Landau damping in a collisionless dipolar Bose gas

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We present a theory for the Landau damping of low-energy quasiparticles in a collisionless, quasi-two-dimensional dipolar Bose gas and produce expressions for the damping rate in uniform and nonuniform systems. Using simple energy-momentum conservation arguments, we show that in the homogeneous system, the nature of the low-energy dispersion in a dipolar Bose gas severely inhibits Landau damping of long wavelength excitations. For a gas with contact and dipolar interactions, the damping rate for phonons tends to *decrease* with increasing dipolar interactions; for strong dipole-dipole interactions, phonons are virtually undamped over a broad range of temperature. The damping rate for maxon-roton excitations is found to be significantly larger than the damping rate for phonons.

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I. INTRODUCTION

The trapping and cooling of highly magnetic atoms such as ^{52}Cr, ^{162}Dy, ^{164}Dy, and ^{168}Er [1–4] has allowed experimentalists to create Bose-Einstein condensates with strong dipole-dipole interactions [5]. The long-range character of these interactions leads to phenomena such as geometry dependent mechanical stability [6], anisotropic superfluid velocity [7], novel topological defects, and vortex lattice configurations upon rotation [8,9]. A key ingredient responsible for some of the new physics of dipolar Bose gases is the low-energy excitation spectrum, which has a phonon-roton character in reduced dimensions [10–12], resembling that of superfluid He-4 [13–15]. Here we present a theory to describe the damping of these excitations at finite temperature, where the damping is provided by interactions between low-lying excitations, mediated by the underlying Bose condensate.

The dynamics of Bose condensed gases with short-range interactions is well studied [16–28] at zero and finite temperature, in homogeneous as well as trapped gases. In the homogeneous case, there is a continuum of excitations characterized by the familiar Bogoliubov dispersion, which contains linearly dispersing phonons at energies smaller than the chemical potential μ, and free-particle-like excitations at energies larger than μ [29]. At low temperatures and low densities, such as those realized in cold atomic gases, the mean free path of the excitations is comparable to the system size and the system enters the "collisionless" regime, where the characteristic excitation frequency is much larger than the typical relaxation rate due to collisions $\omega\tau \gg 1$. In this limit, interactions between quasiparticles are primarily mediated by the Bose condensate. A given excitation can decay into two excitations with lower energy, a process known as Beliaev damping [16], which dominates at low temperatures $k_B T \ll \mu$

[30]. At higher temperatures, an excitation can also decay by near resonant coupling to transitions between other elementary excitations, a mechanism known as Landau damping. In the context of the dilute Bose gas, Landau damping was first discussed in the low temperature limit by Hohenberg and Martin [31], and at higher temperatures by Szépfalusy and Kondor [17]. A systematic theory of Landau damping in a dilute Bose gas covering both temperature regimes was later provided by Pitaevskii and Stringari [24]. In the opposite limit, where the typical collision time is small, $\omega\tau \ll 1$, the damping of collective excitations is governed by two-fluid hydrodynamics [32].

In trapped systems, the spectrum of low-lying modes is discrete, and the frequencies obtained experimentally match theoretical predictions based on superfluid hydrodynamics [33–36]. At higher energies and temperatures, the dynamics are collision dominated, and the temperature dependence of the excitation frequencies and the associated damping rates has been studied in Refs. [18,26,27,37]. At lower energies, the discretization of the collective modes forbids Beliaev damping altogether, and the temperature dependence of the damping rates observed experimentally [36,37] has been found to be consistent with Landau damping [20,21,23,25].

In recent years, the focus has shifted to understanding the properties of Bose condensed gases with long-range interactions. At low temperatures, the dynamics of such a condensate is well described by a nonlocal Gross-Pitaevskii equation, which was successfully employed in describing the anisotropic collapse of a dipolar ^{52}Cr condensate [6]. The spectrum of excitations in the dipolar gas has been established in homogeneous and harmonically confined systems [10–12,38–40].

In this work, we study the Landau damping of excitations in a quasi-two-dimensional (quasi-2D) dipolar Bose gas. We follow the time-dependent mean-field formalism developed by Giorgini [19,20], which describes the coupled dynamics of the condensed and noncondensed clouds interacting with short-range interactions. Generalizing this formalism to include

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long-range interactions, we obtain the Landau damping rate in a dipolar gas. We remark that the zero-temperature version of this theory yields the Beliaev damping rate, which is studied in detail in Refs. [41,42].

We find that the Landau damping of low-energy, long wavelength excitations is *exponentially* suppressed at low temperatures. This is in departure from the T^2 dependence of the damping rate expected in 2D systems with short-range interactions [22]. Surprisingly we find that the phonon damping rate *decreases* with increasing dipolar interactions, and for sufficiently large dipolar interactions, phonons are virtually undamped at finite temperature. By contrast, the damping rate for roton-maxon excitations is significantly higher than the phonon damping rate.

This paper is organized as follows: In Sec. II we describe the time-dependent mean-field scheme used to compute the damping rates, and obtain formulas for the Landau damping rate in homogeneous and trapped systems in the presence of long-range interactions. In Sec. III, we specialize to the case of a homogeneous, quasi-2D dipolar gas, and compute the Landau damping rate for phonons at low and high temperatures. To highlight the role of the low-energy dispersion in determining the damping rates, we compare the phonon damping rate for a purely dipolar gas with that of a gas with purely contact interactions. We then consider the case where both dipolar and contact interactions are present. In Sec. IV we numerically calculate the Landau damping rate at intermediate momenta, where the low-energy dispersion relation develops a maxon-roton feature. We discuss the experimental implications of our work in Sec. V, and summarize our results in Sec. VI.

II. FORMALISM

The Hamiltonian for a Bose gas interacting with a potential $V_{\text{tot}}(\mathbf{r} - \mathbf{r}')$ is given by

$$
\mathcal{H} = \int d\mathbf{r} \, \Psi^\dagger(\mathbf{r},t) \left[-\frac{\hbar^2 \nabla^2}{2m} - \mu(\mathbf{r}) \right] \Psi(\mathbf{r},t)
$$
$$
+ \frac{1}{2} \int d\mathbf{r} \, d\mathbf{r}' V_{\text{tot}}(\mathbf{r} - \mathbf{r}') \Psi^\dagger(\mathbf{r},t) \Psi^\dagger(\mathbf{r}',t) \Psi(\mathbf{r}',t) \Psi(\mathbf{r},t),
$$
$$(1)$$

where $\Psi(\mathbf{r},t)$ is the bosonic annihilation operator at position $\mathbf{r}$ and time t, m is the mass, and $\mu(\mathbf{r})$ is the spatially varying chemical potential. In the presence of a trapping potential $U(\mathbf{r})$, $\mu(\mathbf{r}) = \mu_0 - U(\mathbf{r})$, where μ_0 is the chemical potential in the center of the trap. The interaction potential is assumed to be a sum of two terms $V_{\text{tot}}(\mathbf{r} - \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}') + V_{\text{lr}}(\mathbf{r} - \mathbf{r}')$, where $g = \frac{4\pi\hbar^2 a}{m}$ parametrizes the contact (short-range) part of the potential with a three-dimensional (3D) *s*-wave scattering length a, and all the long-range terms are included in $V_{\text{lr}}(\mathbf{r} - \mathbf{r}')$. Throughout we will assume that the potential satisfies $V_{\text{tot}}(\mathbf{r}) = V_{\text{tot}}(-\mathbf{r})$. For dipolar bosons with dipole moments d, which are oriented along the z axis [5],

$$
V_{\text{lr}}(\mathbf{r} - \mathbf{r}') = d^2 \frac{1 - 3\cos^2(\theta)}{|\mathbf{r} - \mathbf{r}'|^3}, \tag{2}
$$

where θ is the angle between the vector $\mathbf{r} - \mathbf{r}'$ and the z axis.

The equation of motion for Ψ is obtained using the Heisenberg prescription:

$$
i\hbar\partial_t \Psi(\mathbf{r},t) = \left[\left(-\frac{\hbar^2 \nabla^2}{2m} - \mu(\mathbf{r}) \right) \right.
$$
$$
\left. + \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r} - \mathbf{r}') |\Psi(\mathbf{r}',t)|^2 \right] \Psi(\mathbf{r},t). \tag{3}
$$

At temperatures below the Bose-condensation temperature, the bosonic annihilation operator can be decomposed as a sum of two parts: $\Psi(\mathbf{r},t) = \phi(\mathbf{r},t) + \psi(\mathbf{r},t)$ where $\phi(\mathbf{r},t) = \langle \Psi(\mathbf{r},t) \rangle$ is a complex order parameter representing the condensate, and $\psi(\mathbf{r},t)$ represents the noncondensed atoms, which by definition have the property $\langle \psi(\mathbf{r},t) \rangle = 0$. The averages here in general denote nonequilibrium averages, as we allow for fluctuations of the condensed and noncondensed atoms. Equilibrium averages will be denoted by $\langle \cdots \rangle_0$.

Decomposing the term proportional to $\Psi^\dagger \Psi^\dagger \Psi$ in Eq. (3) into condensate and noncondensate contributions, the equation of motion for the condensate field reads

$$
i\hbar\partial_t \phi(\mathbf{r},t) = -\left(\frac{\hbar^2 \nabla^2}{2m} + \mu(\mathbf{r}) \right) \phi(\mathbf{r},t) + \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r} - \mathbf{r}')
$$
$$
\times [|\phi(\mathbf{r}',t)|^2 \phi(\mathbf{r},t) + \phi(\mathbf{r}',t) n(\mathbf{r}',\mathbf{r},t)
$$
$$
+ \phi^*(\mathbf{r}',t) m(\mathbf{r}',\mathbf{r},t) + n(\mathbf{r}',t)\phi(\mathbf{r},t)], \tag{4}
$$

where we have introduced normal and anomalous densities

$$
n(\mathbf{r}',\mathbf{r},t) = \langle \psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t) \rangle,
$$
$$
m(\mathbf{r}',\mathbf{r},t) = \langle \psi(\mathbf{r}',t)\psi(\mathbf{r},t) \rangle, \tag{5}
$$

and we use the shorthand notation $n(\mathbf{r},t) = \langle \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) \rangle$ and $m(\mathbf{r},t) = \langle \psi(\mathbf{r},t)\psi(\mathbf{r},t) \rangle$. Setting n and m to zero in Eq. (4), one obtains a nonlocal Gross-Pitaevskii equation (GPE), which describes the dynamics of a Bose condensate at zero temperature interacting with long-range interactions [5,6].

In general we are interested in small, time-dependent fluctuations of the condensate about its equilibrium value, which is obtained by solving the GPE in equilibrium. To describe the fluctuations, we expand $\phi(\mathbf{r},t) = \phi_0(\mathbf{r}) + \delta\phi(\mathbf{r},t)$, where $\phi_0(\mathbf{r})$ is the equilibrium condensate density, which we assume without any loss of generality to be real [43], and $\delta\phi(\mathbf{r},t)$ corresponds to small, time-dependent fluctuations about the stationary state. At finite temperature, the fluctuations of the condensate are coupled to fluctuations of the noncondensate densities, which can be written in the form $n(\mathbf{r},\mathbf{r}',t) = n_{\text{eq}}(\mathbf{r},\mathbf{r}') + \delta n(\mathbf{r},\mathbf{r}',t)$ and $m(\mathbf{r},\mathbf{r}',t) = \delta m(\mathbf{r},\mathbf{r}',t)$. Here we assume that the anomalous density of the thermal cloud is zero in equilibrium [$m_{\text{eq}}(\mathbf{r},t) = \langle \psi(\mathbf{r},t)\psi(\mathbf{r},t) \rangle_0 = 0$], an approximation due to Popov [30]. As discussed by Giorgini [19,20], this approximation yields an inadequate result for the *real* part of the phonon frequency shift, but leaves the imaginary part (which contributes to damping) unchanged. A detailed description of the Popov approximation and its implications can be found in Ref. [32].

Linearizing Eq. (4) in the Popov approximation, the equation of motion for the fluctuations of the condensate

reads

$$i\partial_t \delta\phi(\mathbf{r},t) = \hat{\mathcal{L}}_0 \delta\phi(\mathbf{r},t) + \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')\{\phi_0(\mathbf{r})\phi_0(\mathbf{r}')$$
$$\times [\delta\phi^*(\mathbf{r}',t) + \delta\phi(\mathbf{r}',t)] + n_0(\mathbf{r}',\mathbf{r})\delta\phi(\mathbf{r}',t)$$
$$+ \phi_0(\mathbf{r}')[\delta n(\mathbf{r}',\mathbf{r},t) + \delta m(\mathbf{r}',\mathbf{r},t)]$$
$$+ \phi_0(\mathbf{r})\delta n(\mathbf{r}',t)\}, \tag{6}$$

where we define the operator

$$\hat{\mathcal{L}}_0 \equiv -\frac{\hbar^2 \nabla^2}{2m} - \mu(\mathbf{r}) + \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')n_{\text{tot}}(\mathbf{r}') \tag{7}$$

and $n_{\text{tot}}(\mathbf{r}) = |\phi_0(\mathbf{r})|^2 + n_{\text{eq}}(\mathbf{r})$ is the total density. In general we define $n_{\text{tot}}(\mathbf{r}',\mathbf{r}) = \phi_0(\mathbf{r}')\phi_0(\mathbf{r}) + n_{\text{eq}}(\mathbf{r}',\mathbf{r})$.

The corresponding Hamiltonian for the noncondensed atoms can be decomposed into two parts:

$$\mathcal{H}_{\text{nc}} = \mathcal{H}_0 + \mathcal{H}_\delta, \tag{8}$$

where

$$\mathcal{H}_0 = \int d\mathbf{r}\, d\mathbf{r}'\{\psi^\dagger(\mathbf{r},t)\hat{\mathcal{L}}_0\psi(\mathbf{r},t) + V_{\text{tot}}(\mathbf{r}-\mathbf{r}')n_{\text{tot}}(\mathbf{r}',\mathbf{r})$$
$$\times [\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t) + \psi^\dagger(\mathbf{r},t)\psi(\mathbf{r}',t)] + \phi_0(\mathbf{r})\phi_0(\mathbf{r}')$$
$$\times [\psi^\dagger(\mathbf{r}',t)\psi^\dagger(\mathbf{r},t) + \psi(\mathbf{r},t)\psi(\mathbf{r}',t)]\} \tag{9}$$

in Popov approximation. The Hamiltonian $\mathcal{H}_0$ couples the noncondensed atoms to the equilibrium condensate and noncondensate densities, and does not contain any condensate fluctuations.

The latter are described by the Hamiltonian $\mathcal{H}_\delta$, containing terms linear in $\delta\phi$:

$$\mathcal{H}_\delta = \frac{1}{2}\int d\mathbf{r}\, d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')[\phi_0(\mathbf{r})\{[\delta\phi(\mathbf{r},t)+\delta\phi^*(\mathbf{r},t)]$$
$$\times \psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r}',t) + \delta\phi(\mathbf{r}',t)\psi^\dagger(\mathbf{r}',t)\psi(\mathbf{r},t)$$
$$+ \delta\phi^*(\mathbf{r}',t)\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r}',t) + \delta\phi(\mathbf{r}',t)\psi^\dagger(\mathbf{r},t)\psi^\dagger(\mathbf{r}',t)$$
$$+ \delta\phi^*(\mathbf{r}',t)\psi(\mathbf{r}',t)\psi(\mathbf{r},t)\} + \mathbf{r} \leftrightarrow \mathbf{r}'], \tag{10}$$

where $\mathbf{r} \leftrightarrow \mathbf{r}'$ above refers to interchanging $\mathbf{r}$ and $\mathbf{r}'$ for every term in the preceding curly brackets. In the weakly interacting gas, the condensate density greatly exceeds the density of the noncondensed atoms ($|\phi_0|^2 \gg n_{\text{eq}}$). As a result, interactions between the noncondensed atoms are mainly mediated by the condensate, and therefore we have ignored terms involving only noncondensed atoms (all terms in $\mathcal{H}_\delta$ containing four noncondensed field operators).

The Hamiltonian $\mathcal{H}_0$ given by Eq. (9) can be diagonalized via the usual Bogoliubov transformation:

$$\psi(\mathbf{r},t) = \sum_i u_i(\mathbf{r})a_i(t) + v_i^*(\mathbf{r})a_i^\dagger(t),$$
$$\psi^\dagger(\mathbf{r},t) = \sum_i u_i^*(\mathbf{r})a_i^\dagger(t) + v_i(\mathbf{r})a_i(t), \tag{11}$$

where $a_i(t)$ denotes the bosonic quasiparticle annihilation operator at time t, and satisfies the usual bosonic commutation relations [29]. The complex functions u_i and v_i obey $\int d\mathbf{r}\, d\mathbf{r}'[u_i^*(\mathbf{r})u_j(\mathbf{r}') - v_i^*(\mathbf{r})v_j(\mathbf{r}')] = \delta_{ij}\delta(\mathbf{r}-\mathbf{r}')$, where δ_{ij} is the Kronecker delta symbol.

The resulting Hamiltonian describes a noninteracting gas of Bogoliubov quasiparticles:

$$\mathcal{H}_0 = \sum_i E_i a_i^\dagger a_i, \tag{12}$$

where E_i are the quasiparticle energies obtained by solving the Bogoliubov equations:

$$E_i u_i(\mathbf{r}) = \hat{\mathcal{L}}[u_i(\mathbf{r})] + \hat{\mathcal{M}}[v_i(\mathbf{r})],$$
$$-E_i v_i(\mathbf{r}) = \hat{\mathcal{L}}[v_i(\mathbf{r})] + \hat{\mathcal{M}}[u_i(\mathbf{r})], \tag{13}$$

where we define

$$\hat{\mathcal{L}}[f(\mathbf{r})] = \hat{\mathcal{L}}_0 f(\mathbf{r}) + \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')n_{\text{tot}}(\mathbf{r}',\mathbf{r})f(\mathbf{r}'),$$
$$\hat{\mathcal{M}}[f(\mathbf{r})] = \int d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')\phi_0(\mathbf{r})\phi_0(\mathbf{r}')f(\mathbf{r}'). \tag{14}$$

A key difference between the Bogoliubov equations above and those for a gas interacting with purely contact interactions arises in the nonlocal term in the definition of $\hat{\mathcal{L}}[f(\mathbf{r})]$ and $\hat{\mathcal{M}}[f(\mathbf{r})]$. This term represents the "exchange" (Fock) contribution to the interaction. For a gas with short-range interactions, this term is identical to the "direct" (Hartree) contribution, and together they give rise to a factor of 2 in the interaction term. In the presence of long-range interactions, these contributions must be treated separately, and the problem of solving the Bogoliubov equations becomes computationally intensive [44–46].

The equation for the fluctuations of the condensate, Eq. (6), can be expressed in terms of the Bogoliubov operators a_i and $a_i^\dagger$ by expanding the normal and anomalous densities as follows:

$$n(\mathbf{r}',\mathbf{r},t) = \sum_{ij} u_i^*(\mathbf{r}')u_j(\mathbf{r})f_{ij} + v_i(\mathbf{r}')v_j^*(\mathbf{r})f_{ji}$$
$$+ u_i^*(\mathbf{r}')v_j^*(\mathbf{r})g_{ij}^* + v_i(\mathbf{r}')u_j(\mathbf{r})g_{ij} + v_i(\mathbf{r}')v_j^*(\mathbf{r}) \tag{15}$$

and

$$m(\mathbf{r}',\mathbf{r},t) = \sum_{ij} u_i(\mathbf{r}')v_j^*(\mathbf{r})f_{ji} + v_i^*(\mathbf{r}')u_j(\mathbf{r})f_{ij}$$
$$+ u_i(\mathbf{r}')u_j(\mathbf{r})g_{ij} + v_i^*(\mathbf{r}')v_j^*(\mathbf{r})g_{ij}^* + u_i(\mathbf{r}')v_j^*(\mathbf{r}), \tag{16}$$

where it is convenient to define the functions:

$$f_{ij}(t) \equiv \langle a_i^\dagger(t)a_j(t)\rangle,$$
$$g_{ij}(t) \equiv \langle a_i(t)a_j(t)\rangle. \tag{17}$$

The equation for the fluctuations of the condensate [Eq. (6)] is thus coupled to the dynamics of the noncondensed atoms via f_{ij} and g_{ij}. Physically, f_{ij} corresponds to the creation and annihilation of a quasiparticle with energies E_i and E_j, and contributes to Landau damping, while g_{ij} (g_{ij}^*) corresponds to the annihilation (creation) of two quasiparticles with energies E_i and E_j, and contributes to Beliaev damping [19]. Henceforth we focus on the dynamics of f_{ij} alone.

The equation for the fluctuations of the condensate [Eq. (6)] can be written in terms of $f_{ij}(t)$ as

$$i\hbar\partial_t\delta\phi(\mathbf{r},t) = \hat{\mathcal{L}}[\delta\phi(\mathbf{r},t)] + \hat{\mathcal{M}}[\delta\phi^*(\mathbf{r})] + \sum_{ij} f_{ij}(t)\int d\mathbf{r}' V_{\text{tot}}(\mathbf{r}-\mathbf{r}')(\phi_0(\mathbf{r}')[u_i^*(\mathbf{r}')u_j(\mathbf{r}) + v_j(\mathbf{r}')v_i^*(\mathbf{r})$$

$$+ v_i^*(\mathbf{r})u_j(\mathbf{r}') + v_i^*(\mathbf{r}')u_j(\mathbf{r})] + \phi_0(\mathbf{r})[u_i^*(\mathbf{r}')u_j(\mathbf{r}') + v_i(\mathbf{r}')v_j^*(\mathbf{r}')]) + \cdots. \tag{18}$$

The $\cdots$ at the end of Eq. (18) refers to the terms proportional to g_{ij} that give rise to Beliaev damping, which we do not study here.

Using Heisenberg's equation of motion, $i\hbar\partial_t\langle\hat{O}\rangle = \langle[O,\mathcal{H}_{\text{nc}}]\rangle$, the equation of motion for f_{ij} reads

$$i\hbar\partial_t f_{ij}(t) = (\epsilon_j - \epsilon_i)f_{ij}(t) + \frac{(n_i^{\text{th}} - n_j^{\text{th}})}{2}\int d\mathbf{r}\, d\mathbf{r}'\, V_{\text{tot}}(\mathbf{r}-\mathbf{r}')[2\phi_0(\mathbf{r})[\delta\phi(\mathbf{r},t) + \delta\phi^*(\mathbf{r},t)]\{u_j^*(\mathbf{r}')u_i(\mathbf{r}') + v_i^*(\mathbf{r}')v_j(\mathbf{r}')\}$$

$$+ 2[\phi_0(\mathbf{r})\delta\phi(\mathbf{r}',t) + \phi_0(\mathbf{r}')\delta\phi^*(\mathbf{r},t)]\{u_j^*(\mathbf{r}')u_i(\mathbf{r}) + v_j(\mathbf{r}')v_i^*(\mathbf{r})\} + [\phi_0(\mathbf{r})\delta\phi(\mathbf{r}',t) + \phi_0(\mathbf{r}')\delta\phi(\mathbf{r},t)]$$

$$\times \{u_j^*(\mathbf{r})v_i(\mathbf{r}') + v_i(\mathbf{r})u_j^*(\mathbf{r}')\} + [\phi_0(\mathbf{r})\delta\phi^*(\mathbf{r}',t) + \phi_0(\mathbf{r}')\delta\phi^*(\mathbf{r},t)]\{u_i(\mathbf{r})v_j^*(\mathbf{r}') + v_j^*(\mathbf{r})u_i(\mathbf{r}')\}] \tag{19}$$

where we define the equilibrium density of the quasiparticles at temperature T, $n_i^{\text{th}} = \langle a_i^\dagger a_i\rangle_0 = \frac{1}{e^{\beta E_i}-1}$, where $\beta = 1/k_B T$.

Thus the fluctuations of the condensate are coupled to the fluctuations of the noncondensed atoms and vice versa. The full self-consistent solution of this system of equations is highly nontrivial. In order to solve the coupled system of equations we assume that the condensate fluctuates with frequency ω as $\delta\phi(\mathbf{r},t) = \delta\phi_1(\mathbf{r})e^{-i\omega t}$ and $\delta\phi^*(\mathbf{r},t) = \delta\phi_2(\mathbf{r})e^{-i\omega t}$ and take the Fourier transform of Eq. (19) to get

$$f_{ij}(\omega) = \frac{(n_i^{\text{th}} - n_j^{\text{th}})}{2[\hbar\omega - (\epsilon_j - \epsilon_i)]}\int d\mathbf{r}\, d\mathbf{r}'\, V_{\text{tot}}(\mathbf{r}-\mathbf{r}')[2\phi_0(\mathbf{r})[\delta\phi_1(\mathbf{r}) + \delta\phi_2(\mathbf{r})]\{u_j^*(\mathbf{r}')u_i(\mathbf{r}') + v_i^*(\mathbf{r}')v_j(\mathbf{r}')\}$$

$$+ 2[\phi_0(\mathbf{r})\delta\phi_1(\mathbf{r}') + \phi_0(\mathbf{r}')\delta\phi_2(\mathbf{r})]\{u_j^*(\mathbf{r}')u_i(\mathbf{r}) + v_j(\mathbf{r}')v_i^*(\mathbf{r})\} + [\phi_0(\mathbf{r})\delta\phi_1(\mathbf{r}') + \phi_0(\mathbf{r}')\delta\phi_1(\mathbf{r})]$$

$$\times \{u_j^*(\mathbf{r})v_i(\mathbf{r}') + v_i(\mathbf{r})u_j^*(\mathbf{r}')\} + [\phi_0(\mathbf{r})\delta\phi_2(\mathbf{r}') + \phi_0(\mathbf{r}')\delta\phi_2(\mathbf{r})]\{u_i(\mathbf{r})v_j^*(\mathbf{r}') + v_j^*(\mathbf{r})u_i(\mathbf{r}')\}]. \tag{20}$$

In the absence of any coupling between the condensed and noncondensed atoms, the fluctuations of the condensate [Eq. (18)] obey the Bogoliubov equations with real frequency ω_0:

$$\hbar\omega_0\begin{pmatrix}\delta\phi_1^0(\mathbf{r})\\\delta\phi_2^0(\mathbf{r})\end{pmatrix} = \begin{pmatrix}\hat{\mathcal{L}} & \hat{\mathcal{M}}\\-\hat{\mathcal{M}}^* & -\hat{\mathcal{L}}^*\end{pmatrix}\begin{pmatrix}\delta\phi_1^0(\mathbf{r})\\\delta\phi_2^0(\mathbf{r})\end{pmatrix} \tag{21}$$

and satisfy the normalization criterion $\int d\mathbf{r}(|\delta\phi_1^0(\mathbf{r})|^2 - |\delta\phi_2^0(\mathbf{r})|^2) = 1$.

In the limit of small noncondensate density, we may treat the condensate-noncondensed atom coupling as a perturbation, and write $\delta\phi_{1,2} = \delta\phi_{1,2}^0 + \delta\phi_{1,2}'$, where the corrections to the oscillation amplitude are chosen to be orthogonal to those in the absence of condensate-noncondensate coupling:

$$\int d\mathbf{r}(\delta\phi_1^0(\mathbf{r})\delta\phi_1^{*\prime}(\mathbf{r}) - \delta\phi_2^0(\mathbf{r})\delta\phi_2^{*\prime}(\mathbf{r})) = 0. \tag{22}$$

Similarly, we write the perturbed eigenfrequency as $\omega = \omega_0 + \delta\omega - i\Gamma_L$, where $\delta\omega$ is the correction to the real part of the normal mode eigenfrequency and Γ_L denotes the Landau damping coefficient.

To lowest order, the equation governing the fluctuations of the condensate in the presence of coupling to the noncondensed atoms reads

$$\hbar\omega\delta\phi_1(\mathbf{r}) = \hat{\mathcal{L}}[\delta\phi_1^0(\mathbf{r})] + \hat{\mathcal{M}}[\delta\phi_2^0(\mathbf{r})] + \sum_{ij} f_{ij}(\omega_0)A_{ij}(\mathbf{r}),$$

$$\tag{23}$$

where we have replaced all the $\delta\phi$ and $\delta\phi^*$ terms in the right-hand side, including in the expression for f_{ij} [Eq. (20)] with

$\delta\phi_1^0$ and $\delta\phi_2^0$, and ω with ω_0. We also introduce the shorthand

$$A_{ij}(\mathbf{r}) = \int d\mathbf{r}'\, V_{\text{tot}}(\mathbf{r}-\mathbf{r}')(\phi_0(\mathbf{r}')[u_i^*(\mathbf{r}')u_j(\mathbf{r}) + v_j(\mathbf{r}')v_i^*(\mathbf{r})$$

$$+ u_i(\mathbf{r}')v_j^*(\mathbf{r}) + u_i(\mathbf{r})v_j^*(\mathbf{r}')]$$

$$+ \phi_0(\mathbf{r})[u_i^*(\mathbf{r}')u_j(\mathbf{r}') + v_i(\mathbf{r}')v_j^*(\mathbf{r}')]). \tag{24}$$

To obtain the damping rate, we multiply Eq. (23) and the corresponding equation for $\delta\phi_2$ by $\delta\phi_1^*$ and $\delta\phi_2^*$, respectively, take the difference of the equations, and integrate over space to find [19]

$$\hbar\omega = \hbar\omega_0 + \sum_{ij} f_{ij}(\omega_0)\int d\mathbf{r}(A_{ij}(\mathbf{r})\delta\phi_1^*(\mathbf{r}) + A_{ji}^*(\mathbf{r})\delta\phi_2^*(\mathbf{r})).$$

$$\tag{25}$$

The damping rate, which is the focus of this work is given by the imaginary part of ω:

$$\Gamma_L(\omega_0) = \frac{\pi}{\hbar}\sum_{ij}(n_i^{\text{th}} - n_j^{\text{th}})\delta(\hbar\omega_0 - (E_j - E_i))|\mathcal{A}_{ij}|^2,$$

$$\tag{26}$$

where we have used Eqs. (20), (24), and (25) to express f_{ij} in terms of A_{ij}, and we define the matrix element:

$$\mathcal{A}_{ij} = \int d\mathbf{r}(A_{ij}(\mathbf{r})\delta\phi_1^*(\mathbf{r}) + A_{ji}^*(\mathbf{r})\delta\phi_2^*(\mathbf{r})). \tag{27}$$

The formula for the Landau damping rate given by Eq. (26) is the main result of this section. We remind the reader that in deriving this result, we have not assumed any particular form of the two-body interaction potential, only that it

is symmetric with respect to the interchange of the two particles. Furthermore, our equations are general and may be applied to study the Landau damping of collective modes in inhomogeneous systems, such as *trapped* atomic gases.

A. Landau damping in a homogeneous gas

In general, solving for the finite temperature damping rates of a trapped gas is numerically intensive [23,26–28]. In the presence of long-range interactions, this is further exacerbated by the exchange interaction, which is nonlocal in nature. As a result, calculating the Bogoliubov coefficients in a trapped dipolar gas is extremely challenging, and practical only in certain simple cases [38,44,46].

In the rest of this work, we focus instead on a *homogeneous* dipolar gas. In a homogeneous system, the condensate density $\phi_0(\mathbf{r}) = \sqrt{n_0}$ is constant throughout space. Furthermore, in order to obtain semianalytic results, we will neglect the exchange contribution to the Bogoliubov equations of motion arising from the noncondensed atoms, and retain only the condensate contribution. In other words, we approximate $n_{tot}(\mathbf{r}',\mathbf{r}) \approx \phi_0(\mathbf{r})\phi_0(\mathbf{r}') = \phi_0^2$. We expect this approximation to be quantitatively accurate at temperatures $k_B T \lesssim \mu$, where the condensate is not significantly depleted. In this limit, the homogeneous Bogoliubov equations of motion can be readily solved in Fourier space, and we will use the resulting solutions in subsequent computations. For sufficiently large systems, our calculation can be readily generalized to include the effects of a trap via a local density approximation [20].

In the homogeneous geometry, the fluctuations of the condensate take the plane-wave form: $\delta\phi(\mathbf{r}) = \frac{1}{\sqrt{V}}e^{-i\mathbf{k}\cdot\mathbf{r}}u_\mathbf{k}$ and $\delta\phi^*(\mathbf{r}) = \frac{1}{\sqrt{V}}e^{-i\mathbf{k}\cdot\mathbf{r}}v_\mathbf{k}$; $u_\mathbf{p} = \frac{1}{\sqrt{V}}u_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{r}}$ and $v_\mathbf{p} = \frac{1}{\sqrt{V}}v_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{r}}$, where $u_\mathbf{k}$ and $v_\mathbf{k}$ are the Bogoliubov coherence factors which read

$$u_\mathbf{k} = \sqrt{\frac{1}{2}\left(\frac{\epsilon_k + V_{tot}(\mathbf{k})}{E_\mathbf{k}} + 1\right)},$$

$$v_\mathbf{k} = -\text{sgn}[V_{tot}(\mathbf{k})]\sqrt{\frac{1}{2}\left(\frac{\epsilon_k + V_{tot}(\mathbf{k})}{E_\mathbf{k}} - 1\right)}, \quad (28)$$

where we denote $V_{tot}(\mathbf{k}) = \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}V_{tot}(\mathbf{r}-\mathbf{r}')$ as the Fourier transform of the two-body interaction potential. Here $\epsilon_k = \hbar^2 k^2/2m$ and the dispersion $E_\mathbf{k} = \sqrt{\epsilon_k[\epsilon_k + 2V_{tot}(\mathbf{k})n_0]}$.

Inserting the expressions for the fluctuations of the condensate and the noncondensed atoms into Eqs. (24) and (26), and integrating over space, the formula for the Landau damping rate for a quasiparticle with momentum $\mathbf{k}$ and energy $\hbar\omega_k$ in a homogeneous system reads

$$\Gamma_L(\mathbf{k}) = -\frac{\pi n_0}{\hbar}\sum_{\mathbf{pq}}\left(n_\mathbf{q}^{th} - n_\mathbf{p}^{th}\right)\delta(\hbar\omega_\mathbf{k} - E_\mathbf{q} + E_\mathbf{p})\mathcal{A}_{\mathbf{pq}}^\mathbf{k}\mathcal{A}_{\mathbf{qp}}^\mathbf{k},$$

$$(29)$$

where

$$\begin{aligned}
\mathcal{A}_{\mathbf{pq}}^\mathbf{k} = \delta_{\mathbf{k},\mathbf{q}-\mathbf{p}}\{&u_\mathbf{k}[V_{tot}(\mathbf{p})(u_\mathbf{p}u_\mathbf{q} + v_\mathbf{p}u_\mathbf{q}) \\
&+ V_{tot}(\mathbf{q})(v_\mathbf{p}u_\mathbf{q} + v_\mathbf{p}v_\mathbf{q})] + v_\mathbf{k}[V_{tot}(\mathbf{q})(u_\mathbf{p}u_\mathbf{q} + v_\mathbf{q}u_\mathbf{p}) \\
&+ V_{tot}(\mathbf{p})(v_\mathbf{p}v_\mathbf{q} + u_\mathbf{p}v_\mathbf{q})] + (u_\mathbf{k} + v_\mathbf{k})V_{tot}(\mathbf{k}) \\
&\times (u_\mathbf{p}u_\mathbf{q} + v_\mathbf{p}v_\mathbf{q})\}. \quad (30)
\end{aligned}$$

We denote by $\mathbf{p},\mathbf{q}$, the momenta of the outgoing quasiparticles, which are determined by the constraints of energy-momentum conservation, enforced by the δ functions in Eqs. (29) and (30). Here $n_\mathbf{k}^{th} = \frac{1}{e^{\beta E_\mathbf{k}} - 1}$ is the Bose occupation factor for a mode with energy $E_\mathbf{k}$.

B. Quasi-2D dipolar gas

A quasi-2D dipolar gas is created experimentally by employing tight confinement along the axial (z) direction, $U(\mathbf{r}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)$, where $\omega_z \gg \omega_x,\omega_y$. Here we assume that the confining direction is the same as the direction in which the dipoles are polarized, although our results can be readily generalized to include an arbitrary polarization angle. In the limit of tight confinement, $\mu \ll \hbar\omega_z$, with no loss of generality, the density can be expressed as $n(\mathbf{r}) = n^{2D}(\rho)\chi(z) = \frac{1}{\sqrt{\pi l_z^2}}n^{2D}(\rho)e^{-z^2/l_z^2}$, where ρ and z are the radial and axial coordinates, respectively, and l_z is a length scale on the order of the harmonic oscillator wavelength in the axial direction $l_z \sim \sqrt{\hbar/m\omega_z}$ [39]. Integrating out the z direction, one obtains an effective quasi-two-dimensional description, which depends on ρ alone. For the homogeneous case we consider here ($\omega_x = \omega_y = 0$), the Fourier transform of the resulting quasi-2D interaction potential reads [7,39]

$$V^{q2D}(k) = g^{q2D} + g_d^{q2D}F\left(\frac{kl_z}{\sqrt{2}}\right), \quad (31)$$

where we define a quasi-two-dimensional interaction parameter $g^{q2D} = \frac{g}{\sqrt{2\pi}l_z}$, $k = \sqrt{k_x^2 + k_y^2}$ is the radial momentum, $g_d^{q2D} = \frac{8\pi}{3\sqrt{2\pi}l_z}d^2$, and $F(x) = 1 - \frac{3}{2}\sqrt{\pi}x\,\text{erfc}(x)e^{x^2}$, where erfc($x$) is the complementary error function. When the dipoles are aligned parallel to one another, the dipole-dipole interaction only depends on the magnitude of the radial momentum.

The Landau damping rate can be computed in quasi-2D simply by replacing $V_{tot}(\mathbf{k})$ with $V^{q2D}(k)$ and n_0 with n_0^{2D} in Eqs. (29) and (30). The integrals are now performed over two-dimensional k space. The Bogoliubov eigenfrequencies and coefficients are $E_k = \sqrt{\epsilon_k[\epsilon_k + 2V^{q2D}(k)n_0^{2D}]}$, $u_k = \sqrt{\frac{1}{2}(\frac{\epsilon_k + V^{q2D}(k)n_0^{2D}}{E_k} + 1)}$ and $v_k = -\text{sgn}[V^{q2D}(k)]\sqrt{\frac{1}{2}(\frac{\epsilon_k + V^{q2D}(k)n_0^{2D}}{E_k} - 1)}$, respectively.

As discussed by Fischer [39], the quasi-2D dipolar Bose gas is mechanically stable even in the absence of short-range repulsive interactions ($g^{q2D} = 0$). Below, we first consider Landau damping of phonons, and derive an energetic criterion for Landau damping in a purely dipolar gas. We contrast the damping in a purely dipolar gas with analogous results in a gas with contact interactions [17,19,21,22,24,25]. We then calculate the Landau damping rate for a gas where both contact and dipolar interactions are present, as is the case for current experiments on dipolar ^{52}Cr, ^{162}Dy, ^{164}Dy, and ^{168}Er [1–4]. Finally, we discuss in detail the nature of Landau damping for strong dipolar interactions, where the low-energy dispersion acquires a roton-maxon character.

III. LANDAU DAMPING OF PHONONS

A. Criterion for Landau damping

We first consider the Landau damping of a phonon with wave vector **k** into two quasiparticles with wave vectors **p** and **q**. As shown by Eq. (29), Landau damping is only allowed provided the conditions for energy and momentum conservation can be simultaneously satisfied. At low energies, the dispersion of the incoming quasiparticle takes the form $E_k \approx \hbar c k$, where c is the phonon velocity in the gas, which will be defined subsequently.

Momentum conservation [enforced by the δ function in Eq. (30)] implies that $\mathbf{q} = \mathbf{k} + \mathbf{p}$, which can be expanded as $q = |\mathbf{q}| \approx p + k \cos(\theta)$ in the long-wavelength limit $\hbar k \ll mc$. Here θ is the angle between the vectors **p** with **k**. Likewise, $E_\mathbf{q} - E_\mathbf{p} \approx \hbar v_g(p) k \cos(\theta)$, where $v_g(p) = 1/\hbar \, \partial E_p/\partial p$ is the group velocity of the excitations [24]. Energetically, Landau damping can occur only if

$$\frac{c}{v_g(p)} \leqslant 1, \qquad (32)$$

i.e., the velocity of the incoming quasiparticle has to be less than the group velocity of the decaying excitations. We note that this criterion is not limited to damping of phonons in Bose condensates; rather it is completely general, and also applies to Landau damping in Fermi liquids or plasmas [47].

At large values of $\hbar p \gg mc$, the dispersion for both dipolar and contact interactions becomes free-particle-like, $E_p \to \hbar^2 p^2/2m$ and thus $v_g(p) \to \hbar p/m$. As $p \to \infty$, Eq. (32) is always satisfied, and Landau damping can occur.

However, at small quasiparticle wave vectors, the dispersion of the dipolar gas is dramatically different from a gas with contact interactions. For a gas with purely contact interactions, the dispersion takes the form $E_p/\hbar \approx cp + \frac{\hbar^2 p^3}{8m^2 c}$, where $c = \sqrt{g^{q2D} n_0^{2D}/m}$ is the phonon velocity. Thus the criterion for energy conservation as $p/mc \to 0$ reads $c/v_g \sim 1 - O(p^2) < 1$ and Landau damping is allowed. By contrast, the dipolar dispersion E_p is given by

$$E_p/\hbar \approx c_d p - \frac{3}{4}\sqrt{\frac{\pi}{2}} l_z c_d p^2 + \frac{\hbar^2 p^3}{8m^2 c_d}, \qquad (33)$$

where we have introduced a speed of sound associated with the dipolar interaction $c_d = \sqrt{g_d^{q2D} n_0^{2D}/m}$. As $p \to 0$, $c_d/v_g \sim 1 + O(p) > 1$, and Landau damping is thus forbidden by energy-momentum conservation.

Landau damping becomes possible when the cubic term in the expansion of Eq. (33) dominates over the quadratic term. Equating the quadratic and cubic terms in Eq. (33), we find that Landau damping occurs only when p exceeds a critical value:

$$p > p_{\text{crit}} = \frac{2\sqrt{2\pi} l_z m^2 c_d^2}{\hbar^2} = \frac{2}{l_z} \frac{E_{\text{int}}}{E_{\text{ho}}}, \qquad (34)$$

where $E_{\text{int}} = g_d^{q2D} n_0^{2D}$ is the mean-field interaction energy and $E_{\text{ho}} = \hbar^2/m l_z^2 = \hbar\omega_z$ is the axial harmonic oscillator energy. In other words, a phonon with wave vector **k** can only Landau damp into excitations with wave vectors **p** and $\mathbf{p} + \mathbf{k}$ for $p > p_{\text{crit}}$.

B. Temperature dependence of damping rate in a purely dipolar gas

To obtain the temperature dependence of the phonon damping rate, we expand Eq. (30) for small values of k. Introducing dimensionless quantities for the interaction strength, momentum, and temperature—$\tilde{c}_d = \sqrt{E_{\text{int}}/E_{\text{ho}}}$, $\tilde{k} = k l_z/\sqrt{2}$, and $\tau = k_B T/E_{\text{ho}}$, respectively—the Landau damping rate for a phonon with momentum k and energy $E_k = \hbar c_d k$ in a quasi-2D dipolar gas reads

$$\frac{\hbar \Gamma_L}{E_k} = \frac{\tilde{c}_d}{4\pi n_0^{2D} l_z^2} \mathcal{I}(\tau), \qquad (35)$$

where the integral $\mathcal{I}(\tau)$ is given by

$$\mathcal{I}(\tau) = \frac{1}{\tau} \int_{\sqrt{2}\tilde{c}_d^2}^{\infty} dx \, x \cot(\theta_c) \frac{1}{(e^{u/2\tau} - e^{-u/2\tau})^2}$$
$$\times \left(\frac{x^2 + \tilde{c}_d^2 F(x)}{u}[1 + F(x)] - \frac{\tilde{c}_d^2}{u} F^2(x) \mathrm{sgn}[F(x)] \right.$$
$$\left. + \frac{\tilde{c}_d^4 F(x)}{v(x)\mathrm{sgn}[F(x)]} \left[\frac{-F(x)v(x) + u\partial_x F(x)}{u\sqrt{u^2 + \tilde{c}_d^4 F(x)^2}} \right] \right)^2,$$
$$(36)$$

where $F(x)$ is defined in Eq. (31), $u = \sqrt{x^2[x^2 + 2\tilde{c}_d^2 F(x)]}$, $v(x) = \partial u/\partial x$, and $\theta_c = \sqrt{2}\tilde{c}_d/v(x)$. For a gas with purely contact interactions, we simply set $F(x) = 1$, and $\tilde{c}_d \to \tilde{c}$ where $\tilde{c}$ is similarly defined, and the integral over x runs from 0 to ∞.

In Fig. 1 we plot the integral $\mathcal{I}(\tau)$ obtained by numerically integrating Eq. (36) at fixed $\tilde{c}_d$. The corresponding damping rate for a gas with purely contact interactions ($\tilde{c} = 1$) is also shown for comparison. The phonon damping rate in a purely dipolar gas remains consistently lower than that for a gas with contact interactions at all temperatures. At high temperature ($\tau > 1$), the damping rates appear to scale linearly with
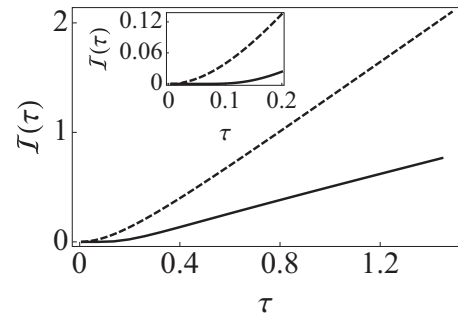

FIG. 1. The integral $\mathcal{I}(\tau)$ [Eq. (36)] plotted as a function of the reduced temperature $\tau = k_B T/E_{\text{ho}}$ in quasi-2D. The solid line is for a purely dipolar gas, while the dashed line is for a gas with contact interactions. We choose the dimensionless speed of sound for the dipolar and contact interactions $\tilde{c}_d = \tilde{c} = 1$ to enable comparison between the two cases. At high temperatures the damping rate scales linearly with T in both cases, with the dipolar damping rate being consistently smaller than that for a gas with contact interactions. Inset is a zoom-in at low temperatures showing the dramatic suppression of the dipolar damping rate compared to that in a gas with contact interactions.

temperature in both cases, while at low temperature ($\tau \ll 1$), the damping rate in a dipolar gas drops significantly faster than the damping rate for a gas with contact interactions. In particular, the dipolar damping rate is found to scale as $e^{-1/\tau}$ for $\tau \ll 1$. This is in stark contrast with the corresponding damping rate in a gas with purely contact interactions, which scales as T^4 in 3D [17,19,24,25] and T^2 in 2D [22].

As discussed previously, for a dipolar gas, modes with energy less than $E_{crit} = \hbar^2 p_{crit}^2/2m$ do not contribute to Landau damping. At low temperatures $k_B T \ll E_{crit}$, the occupation of modes with energy $E > E_{crit}$ scales as $n_{k>k_{crit}}^{th} \sim e^{-\beta E_k}$, and subsequently the damping is suppressed. As shown in the inset of Fig. 1, the low-temperature damping rate in a purely dipolar gas grows much more slowly than the T^2 scaling for a gas with contact interactions [22]. Consequently, phonons in a purely dipolar gas are virtually undamped at low temperatures.

At high temperatures, $k_B T \gg E_{crit}$, the damping rate scales linearly with temperature as in the purely contact case, however the magnitude of the damping rate at fixed $\tilde{c}_d$ is smaller than that of a gas with the purely contact interactions, as phonons cannot damp into excitations with momentum less than p_{crit}.

C. Landau damping with contact and dipolar interactions

In Fig. 2 we show the temperature dependence of the Landau damping rate for phonons in a gas with contact and dipolar interactions. Once again the damping rate can be
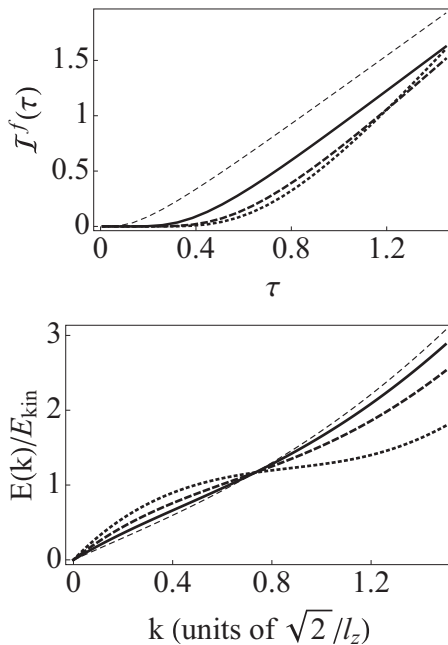

FIG. 2. Top: Temperature dependence of the phonon damping rate plotted versus reduced temperature $\tau = k_B T/E_{ho}$ in a quasi-2D gas with dipolar and contact interactions present. The contact interaction is held fixed ($\tilde{c} = 1$) in all the curves and the dimensionless ratio $\tilde{g} = g_d^{q2D}/g^{q2D}$ is varied. From top to bottom: (thin, dashed) $\tilde{g} = 0.1$, (thick, solid) $\tilde{g} = 1$, (thick, dashed) $\tilde{g} = 2.5$, (thick, dotted) $\tilde{g} = 5$. Bottom: The energy momentum dispersion corresponding to the same values of $\tilde{g}$ and $\tilde{c}$ as in the top figure.

expressed as $\frac{\hbar \Gamma_L}{E_k} = \frac{\tilde{c}}{4\pi n_0^{2D} l_z^2} \mathcal{I}^f(\tau)$, where $\tilde{c} = \sqrt{g^{q2D} n_0^{2D}/E_{ho}}$ parametrizes the contact part of the interaction, $E_k = \hbar c k$, and the integrand $\mathcal{I}^f(\tau)$ contains the temperature dependence. The full expression for $\mathcal{I}^f(\tau)$ is rather cumbersome, and is not shown here. We also introduce a dimensionless ratio $\tilde{g} = g_d^{q2D}/g^{q2D}$, which parametrizes the relative strength of the dipole-dipole and the contact interaction.

In Fig. 2, $\tilde{c}$ is held fixed and we vary $\tilde{g}$. The corresponding energy-momentum dispersion, normalized to E_{ho}, is shown on the bottom. Somewhat counterintuitively, we find that at fixed temperature, increasing the dipolar interaction *decreases* the phonon damping rate. As the dipolar interaction strength is increased, p_{crit} increases, and the number of available modes for Landau damping decreases. When the interactions become large enough such that $E_{crit} = \hbar^2 p_{crit}^2/2m \gg k_B T$, the damping rate is strongly suppressed as the available modes for Landau damping are largely unoccupied. For strong dipole-dipole interactions, phonons are virtually undamped at low temperatures. At higher temperatures $\tau \sim 1$, the damping rate again scales linearly with T.

Before discussing the damping of the roton-maxon excitations, we briefly comment on the validity of the results presented in this section. Strictly speaking, the quasi-2D ansatz we employ here is only valid provided that $\eta = m(g^{q2D} + g_d^{q2D})/4\pi\hbar^2 \ll 1$. As the interaction strength is increased, the Gaussian ansatz for the axial wave-function needs to be modified, for example by replacing the width of the Gaussian l_z with a variational parameter that depends on the interaction strength [48]. While this will quantitatively change our results, it will not affect the qualitative physics. We also expect our results to hold as long as the temperature $T \leqslant \mu$. At higher temperatures and strong interactions, the condensate becomes significantly depleted, our starting point of treating the coupling between the condensed and non-condensate atoms as a weak perturbation to Bogoliubov theory is no longer valid. Moreover, exchange interactions between noncondensed atoms alter the Bogoliubov equations of motion. A systematic understanding of how these effects modify the physics of the dipolar gas is currently being developed [49].

IV. LANDAU DAMPING IN THE MAXON-ROTON REGIME

Thus far we have focused on the damping of low-energy phonons in a quasi-2D dipolar gas. One of the novel features of a dipolar gas is that for sufficiently strong dipole-dipole interactions, the dispersion develops a roton-maxon character [10,39,50] at intermediate momenta. We now discuss the Landau damping of these excitations.

In Fig. 3 we plot the Landau damping rate at intermediate momenta for strong dipole-dipole interactions. The damping rate is normalized to $\hbar\Gamma_0 = \sqrt{2/\pi} E_{ho} \tilde{c}^2(a/l_z)$. The curves are obtained by numerically integrating Eq. (29) using Eq. (30). As before, we fix the contact part of the interaction ($\tilde{c} = 1$) and vary the ratio $\tilde{g}$. The corresponding phonon damping rate for the same value of $\tilde{g}$ is shown for comparison. At the value of $\tilde{g}$ chosen here, the critical momentum at which Landau damping of phonons is allowed is $p_{crit} \sim 2\hbar\sqrt{2}/l_z$ [Eq. (34)]. Phonon damping is therefore highly suppressed at strong dipole-dipole
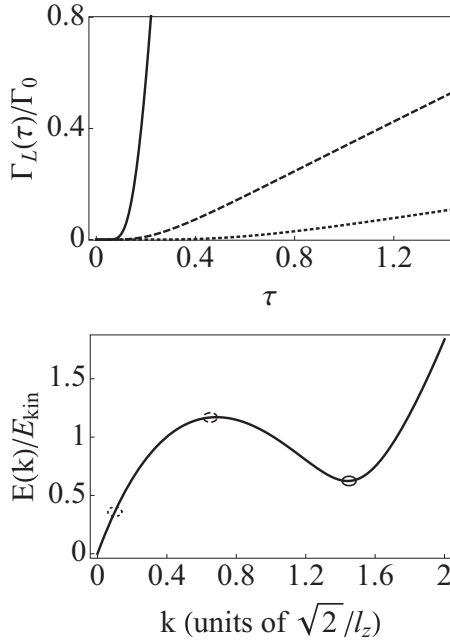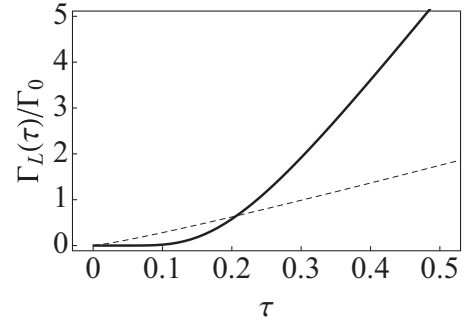
FIG. 4. Temperature dependence of the Landau damping rate at $\tilde{g} = 7.2$ and $k = 1.45$ (corresponding to the roton minimum on the plot in Fig. 3 [normalized to $\hbar\Gamma_0 = \sqrt{2/\pi}\, E_{\text{ho}}\tilde{c}^2(a/l_z)$] for a quasi-2D gas with contact and dipolar interactions. The dashed curve shows the damping rate for the same value of momentum, but with no dipolar interactions $\tilde{g} = 0$. For our parameters, the roton instability (where the roton minimum is at zero energy) occurs at $\tilde{g} = 7.5$.

FIG. 3. Top: Temperature dependence of the Landau damping rate at three different momenta corresponding to the phonon (dotted), maxon (dashed), and roton (solid) for a quasi-2D gas with contact and dipolar interactions. The damping rate is normalized to $\hbar\Gamma_0 = \sqrt{2/\pi}\, E_{\text{ho}}\tilde{c}^2(a/l_z)$. The contact interaction parametrized by $\tilde{c} = \sqrt{g^{\text{q2D}} n_0^{\text{2D}}/E_{\text{ho}}} = 1$ and the dimensionless ratio $\tilde{g} = g_d^{\text{q2D}}/g^{\text{q2D}} = 7.2$. Bottom: The dispersion for $\tilde{c} = 1$ and $\tilde{g} = 7.2$ plotted versus k. The solid, dashed, and dotted circles correspond to the values of the momenta used in the top plot.

interactions as virtually none of the modes that are available for Landau damping are thermally occupied.

To understand the damping rate at intermediate momenta, first we show that rotons cannot scatter off of phonons. To see this, consider an incoming roton with momentum **k**, scattering off of a phonon with momentum **p**. The criterion for Landau damping now reads $c/v_g^{\text{r}}(k) \leqslant 1$, where $v_g^{\text{r}}(k)$ is the group velocity of the rotons. Near the roton minimum, the excitations are free-particle-like with a small gap, and the group velocity scales as $|k - k_r|$, where k_r corresponds to the momentum at the roton minimum. As $k \to k_r$, $v_g^{\text{r}} \to 0$, and Landau damping is thus forbidden.

Solving the energy conservation criterion numerically, we find that Landau damping of rotons turns on at intermediate momenta, $p_{\text{crit}}^{\text{r}} \geqslant 0.5\sqrt{2}\hbar/l_z$, which is consistent with the location of the maxon peak in the excitation spectrum. This critical momentum is considerably smaller than p_{crit} for phonon damping, and as a result, at any given temperature, a larger number of modes contribute to the Landau damping of rotons. Hence, the roton damping rate is significantly higher than the damping rate for phonons. The damping rate for the maxon mode lies between the damping rates for the phonon and roton modes.

Finally, in Fig. 4, we compare the damping rate at fixed momentum for a gas with strong and weak dipole-dipole interactions. Recall that in the long-wavelength limit, we found that the damping rate *decreases* with increasing dipole-dipole

interactions over the temperature range $0 < \tau \lesssim 1$ (see Fig. 2). By contrast, here we find that the damping rate at intermediate momenta *increases* with increasing dipolar interactions, except at very low temperatures. At very low temperatures, modes with energies $\sim E_{p_{\text{crit}}^{\text{r}}}$ are unoccupied, and the rotons are undamped. At intermediate temperatures, the damping rate grows linearly with T but the slope is much larger for the rotons (solid curve) than for the excitations in a nondipolar gas at the same momentum (dashed curve).

The softening of the roton mode $E_{k_r} \to 0$ is associated with a divergence of the Bogoliubov amplitudes u_k and v_k. As the damping rate is directly proportional to the Bogoliubov amplitudes, via the matrix elements $A_{\mathbf{kq}}$ in Eq. (30), the damping rate also increases near the roton minimum for strong dipole-dipole interactions. We note, however, that our approach breaks down well before the roton instability is reached, as the diverging Bogoliubov amplitudes imply significant condensate depletion, and our assumption of $n_{\text{tot}} \sim \phi_0^2$ becomes invalid.

Boudgemaa and Shlyapnikov [51] argue that for low temperatures $T \ll \Delta_r$, where Δ_r is the roton gap, the Bogoliubov approach is valid as long as $\eta \ll 1$. At higher temperatures or stronger interactions where the roton gap is small $T \gg \Delta_r$, thermal effects begin to dominate. For the interaction parameters of Fig. 3, the roton gap $\Delta_r \sim 0.65 E_{\text{kin}}$ (see bottom panel in Fig. 2). Somewhat surprisingly, the damping rate for the roton mode is large even at temperatures smaller than Δ_r suggesting that roton excitations may be short lived at finite temperatures.

V. EXPERIMENTAL SIGNIFICANCE

We now briefly discuss the possibility of observing the physics described above in experiments. Experimentalists can create a single 2D pancake trap by confining atoms in an optical dipole trap with frequencies $\omega_r \sim 2\pi \times 10$ Hz and $\omega_z \sim 2\pi \times 10^3$ Hz [52]. For bosonic ^{164}Dy, which was recently Bose condensed by Lu and co-workers [2], this yields a transverse confinement length $l_z = \sqrt{\hbar/m\omega_z} = 0.1\ \mu$m. Experiments typically operate at temperatures $T \sim 10$ nK,

which yields a $\tau = k_B T / E_{\rm ho} \sim 0.1$, and corresponding 2D densities of $n_0^{\rm 2D} \sim 10^{14}$ m^{-2} [52]. This implies that $\tilde{c} = \sqrt{g^{\rm q2D} n_0^{\rm 2D} / E_{\rm ho}} \sim 1$ can be achieved for scattering lengths $a/l_z \sim 0.02 \ll 1$ [51]. Choosing a typical scattering length of $a \sim 100 a_B$, $\hbar \Gamma_0 \sim$ 1–10 Hz.

Damping of excitations in the nondipolar gas was observed by Katz *et al.* [53] who used Bragg spectroscopy to study the momentum dependence of the Beliaev damping rate of phonons. By tuning the energy difference and the angle between the Bragg beams, experimentalists excited quasiparticles at different energies and momenta. After a short time of flight, a halo of scattered atoms was observed at momenta intermediate between the condensate and the momentum of the atoms excited initially. By measuring the ratio of the number of scattered atoms to the initial cloud of atoms excited by the probe beams, as a function of energy and momentum, Katz and co-workers were able to extract the damping rate [53]. A similar technique may be used to detect Landau damping in the dipolar gas.

The energy resolution achieved in the experiment was $\sim$kHz, and the momentum resolution was $\sim 2\pi/\zeta$, where $\zeta \sim 0.25$ μm is the coherence length of the Bose condensate. The roton mode is located at $k l_z \sim 1.5$, and hence lies above the typical momentum resolution in experiments. In addition to spectroscopic probes, Landau damping also influences the decay of collective excitations in trapped gases [35,37,54,55].

VI. SUMMARY AND CONCLUSIONS

In summary, we have developed a theory of damping of low-energy excitations in a Bose gas interacting with long-range interactions, generalizing previous works on Bose gases interacting with short-range forces [19,22]. We work in the collisionless limit, where the damping is provided by scattering of a quasiparticle with other excitations, mediated by the Bose condensate. Following the time-dependent mean-field approach of Giorgini [19], we derive a coupled system of

equations describing the dynamics of the condensed and noncondensed atoms, within the Popov approximation. We then solve these equations perturbatively in the interaction strength, to obtain the expression for the Landau damping rate in a Bose gas with long-range interactions.

Focusing on the homogeneous gas and neglecting exchange interactions between noncondensed atoms, we find that the nature of the low-energy quasi-2D dipolar dispersion forbids a phonon from decaying into modes with momenta lower than a certain threshold. This leads to a dramatic suppression of Landau damping at low temperatures. For stronger dipole-dipole interactions, we find that phonons are virtually undamped over a large temperature regime. By contrast, at intermediate momenta and strong interactions, where the dispersion develops a roton-maxon feature, we find much higher damping rates.

Finally, we remark that experiments on collective modes in trapped gases typically also measure the temperature dependence of the frequency shift of the collective modes, in addition to the damping rates [35,37]. These frequency shifts are related to many-body effects which go beyond the Popov approximation [15,18,20,26,27]. A systematic study of these effects and how they affect the frequencies of the collective excitations in the dipolar gas has not yet been performed, and is a promising direction for future study [56].

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