt{5}$ (this equality is valid only when m is an integer). This latter condition is only met when $S=1$ with $m=0$.

On the other hand, $\mathbb{P}_2$ attains its minimum value

$$\mathbb{P}_{2,\min}(|S,m\rangle) = \sqrt{\frac{9+18S+8S^2}{80S^2}} \simeq \frac{1}{\sqrt{10}}, \tag{5.9}$$

whenever $m = \pm\sqrt{1+2S+2S^2}/\sqrt{10}$. In Fig. 5 we outline these facts.

VI. CONCLUDING REMARKS

Multipolar expansions are a commonplace and a formidable tool in many branches of physics. We have applied such an expansion to the polarization density matrix, showing how the corresponding state multipoles quantify higher-order fluctuations in the Stokes variables. In this way we have provided a systematic characterization of quantum polarization fluctuations that, paradoxically, was missing in the realm of quantum optics.

Moreover, the formalism can be manifestly extended to other systems in which SU(2) symmetry plays a crucial role (such as in Bose-Einstein condensates and spin chains) and to other unitary symmetries, such as SU(3) (which is pivotal to understanding the polarization properties of the near field). This is more than an academic curiosity, and work in this direction is ongoing in our group.

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APPENDIX: POLARIZATION QUASIDISTRIBUTIONS

The discussion in this paper suggests that polarization must be specified by a probability distribution of polarization states. As a matter of fact, such a probabilistic description is unavoidable in quantum optics from the very beginning, since $\{\hat{S}_k\}$ do not commute and thus no state can have a definite value of all of them simultaneously.

The SU(2) symmetry inherent in the polarization structure of quantum fields allows us to take advantage of the pioneering work of Stratonovich [37] and Berezin [38], who worked out quasiprobability distributions on the sphere satisfying all the pertinent requirements. This construction was later generalized by others [39–43] and has proved to be very useful in visualizing properties of spinlike systems [44–46].

For each partial $\hat{\varrho}^{(S)}$, one can define r-parametrized SU(2) quasidistributions as

$$W_r^{(S)}(\theta,\phi) = \frac{\sqrt{4\pi}}{\sqrt{2S+1}} \sum_{K=0}^{2S} \sum_{q=-K}^{K} \left(C_{SS,K0}^{SS}\right)^{-r} \varrho_{Kq}^{(S)}\, Y_{Kq}^{*}(\theta,\phi). \tag{A1}$$

For $r=0$ this is the Wigner function, while $r=+1$ and -1 lead to the P and Q functions, respectively. Note also that the Clebsch-Gordan coefficient $C_{SS,K0}^{SS}$ has a very simple analytical form [28]:

$$C_{SS,K0}^{SS} = \frac{\sqrt{2S+1}(2S)!}{\sqrt{(2S-K)!\,(2S+1+K)!}}. \tag{A2}$$

While, for spins, S is typically a fixed number, in quantum optics most of the states involve a full polarization sector and one should sum over the subspaces contributing to the state.

The integral

$$\Sigma = \frac{1}{\int d\Omega \left[W_r^{(S)}(\theta,\phi)\right]^2}, \tag{A3}$$

extended to the whole sphere, can be interpreted as the effective area where the corresponding quasidistribution is different from zero. In other words, Σ is a measure of the number of polarization states contained in a given field state. This and similar definitions have already been used as measures of localization and uncertainty in different contexts [47].

Using the explicit form of Eq. (A1) we immediately get

$$\int d\Omega \left[W_r^{(S)}(\theta,\phi)\right]^2 = \frac{4\pi}{2S+1} \sum_{K=0}^{2S} \left(C_{SS,K0}^{SS}\right)^{-2r} \mathscr{W}_K^{(S)}. \tag{A4}$$

We can appreciate a deep connection (except for the unessential Clebsch-Gordan coefficient) between the distribution $\{\mathscr{W}_K^{(S)}\}$ and the notion of localization in phase space. In particular, for the Wigner function $r = 0$, the right-hand side of Eq. (A4) is giving information about the measured $\{\mathscr{W}_K^{(S)}\}$.

For the sake of completeness, we briefly recall the definition of the SU(2) coherent states (also known as spin or atomic coherent states), which reads [48,49]

$$|S;\theta,\phi\rangle = \hat{D}(\theta,\phi)|S,-S\rangle. \tag{A5}$$

Here $\hat{D}(\theta,\phi) = \exp(\xi\hat{S}_+ - \xi^*\hat{S}_-)$ [with $\xi = (\theta/2)\exp(-i\phi)$ and (θ,ϕ) being spherical angular coordinates] plays the role of a displacement on the Poincaré sphere of radius S.

The ladder operators $\hat{S}_\pm = \hat{S}_1 \pm i\hat{S}_2$ select the fiducial state $|S,-S\rangle$ as usual: $\hat{S}_-|S,-S\rangle = 0$. This definition closely mimics its standard counterpart for position and momentum.

Note that these coherent states are eigenstates of the measured operator $\hat{S}_{\mathbf{n}} = \mathbf{n}\cdot\hat{\mathbf{S}}$,

$$\hat{S}_{\mathbf{n}}|S;\theta,\phi\rangle = S|S;\theta,\phi\rangle, \tag{A6}$$

and they saturate the uncertainty relation Eq. (2.3), so they are the minimum uncertainty states in polarization optics.

The two-mode quadrature coherent states $|\alpha_H,\alpha_V\rangle$ can be expressed as a Poissonian superposition of SU(2) coherent states:

$$|\alpha_H,\alpha_V\rangle = \sum_S \frac{\bar{N}^{2S}e^{-\bar{N}}}{(2S)!}|2S,\theta,\phi\rangle, \tag{A7}$$

where $\bar{N} = |\alpha_H|^2 + |\alpha_V|^2$ is the average number of photons.

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