Speeding up and slowing down the relaxation of a qubit by optimal control

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We consider a two-level quantum system prepared in an arbitrary initial state and relaxing to a steady state due to the action of a Markovian dissipative channel. We study how optimal control can be used for speeding up or slowing down the relaxation towards the fixed point of the dynamics. We analytically derive the optimal relaxation times for different quantum channels in the ideal ansatz of unconstrained quantum control (a magnetic field of infinite strength). We also analyze the situation in which the control Hamiltonian is bounded by a finite threshold. As by-products of our analysis, we find that (i) if the qubit is initially in a thermal state hotter than the environmental bath, quantum control can not speed up its natural cooling rate; (ii) if the qubit is initially in a thermal state colder than the bath, it can reach the fixed point of the dynamics in finite time if a strong control field is applied; (iii) in the presence of unconstrained quantum control, it is possible to keep the evolved state indefinitely and arbitrarily close to special initial states which are far away from the fixed points of the dynamics.

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I. INTRODUCTION

If a quantum system is not perfectly isolated from the environment, it is subject to dissipation and decoherence and its dynamics is often well approximated by a Markovian quantum channel [1,2]. In this case, a given arbitrary initial state will usually converge towards a steady state and this process is called *relaxation*. The steady state can be the thermal state if the bath is in equilibrium; more generally, it will be a fixed point of the quantum channel describing the nonunitary evolution of the system. Depending on the situation, such a relaxation process can be advantageous or disadvantageous. If, for example, we want to cool a system by placing it into a refrigerator (or if we want to initialize a qubit), a fast thermalization is desirable. On the other hand, especially in quantum computation or communication, decoherence during processing is a detrimental effect and in this case a slow relaxation is preferable. The goal of this paper is to investigate how quantum control can be used to increase or decrease the relaxation time of a qubit towards a fixed point of the dynamics. The theory of optimal quantum control is well established and has been studied in a large variety of settings and under different perspectives (for a recent review see, e.g., [3]). For example, the application of optimal control to open systems is discussed in Refs. [4] (cooling of molecular rotations), [5] (using measurement), [6] (in the context of NMR), [7,8] (in N-level systems), [9,10] (non-Markovian dynamics), and [11] (for a review). In particular, time-optimal quantum control has been extensively discussed for one-qubit systems in a dissipative environment [12–21], a variational principle for constrained Hamiltonians in open systems can be found in [22,23], while a comparison of several numerical algorithms is given in [24]. The controllability properties of finite-dimensional Markovian master equations has also been extensively discussed (see, e.g., [25]). On the other hand, studies in closed [26] as well as open quantum systems [27] pointed to the existence of upper bounds in the speed with which a quantum system can evolve in the Hilbert space (the "quantum speed limit", or QSL), and several applications of quantum control theory to achieve the QSL can be found in [28]. An analysis of sideband cooling is given in [29,30], while superfast cooling with laser schemes has proven to be advantageous [31]. More recently, the engineering of multipartite entangled quantum states via a quasilocal Markovian quantum dynamics has also been studied depending upon the available local Hamiltonian controls and dissipative channels (see, e.g., [32] and references therein). Time-optimal quantum control has also been successfully applied in quantum thermodynamics [33], e.g., to describe the fast cooling of harmonic traps [34] or to maximize the extraction of work [35].

This work provides both analytical and numerical results. In the case in which the strength of the optimal control is allowed to be arbitrarily large, we give analytical expressions for the minimum and maximum relaxation times of a qubit subject to three prototypical classes of dissipative channels: generalized amplitude damping, depolarization, and phase damping. For the amplitude-damping channel, we also analytically derive the results in the limit of a weak control field, as well as numerically optimize the relaxation time for different strengths of the control field using the chopped random basis (CRAB) optimization algorithm [36]. We find that for initial hot thermal states the optimal path is a straight line towards the fixed point. This implies that it is impossible to speed up the cooling process of a thermal qubit in a cold bath by optimal control. However, optimal control can be advantageous if we want to heat a thermal qubit in the presence of a hot bath. Furthermore, in the limit of infinitesimal strength m of a generic control Hamiltonian, the minimum time taken by a qubit to reach its fixed point decreases linearly with m, with the slope depending on the explicit form of the control Hamiltonian. We also consider a different optimization task: to determine the maximum time for which one can keep the state of a qubit inside a ball of radius ϵ centered around the initial state. We show that, even if dynamical decoupling can not be applied because the bath is Markovian, there exist special states for which the dissipative dynamics can be stopped by optimal control. In deriving our results, we assume that the unitary (represented by a Hamiltonian) and the dissipative (represented by Lindbladians) parts act separately in the master equation governing the time evolution of the qubit. The Hamiltonian driving the qubit in the Bloch sphere can be controlled, subject to some constraints, in order to achieve our desired optimization task. However, the Lindbladians appearing in the master equation are fixed, time independent, and not affected by any change in the system Hamiltonian. This is a reasonable assumption in the limit of very small changes in the strength of the system Hamiltonian, as well as in the opposite limit of an infinitely strong system Hamiltonian when any unitary evolution takes place almost instantaneously, during which time we can neglect the nonunitary part. Furthermore, we do not allow any feedback in our quantum control.

The paper is organized as follows: In Sec. II, we review the master equation describing the dynamics of a general dissipative and Markovian process and apply it to the case of two-level quantum systems whose state is represented in the Bloch sphere. We also introduce the problem of controlled time-optimal evolution up to an arbitrarily small distance from the target. In Sec. III, we discuss in more details the generalized amplitude-damping channel. In Sec. IIIA, we analytically study how optimal control can speed up the relaxation of a qubit. In particular, Sec. IIIA1 is devoted to the case of unconstrained coherent control, while Section IIIA2 is devoted to the case of controls with constrained amplitude (with analytical results in the limit of small magnetic fields, and numerical results for arbitrary control amplitudes). Then, the situation in which the control slows down the relaxation is treated in Sec. IIIB. Section IV deals with similar analytical studies of optimal control in the depolarizing channel, while Sec. V is devoted to the analysis of the phase-damping channel. Finally, we provide some discussion of the results in Sec. VI. The general expression for the speed of change of purity of a qubit is given in the Appendix.

II. CONTROLLING THE MARKOVIAN DYNAMICS OF A QUBIT

A general dissipative and Markovian process can be described by the time-local master equation [1,2]

$$\dot{\rho} = -i\,[H,\rho] + \mathcal{L}(\rho)\,, \qquad (1)$$

where $\rho(t)$ is the density operator representing the quantum system and $\dot{\rho} := \partial\rho/\partial t$. Having set $\hbar = 1$ for convenience, the Hermitian operator $H(t)$ describes the Hamiltonian of the system, which drives the unitary part of the quantum evolution. The superoperator $\mathcal{L}(\rho(t))$ instead is the dissipator, which is responsible for the decoherent part of the quantum evolution, and which can be expressed in terms of a collection of (in general non-Hermitian) operators L_a (the Lindblad operators) as in

$$\mathcal{L}(\rho(t)) := \sum_a \left[L_a \rho L_a^\dagger - \frac{1}{2}(L_a^\dagger L_a \rho + \rho L_a^\dagger L_a) \right]. \qquad (2)$$

For a two-level quantum system, a qubit, the representation (2) can always be defined in terms of no more than three Lindblad operators L_a $(a = 1,2,3)$, which, exploiting the gauge freedom inherent to the master equation (1), can be chosen to be traceless, i.e.,

$$L_a := \sqrt{\gamma_a}\mathbf{l}_a \cdot \boldsymbol{\sigma}\,, \qquad (3)$$

with $\boldsymbol{\sigma} := (\sigma_x,\sigma_y,\sigma_z)$ being the vector formed by the Pauli matrices $\{\sigma_i, i = x,y,z\}$. In this expression, $\mathbf{l}_a := (l_{ax},l_{ay},l_{az})^\top$ are (possibly complex) three-dimensional vectors, fulfilling the orthonormalization condition $\mathbf{l}_a \cdot \mathbf{l}_b^* = \delta_{ab}$, while the nonnegative parameters γ_a define the decoherence rates of the system. Analogously, without loss of generality, the Hamiltonian H can be written as

$$H(t) := \mathbf{h} \cdot \boldsymbol{\sigma}\,, \qquad (4)$$

with $\mathbf{h}(t)$ being a three-dimensional real vector. {Since the master equation (1) for the generalized amplitude-damping channel discussed in Sec. III is invariant under rotations about the z axis of the Bloch sphere, actually only two independent controls [i.e., two nonzero components of $\mathbf{h}(t)$] are enough to determine the unitary dynamics in the case of a control with infinite strength. Three independent controls are instead needed for more general (nonsymmetric) channels.} Accordingly, Eq. (1) reduces to the following differential equation:

$$\dot{\mathbf{r}} = 2\left[\mathbf{h} \wedge \mathbf{r} + \sum_a \gamma_a \{\mathrm{Re}[(\mathbf{l}_a \cdot \mathbf{r})\mathbf{l}_a^*] - \mathbf{r} + i(\mathbf{l}_a \wedge \mathbf{l}_a^*)\} \right], \qquad (5)$$

where $\mathbf{r}(t) := (r_x,r_y,r_z)^\top$ is the three-dimensional, real vector that represents the qubit density matrix ρ in the Bloch ball, i.e.,

$$\rho(t) = \tfrac{1}{2}\,(I + \mathbf{r} \cdot \boldsymbol{\sigma}) \qquad (6)$$

(I being the identity operator). For future reference, it is worth reminding that while the Hamiltonian H only induces rotations of the Bloch vector $\mathbf{r}$, the action of $\mathcal{L}$ typically will modify also its length $r = |\mathbf{r}|$, i.e., the purity $P := \mathrm{Tr}[\rho^2] = (1 + r^2)/2$ of the associated state ρ.

The main aim of our work is to study the time-optimal, open-loop, coherent quantum control of the evolution of one qubit state under the action of the master equation (5). The coherent (unitary) control is achieved via the effective magnetic field $\mathbf{h}(t)$ of Eq. (4). On the contrary, we assume the dissipative part of the quantum evolution (2) fixed and assigned. We also exclude the possibility of performing measurements on the system to update the quantum control during the evolution, i.e., no feedback is allowed [notice, however, that complete information on the initial state of the qubit $\rho(t = 0) := \rho_i = (I + \mathbf{r}_i \cdot \boldsymbol{\sigma})/2$ is assumed].

Within this theoretical framework, we analyze how to evolve the system towards a target state $\rho_f := (I + \mathbf{r}_f \cdot \boldsymbol{\sigma})/2$ in the shortest possible time. Specifically, we take as ρ_f a fixed point of the dissipative part of the master equation, i.e., a state $\rho_{\mathrm{fp}} := (I + \mathbf{r}_{\mathrm{fp}} \cdot \boldsymbol{\sigma})/2$ fulfilling the condition $\mathcal{L}(\rho_{\mathrm{fp}}) = 0$, or

$$\sum_a \gamma_a \{\mathrm{Re}[(\mathbf{l}_a \cdot \mathbf{r}_{\mathrm{fp}})\mathbf{l}_a^*] - \mathbf{r}_{\mathrm{fp}} + i(\mathbf{l}_a \wedge \mathbf{l}_a^*)\} = 0. \qquad (7)$$

Equation (7) identifies stationary solutions (i.e., $\dot{\rho} = 0$) of the master equation (5) when no Hamiltonian is present. They represent attractor points for the dissipative part of evolution, i.e., states where noise would typically drive the system. By setting $\rho_f = \rho_{\rm fp}$ in our time-optimal analysis we are hence effectively aiming at speeding up relaxation processes that would naturally occur in the system even in the absence of external control. In addressing this issue, we do not require perfect unit fidelity, i.e., we tolerate that the quantum state arrives within a small distance from the target, fixed *a priori*. More precisely, given $\epsilon \in [0,1]$ we look for the minimum value of time $T_{\rm fast}$ which thanks to a proper choice of $H(t)$ allows us to satisfy the constraint

$$2D[\rho(T_{\rm fast}),\rho_f] = |\mathbf{r}(T_{\rm fast}) - \mathbf{r}_{\rm fp}| = \epsilon, \tag{8}$$

with $D(\rho,\rho') := {\rm Tr}|\rho - \rho'|/2$ being the trace distance between the quantum states ρ and ρ' [37].

A second problem we address is the exact counterpart of the one detailed above: namely, we focus on keeping the system in its initial state ρ_i (or at least in its proximity) for the longest possible time. In other words, we try to slow down the relaxation which is naturally induced by $\mathcal{L}$ through the action of the control Hamiltonian H.

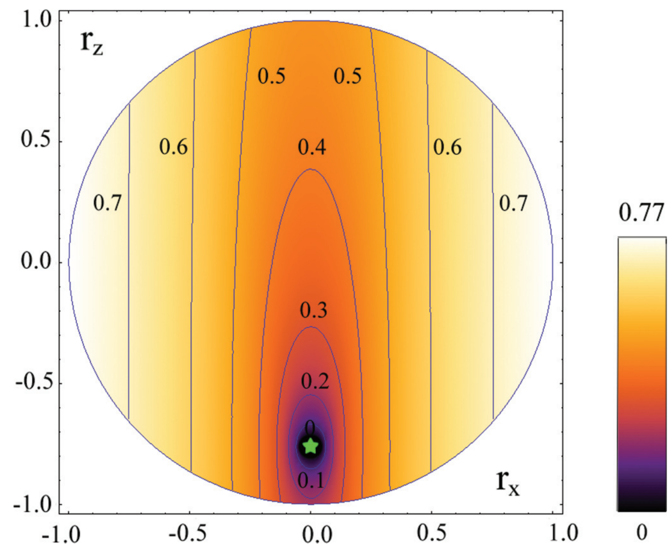

FIG. 1. (Color online) Density plot of $T^{\rm AD}_{\rm free}(\mathbf{r}_i,\epsilon)$ of Eq. (13) as a function of the initial state $\mathbf{r}_i = (r_{ix},r_{iy},r_{iz})$. As the system is invariant under rotations around the z axis, we set $r_y = 0$ without loss of generality. Here, $\epsilon = 0.04$ and the noise parameters have been set equal to $\beta = 2$ and $\gamma = e^\beta - 1 \approx 6.39$. The fixed point is indicated with a green star.

III. GENERALIZED AMPLITUDE-DAMPING CHANNEL

Here, we analyze both the speeding up and the slowing down of relaxation problems detailed in the previous section under the assumption that the dissipative dynamics (2) which is affecting the system is a generalized amplitude-damping channel [37]. The latter is described by the Lindblad operators

$$(L_1)_{\rm AD} = \sqrt{\frac{\gamma}{e^\beta - 1}}\sigma_+; \quad (L_2)_{\rm AD} = \sqrt{\frac{\gamma e^\beta}{e^\beta - 1}}\sigma_-, \tag{9}$$

where $\sigma_\pm := (\sigma_x \pm i\sigma_y)/2$, and where the non-negative quantities γ and β, respectively, describe the decoherence rate of the system and the effective inverse temperature of the environmental bath. In the absence of the Hamiltonian control, the associated superoperator $\mathcal{L}$ induces a dynamical evolution, which in the Cartesian coordinates representation (5) is given by

$$\dot{\mathbf{r}} = -\frac{\gamma}{2r_{\rm fp}}(r_x,r_y,2r_z)^\top - \gamma(0,0,1)^\top, \tag{10}$$

with $r_{\rm fp} := (e^\beta - 1)/(e^\beta + 1)$. For an initial state $\mathbf{r}_i := (r_{ix},r_{iy},r_{iz})^\top$, Eq. (10) admits a solution of the form

$$\mathbf{r}(t) = e^{-\frac{\gamma t}{2r_{\rm fp}}}\left(r_{ix},r_{iy},e^{-\frac{\gamma t}{2r_{\rm fp}}}[r_{iz} + r_{\rm fp}] - e^{\frac{\gamma t}{2r_{\rm fp}}}r_{\rm fp}\right)^\top, \tag{11}$$

which for sufficiently large t converges to the unique fixed point (7) of the problem

$$\mathbf{r}_{\rm fp} = (0,0,-r_{\rm fp})^\top. \tag{12}$$

From these expressions we can also compute the minimal time $T^{\rm AD}_{\rm free}(\mathbf{r}_i,\epsilon)$ required for the initial state $\mathbf{r}_i$ to reach the target $\mathbf{r}_{\rm fp}$ within a fixed trace distance ϵ without the aid of any external control, i.e.,

$$T^{\rm AD}_{\rm free}(\mathbf{r}_i;\epsilon) = \frac{r_{\rm fp}}{\gamma}\ln\left\{\frac{(r_{ix}^2 + r_{iy}^2)}{2\epsilon^2} \times\left[1 + \sqrt{1 + \left[\frac{2(r_{iz} + r_{\rm fp})\epsilon}{(r_{ix}^2 + r_{iy}^2)}\right]^2}\right]\right\} \tag{13}$$

(see Fig. 1). This function sets the benchmark that we use to compare the performance of our time-optimal control problem.

A. Speeding up relaxation

In this section, we address the problem of speeding up the transition of the system from ρ_i towards the fixed-point state $\rho_{\rm fp}$ with a proper engineering of the quantum control Hamiltonian $H(t)$ to see how much one can gain with respect to the "natural" time $T^{\rm AD}_{\rm free}(\mathbf{r}_i,\epsilon)$ of Eq. (13). Clearly, the result will depend strongly on the freedom we have in choosing the functions $\mathbf{h}(t)$ of Eq. (4).

1. *Unconstrained Hamiltonian control*

For a coherent control where the choice of the possible functions $\mathbf{h}(t)$ is unconstrained, the problem essentially reduces to finding the maximum of the modulus of the speed of purity change, at any given purity, for the amplitude-damping channel. In fact, given any arbitrary initial state of the qubit (i.e., given an initial Bloch vector $\mathbf{r}_i$), one can always unitarily and instantaneously (since we may take a control with infinite strength) rotate the Bloch vector from the initial point along the surface of a sphere of radius r_i until one reaches the new position of spherical coordinates $(r_i,\theta_{\rm ext},\varphi_{\rm ext})$ where the speed

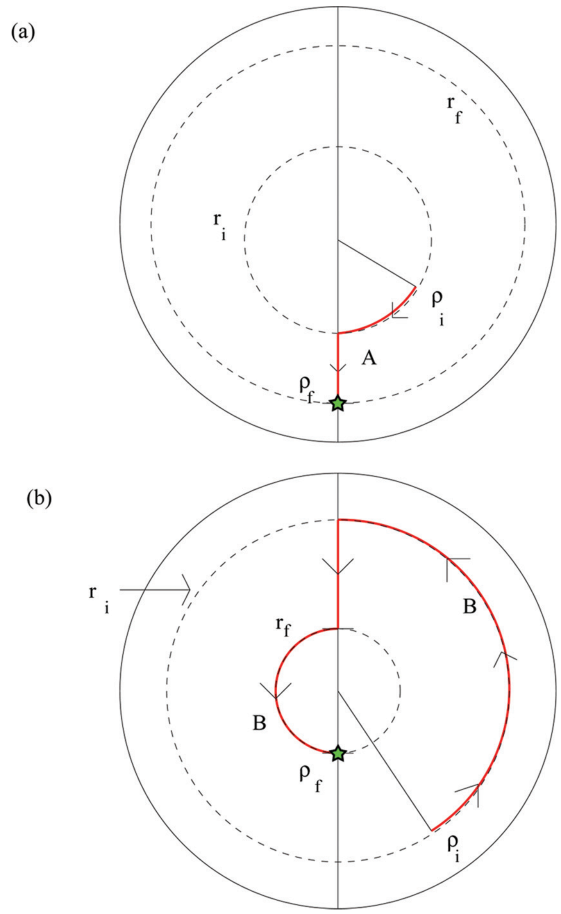

FIG. 2. (Color online) Schematic diagram showing the optimal paths in the case of (a) cooling (path A) and (b) heating (path B) on the x-z plane of the Bloch sphere. We start from an initial state ρ_i with radius r_i. The fixed point is given by ρ_f with radius r_f (green star). The solid vertical line is the z axis.

of purity change induced by the dissipator, i.e., the quantity

$$v[\mathbf{r}(P)] := \frac{dP}{dt} = 2\,\mathrm{Tr}[\rho\mathcal{L}(\rho)], \tag{14}$$

is extremal for fixed radius r_i. Then, one can switch off the control and let the system decohere for a time $T_{\rm fast}$ until the radius $r(T_{\rm fast})$ which satisfies the trace distance condition (8) is reached. Finally, one can switch the (magnetic field) quantum control on again and unitarily rotate the Bloch vector from the position $[r(T_{\rm fast}),\theta_{\rm ext},\varphi_{\rm ext}]$ to a point within tolerable distance from the target at $[r(T_{\rm fast}),\theta_{\rm fp},\varphi_{\rm fp}]$. Two examples of such a time-optimal control strategy are depicted in Figs. 2(a) and 2(b), respectively, for the cases $r_i < r_{\rm fp}$ and $r_i > r_{\rm fp}$.

From Eqs. (9) and (A1) and (A2) of the Appendix, the speed of purity change in spherical coordinates induced by the generalized amplitude-damping channel is easily shown to be independent of the azimuthal angle φ and given by

$$v_{\rm AD}(r,\theta) = -\gamma r\left[\cos\theta + \frac{r}{2r_{\rm fp}}(1+\cos^2\theta)\right]. \tag{15}$$

The optimal values of the speed for a given radius r are determined by the equation $\partial_\theta v_{\rm AD}|_r = 0$. In the case of cooling, i.e., when we want to reach $r_{\rm fp}$ starting from $r_i < r_{\rm fp}$, we find that the speed $v_{\rm AD}$ is monotonically increasing from a negative minimum at $\theta_0 = 0$ (which corresponds to a global maximum of $|v_{\rm AD}|$) up to a positive maximum at $\theta_1 = \pi$ (which corresponds to a local maximum of $|v_{\rm AD}|$). Therefore, the optimal cooling is achieved at $\theta_1 = \pi$, where

$$v_{\rm fast}^{\rm AD,cool}(r,\pi) = \gamma r\left(1 - \frac{r}{r_{\rm fp}}\right), \quad r < r_{\rm fp}. \tag{16}$$

Incidentally, this is consistent with the zero-temperature result considered in [4]. On the other hand, in the heating case, i.e., when we want to reach the thermal state $r_{\rm fp}$ starting from $r_i > r_{\rm fp}$, the speed $v_{\rm AD}$ is always negative, it starts from a global minimum at $\theta_0 = 0$ (which again corresponds to a global maximum of $|v_{\rm AD}|$), grows up to a maximum at $\theta_2 = \arccos(-r_{\rm fp}/r)$ (which corresponds to a global minimum of $|v_{\rm AD}|$), and then decreases to a local minimum at $\theta_1 = \pi$ (which corresponds to a local maximum of $|v_{\rm AD}|$). Therefore, the optimal heating is obtained by starting from $\theta_0 = 0$ where

$$v_{\rm fast}^{\rm AD,heat}(r,0) = -\gamma r\left(1 + \frac{r}{r_{\rm fp}}\right), \quad r > r_{\rm fp}. \tag{17}$$

We remark here that, even if the above reasoning is valid in the regime of infinite strength of the control, nevertheless it gives also a no-go result for the task of cooling a thermal hot state embedded in a cold bath. Since in this case the initial state is already along the negative z axis, we can not increase the cooling time by optimal control and the fastest strategy is to just let the system thermalize with the bath.

We can finally proceed to compute the optimal-time duration of the quantum controlled evolutions. Using Eq. (14) and recalling the relationship between the purity and the Bloch vector of a given state, one can evaluate the required optimal time from the optimal speeds, Eqs. (16) and (17), by the formula

$$T_{\rm fast}^{\rm AD}(\mathbf{r}_i;\epsilon) := \begin{cases} \int_{r_i}^{r_{\rm fp}-\epsilon} \frac{r\,dr}{v_{\rm fast}^{\rm AD,cool}(r)} & \text{for } r_i < r_{\rm fp}-\epsilon, \\ 0 & \text{for } |r_i - r_{\rm fp}| \leqslant \epsilon, \\ \int_{r_i}^{r_{\rm fp}+\epsilon} \frac{r\,dr}{v_{\rm fast}^{\rm AD,heat}(r)} & \text{for } r_i > r_{\rm fp}+\epsilon, \end{cases} \tag{18}$$

where we used $dP = r\,dr$.

In particular, in the case of cooling, i.e., when we want to reach the target $r_{\rm fp}$ starting from $r_i < r_{\rm fp} - \epsilon$, we obtain

$$T_{\rm fast}^{\rm AD,cool}(\mathbf{r}_i;\epsilon) = \frac{r_{\rm fp}}{\gamma}\ln\left[\frac{(r_{\rm fp}-r_i)}{\epsilon}\right], \tag{19}$$

which, analogously to the free relaxation time (13), diverges for $\epsilon \to 0$. In the case of heating, i.e., when we want to reach the target $r_{\rm fp}$ starting from $r_i > r_{\rm fp} + \epsilon$ we obtain

$$T_{\rm fast}^{\rm AD,heat}(\mathbf{r}_i;\epsilon) = \frac{r_{\rm fp}}{\gamma}\ln\left[\frac{(r_{\rm fp}+r_i)}{(2r_{\rm fp}+\epsilon)}\right]. \tag{20}$$

This time is finite even in the limit of $\epsilon \to 0$, and it clearly represents an advantage with respect to the action of simply letting the system evolve without any control from the initial state [cf. Eq. (13) for $\epsilon \to 0$, also see Figs. 1 and 3].

We notice finally that, to the most significant order in an expansion in ϵ, the function (18) reaches its maximum for

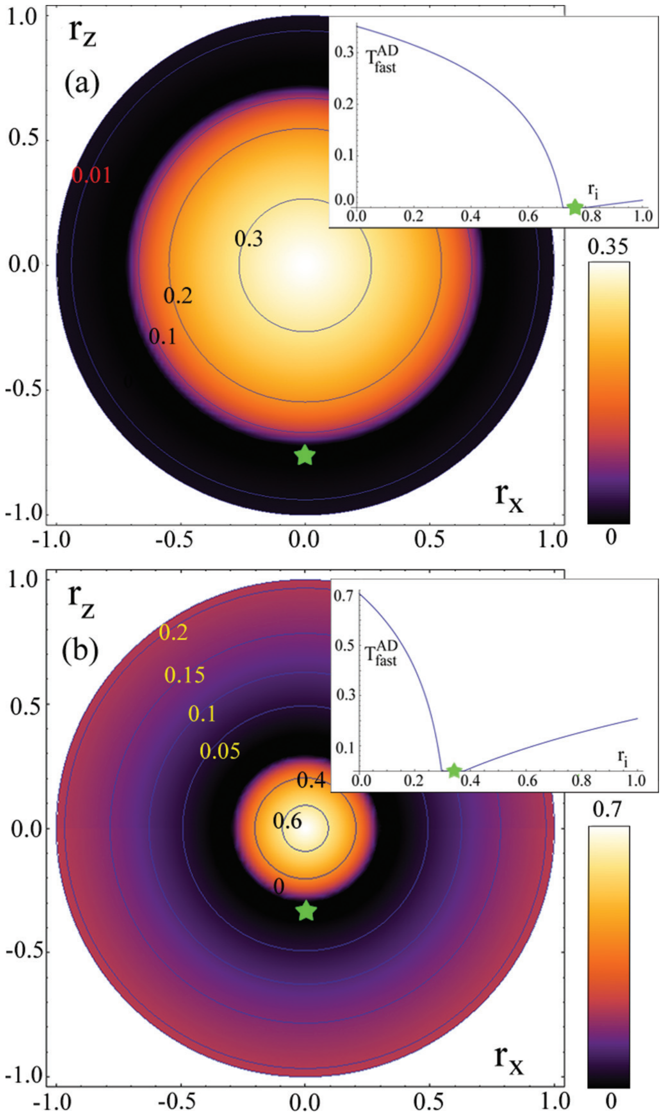

FIG. 3. (Color online) Density plots of the minimal time $T_{\text{fast}}^{\text{AD}}(\mathbf{r}_i;\epsilon)$ of Eq. (18) for a generalized amplitude-damping channel as a function of the initial state $\mathbf{r}_i=(r_{ix},r_{iy},r_{iz})$. The parameters β and ϵ are as in Fig. 1 in (a), while $\beta=0.7$ and $\epsilon=0.04$ in (b). The fixed point is indicated by a green star. The inset shows a section of the density plot along the x axis.

$\mathbf{r}_i=0$, i.e.,

$$\max_{\mathbf{r}_i}\ T_{\text{fast}}^{\text{AD}}(\mathbf{r}_i;\epsilon)\simeq\frac{r_{\text{fp}}}{\gamma}|\ln\epsilon|. \tag{21}$$

This is the optimal time one would have to wait in the worst possible scenario (of choice of initial conditions) in order to bring the system close to the target in the case of unconstrained control. By comparing it with the maximum of the function (13), i.e.,

$$\max_{\mathbf{r}_i} T_{\text{free}}^{\text{AD}}(\mathbf{r}_i;\epsilon)\simeq 2\frac{r_{\text{fp}}}{\gamma}|\ln\epsilon| \tag{22}$$

(reached by a pure state along the equator of the Bloch ball), we notice that the optimal quantum control yields a shortening of a factor 2 in the evolution time.

2. *Optimal control with constrained magnetic field intensity*

The results of the previous section have been obtained under the assumption of an unconstrained Hamiltonian control. Of course, this is a highly idealized scenario which may not be approached in realistic experimental setups. On the contrary, the effective magnetic field $\mathbf{h}(t)$ entering in Eq. (4) contains an uncontrollable, fixed part $\mathbf{h}^D(t)$ (*drift* contribution) which can be only in part compensated via the application of some controlling pulse $\mathbf{h}^C(t)$ whose maximum intensity is bounded by a fixed, finite value m, i.e.,

$$\mathbf{h}(t):=\mathbf{h}^D(t)+m\ \mathbf{h}^C(t),\quad |\mathbf{h}^C(t)|\leqslant 1. \tag{23}$$

Discussing the speeding up of relaxation under these conditions is a rather complex task for which at present we do not have an analytical solution (apart from the special case where m is small, see following). Still, in the following we present a numerical analysis that allows us to gain some insight into the problem. In particular, we focus on the case where the initial state of the system ρ_i is characterized by a Bloch vector of length $r_i=0.41$ [specifically, we take $\mathbf{r}_i=(0.38,-0.22,-0.46)$ and take $\beta=2$ and $\gamma=e^\beta-1$ as parameters for the generalized amplitude-damping channel]. Accordingly, this corresponds to have Lindblad generators (9) equal to $(L_1)_{\text{AD}}=\sigma_+$, $(L_2)_{\text{AD}}=e\sigma_-$, and a fixed point (12) with $r_{\text{fp}}\simeq 0.76$. For the Hamiltonian (23), moreover, we take

$$\mathbf{h}^D(t)=\frac{\omega}{2}\,\mathbf{e}_z+\frac{t}{\tau}(\mathbf{e}_x+\mathbf{e}_y+\mathbf{e}_z), \tag{24}$$

$$\mathbf{h}^C(t)=\frac{t}{\tau N_c}\sum_{n=1}^{N_c}\sum_{\mu=x,y,z}h_{\mu,n}\sin\left(\frac{2\pi nt}{\tau}\right)\mathbf{e}_\mu, \tag{25}$$

where $\{\mathbf{e}_\mu,\mu=x,y,z\}$ are the Cartesian unit vectors. The control term $\mathbf{h}^C(t)$ is chosen following the methods of CRAB [36]. The drift term $\mathbf{h}^D(t)$ contains two contributions: a constant term which sets the energy scale for the qubit and a time-dependent term describing side effects of the control process (in particular, we model it as an isotropic increase of the magnetic field over the duration time of the evolution). The control pulses to be optimized are finally represented in terms of a truncated Fourier expansion containing N_c terms whose coefficients are subject to the constraints $-1<h_{x,n},h_{y,n},h_{z,n}<1$, for all n. For a given value of the intensity bound m, we then use a simplex method [36] to numerically optimize $h_{\mu,n}$ so that the system, starting from ρ_i, will get to a (trace) distance $\epsilon=0.04$ from the fixed point ρ_{fp} in the shortest possible time T_m. Results are reported in Fig. 4: as expected, T_m decreases monotonically with m, converging to a constant value T_m^∞ at large m. As we are simulating a cooling process (r_i being smaller than r_{fp}), the latter should be compared with the analytic value of $T_{\text{fast}}^{\text{AD,cool}}(\mathbf{r}_i,\epsilon)$ of Eq. (19) where an unbounded (both in the intensity m and in the frequency domain) Hamiltonian control was explicitly assumed. The value of $T_{\text{fast}}^{\text{AD,cool}}(\mathbf{r}_i,\epsilon)$ is represented by the dashed line of Fig. 4: the discrepancy between T_m^∞ and the quantum speed limit $T_{\text{fast}}^{\text{AD,cool}}(\mathbf{r}_i,\epsilon)$ is expected to saturate in the limit of a large m and a large number N_c of frequencies in Eq. (25). We note that with a large number of parameters, the search for the optimal T_m is slower. However, previous studies (see,

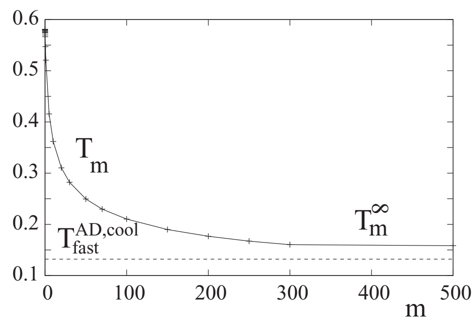

FIG. 4. Plot of the optimal-time evolution T_m needed to bring the initial state ρ_i with $\mathbf{r}_i = (0.38, -0.22, -0.46)$ towards the fixed point of a generalized amplitude-damping channel with $\beta = 2$ and $\gamma = e^{\beta} - 1$. Data obtained via numerical optimization of the control parameters $h_{\mu,n}$ of Eq. (25) setting $\tau = N_c = 10$ and $\epsilon = 0.04$. In the limit of $m \to \infty$ and of $N_c \to \infty$, we expect T_m to saturate to the corresponding value of the function $T_{\rm fast}^{\rm AD,cool}(\mathbf{r}_i,\epsilon)$ given in Eq. (19) (dashed line).

e.g., [36]) have shown that the fidelity with respect to the target state after a fixed time of evolution, as obtained by CRAB, converges exponentially with the number of frequencies, allowing for good results with a reasonable dimension of parameter space. Therefore, we have chosen an intermediate value of $N_c = 10$, which produces reasonably good results, as shown in Fig. 4.

Now, let us focus on the small-m limit. To do so, we find it convenient to write the master equation (5) for the generalized amplitude-damping channel in terms of the spherical coordinates $[r(t),\theta(t),\varphi(t)]$ of the vector $\mathbf{r}(t)$, i.e.,

$$\begin{aligned}\dot{r} &= -\frac{[r(1+\cos^2\theta) + 2r_{\rm fp}\cos\theta]}{(1-r_{\rm fp})},\\ \dot{\theta} &= \frac{\sin\theta(r\cos\theta + 2r_{\rm fp})}{r(1-r_{\rm fp})} + 2(-h_x\sin\varphi + h_y\cos\varphi),\\ \dot{\varphi} &= -2[(h_x\cos\varphi + h_y\sin\varphi)\cot\theta - h_z],\end{aligned} \tag{26}$$

where $h_x(t)$, $h_y(t)$, and $h_z(t)$ are the Cartesian components of the Hamiltonian vector (23). For the moment, let us consider the case $m = 0$ (no control). When $r < r_{\rm fp}$, from Eq. (26) we have that $\dot{\theta} > \sin\theta(\cos\theta + 2)/(1 - r_{\rm fp}) > 0$ for any t, i.e., θ increases monotonically in the cooling case. On the other hand, in the case of heating, even though $r > r_{\rm fp}$ implies that $\dot{\theta}$ can be negative at small times (when the system is far away from the fixed point), at large times when $r \approx r_{\rm fp}$ we have $\dot{\theta} \approx \sin\theta(\cos\theta + 2)/(1 - r_{\rm fp})$, and thus again θ increases monotonically. These behaviors will be maintained also for $m \neq 0$ as long as m is sufficiently small. Therefore, as θ is almost monotonic in time for all possible choices of the input state (the only exceptions being for heating processes), we can use it to parametrize the trajectories of the system. This allows us to write the time T_m taken by the qubit to move from the initial state to a state within trace distance ϵ of the fixed point as

$$T_m = \int_{\theta_i}^{\theta_m} \frac{d\theta}{\dot{\theta}} = \int_{\theta_i}^{\theta_m} \frac{d\theta}{\dot{\theta}_0 + m\,\Gamma}, \tag{27}$$

where $\Gamma(t(\theta)) := 2[-h_x^C\sin\varphi + h_y^C\cos\varphi]$, $\dot{\theta}_0(t(\theta)) := \dot{\theta}$ at $m = 0$, and (r_m,θ_m,φ_m) [respectively $(\bar{r},\bar{\theta},\bar{\varphi})$] are the coordinates of the final state for $m \neq 0$ (respectively $m = 0$). In the limit $m\Gamma \ll \dot{\theta}_0$ and expanding for small m, we get

$$T_m \approx \bar{T} - mA, \tag{28}$$

where

$$\bar{T} := \int_{\theta_i}^{\bar{\theta}} \frac{d\theta}{\dot{\theta}_0} \tag{29}$$

is the time taken to reach the fixed point at $m = 0$ and

$$A := \int_{\theta_i}^{\bar{\theta}} \frac{\Gamma(\theta)}{\dot{\theta}_0^2}d\theta - \left[\frac{1}{\dot{\theta}_0}\right]_{\bar{\theta}} \left[\frac{\partial\theta_m}{\partial m}\right]_{m=0}. \tag{30}$$

Assuming that $\dot{r} = \dot{\bar{r}}_0 := \dot{r}(t = \bar{T}, m = 0)$ is a constant for $T_m \leqslant t \leqslant \bar{T}$, and using the trace distance criteria (8), it can be shown that

$$A = \frac{1}{(1-D)}\left[\int_{\theta_i}^{\bar{\theta}} \frac{\Gamma(\theta)}{\dot{\theta}_0^2}d\theta\right], \tag{31}$$

where $D = \dot{\bar{r}}_0(\bar{r} + r_{\rm fp}\cos\bar{\theta})/(\dot{\bar{\theta}}_0\bar{r}r_{\rm fp}\sin\bar{\theta})$ and $\dot{\bar{\theta}}_0$ is $\dot{\theta}_0$ at $\theta = \bar{\theta}$. Equations (28) and (30) clearly show that, in the limit in which the magnetic field used for quantum control has small amplitude, the optimal time to reach the target fixed point within trace distance ϵ decreases linearly with m for the qubit in the amplitude-damping channel. To validate the above analysis, we have again adopted numerical techniques assuming a temporal dependence for $\mathbf{h}^C(t)$ as in Eq. (25) (results are reported in Fig. 5). In these simulations, the value of $h_{\mu,n}$ is fixed at the beginning of an iteration and it can not change during the course of the evolution. Therefore, $|h_\mu^c|$ can take its maximum possible value of $\alpha(t) = \frac{1}{N_c}\frac{t}{\tau}\sum_{n=1}^{N_c}|\sin(2\pi nt/\tau)|$ only if $\sin(2\pi nt/\tau)$ has the same sign for any t and for a particular n, i.e., $2\pi N_c T_m/\tau \leqslant \pi$. Again, from the definition of Γ of Eq. (27), we get $\Gamma \leqslant 2(|\sin\varphi| + |\cos\varphi|)\alpha$. Therefore, using Eq. (31)

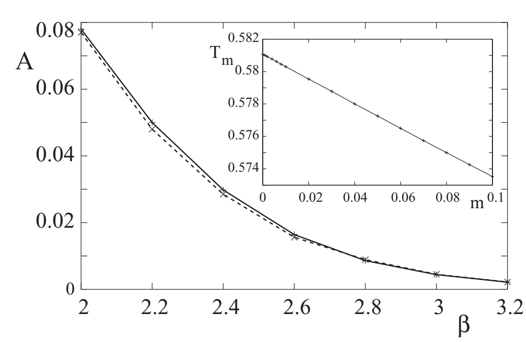

FIG. 5. Comparison between the numerical (solid line) and the analytical bound (32) (dashed line) values of the slope A as a function of β for $N_c = \tau = 10$ and $\epsilon = 0.04$. The initial point $\mathbf{r}_i$ is the same as in Fig. 4. $m/\dot{\theta}_0$ decreases for larger values of β, thus resulting in a better match between the numerical and analytical values in this regime. Inset: variation of T_m as a function of m for small m for $\tau = N_c = 10$, $\beta = 2$, and $\epsilon = 0.04$. As expected, T_m decreases linearly with m.

we finally arrive at an upper bound for the slope A, given by

$$A \leqslant \frac{2}{N_c \tau (1-D)} \sum_{n=1}^{N_c} \times \int_{\theta_i}^{\bar{\theta}} \frac{t\,(|\sin\varphi| + |\cos\varphi|)}{\dot{\theta}_0^2} \left| \sin\left[\frac{2\pi n t}{\tau}\right]\right| d\theta. \tag{32}$$

B. Slowing down relaxation

Here, we are interested in the opposite problem to that analyzed so far. In other words, we would like to find out for how long a qubit subject to amplitude damping can be kept, with the aid of a quantum control represented by a magnetic field of infinite maximum strength, arbitrarily close to a given initial state $\mathbf{r}_i$. Again, one can quantify the notion of closeness by imposing that the trace distance between the evolved state and the initial state is arbitrarily small. In other words, we are interested in applying the optimal control such that $|\mathbf{r}_i - \mathbf{r}(t)| \leqslant \epsilon$ for the maximum time duration $T^{\rm AD}_{\rm slow}$. On the one hand, we are free to control the Bloch vector of the qubit unitarily and instantaneously in the directions tangent to the sphere of radius r_i. On the other hand, the qubit will be subject to uncontrollable decoherence along the radial direction, with its purity changing at speed v. Here, we confine ourselves to the explicit analysis of the case in which the relaxation dynamics can be controlled for an indefinitely long time. (We note that one could define the problem in other ways, namely, one could allow for the quantum state to evolve along a trajectory which crosses the ϵ ball around $\mathbf{r}_i$ several times before finally returning inside it, and calculate the maximal time for which this dynamics is possible.)

For the amplitude-damping channel, in the case of an initial state with $r_i < r_{\rm fp}$ we can see that the speed $v_{\rm AD}$ [and equivalently $\dot{r}(t)$] becomes zero as we approach the angle (see Fig. 6)

$$\theta_3 := \arccos\left[\frac{r_{\rm fp}}{r_i}\left(\sqrt{1-\frac{r_i^2}{r_{\rm fp}^2}} - 1\right)\right]. \tag{33}$$

Thus, if the quantum state of the qubit happens to have initial polar angle θ_3, quantum control with infinite strength will be able to keep the qubit there indefinitely, i.e., $T^{\rm AD}_{\rm slow} \to \infty$ for these initial states. This is because for any point along the ellipsoid defined by Eq. (33), the velocity $\dot{\mathbf{r}}$ is orthogonal to the Bloch vector and therefore it can be controlled by unitaries. In a sense, one could say that unbounded coherent control has allowed us to extend the set of fixed points by adding the set of points with $v = 0$.

IV. DEPOLARIZING CHANNEL

In this section, we address the problem of quantum control of the relaxation when the dissipative process affecting the system is a depolarizing channel [37]. The latter is characterized by the three Lindblad operators

$$(L_1)_{\rm DP} = \sqrt{\gamma_x}\sigma_x; \quad (L_2)_{\rm DP} = \sqrt{\gamma_y}\sigma_y; \quad (L_3)_{\rm DP} = \sqrt{\gamma_z}\sigma_z \tag{34}$$

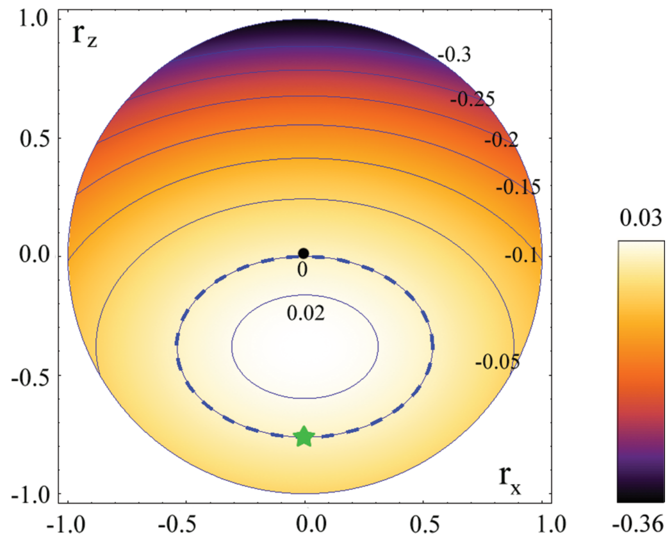

FIG. 6. (Color online) Plot of the speed $v_{\rm AD}$ on the x-z plane of the Bloch sphere when the fixed point is the thermal state corresponding to $\beta = 2$. The curve $v_{\rm AD} = 0$ (dashed line) is an ellipse which passes through the origin of coordinates (black dot) and the fixed point (green star).

and it admits as unique fixed point the fully mixed state $\rho_{\rm fp} = I/2$, i.e., $\mathbf{r}_{\rm fp} = 0$. In the absence of unitary control, the associated master equation (5) is given by

$$\dot{\mathbf{r}} = -2(\Gamma_x r_x, \Gamma_y r_y, \Gamma_z r_z)^\top, \tag{35}$$

where $\Gamma_x := \gamma_y + \gamma_z$, $\Gamma_y := \gamma_x + \gamma_z$, $\Gamma_z := \gamma_x + \gamma_y$, with solution, for the initial condition $\mathbf{r}_i := (r_{ix}, r_{iy}, r_{iz})^\top$,

$$\mathbf{r}(t) = (e^{-2\Gamma_x t} r_{ix}, e^{-2\Gamma_y t} r_{iy}, e^{-2\Gamma_z t} r_{iz})^\top. \tag{36}$$

The relaxation time $T^{\rm DP}_{\rm free}(\mathbf{r}_i;\epsilon)$ from an arbitrary initial state r_i to the fixed point in the absence of quantum control can be found from the trace distance condition (8) and from the solution (36) by solving the implicit equation

$$\left|\mathbf{r}\left[T^{\rm DP}_{\rm free}(\mathbf{r}_i;\epsilon)\right]\right| = \epsilon. \tag{37}$$

Moreover, from Eqs. (34) and (A1) and (A2) of the Appendix, the speed of purity change in spherical coordinates reads as

$$v_{\rm DP}(r,\theta,\varphi) = -r^2\{2\Gamma_z + [(\Gamma_x + \Gamma_y - 2\Gamma_z) + (\Gamma_x - \Gamma_y)\cos 2\varphi]\sin^2\theta\}. \tag{38}$$

This velocity is always negative and it is easy to check that its absolute value is maximum at the intersection of the sphere of radius r with the coordinate axis associated with the minimum value among γ_x, γ_y, and γ_z. The optimal heating velocity is then

$$v^{\rm DP,heat}_{\rm fast}(r) = -2\Gamma_M r^2, \tag{39}$$

where Γ_M is the largest among Γ_x, Γ_y, and Γ_z. Note that, in the special case when any two of the decay rates are equal, one has families of optimal solutions along the circle that is the intersection between the sphere of radius r and the plane of coordinates corresponding to the equal decay rates. Moreover, in the completely symmetric case of $\gamma_x = \gamma_y = \gamma_z := \gamma_0$, the heating speed is given by $v_{\rm fast}^{\rm DP,heat} = -4\gamma_0 r^2$ for all angles θ and φ. Therefore, in this case any control is useless.

Inserting the maximal speed (39) into Eq. (18) we obtain the optimal time

$$T_{\rm fast}^{\rm DP,heat}(\mathbf{r}_i;\epsilon) = \frac{1}{2\Gamma_M}\ln\left[\frac{r_i}{\epsilon}\right]. \quad (40)$$

The function (40) reaches its maximum for a pure state along one of the coordinate axes, i.e.,

$$\max_{\mathbf{r}_i}\ T_{\rm fast}^{\rm DP}(\mathbf{r}_i;\epsilon) = \frac{|\ln\epsilon|}{2\Gamma_M}. \quad (41)$$

This is the largest time one would need to wait in order to bring the system close to the target in the case of unconstrained control. By comparing it with the maximum for the free relaxation time obtainable from Eq. (37), i.e.,

$$\max_{\mathbf{r}_i} T_{\rm free}^{\rm DP}(\mathbf{r}_i;\epsilon) = \frac{|\ln\epsilon|}{2\Gamma_m}, \quad (42)$$

where Γ_m is the smallest among the Γ_x, Γ_y, and Γ_z (reached for a pure state along one of the axes) we notice that the optimal-time control yields a shortening by a factor Γ_m/Γ_M in the evolution time.

In this case, the set of points with $v_{\rm DP} = 0$ coincides with the set of fixed points and, therefore, any control is useless for stopping the relaxation.

V. PHASE-DAMPING CHANNEL

The phase-damping channel is a dissipative process characterized by a single Lindblad operator

$$(L_1)_{\rm PD} = \sqrt{\hat{\gamma}}\sigma_z, \quad (43)$$

where $\hat{\gamma}$ is the decoherence rate. In this case, the master equation in Cartesian coordinates reads as

$$\dot{\mathbf{r}} = -2\hat{\gamma}(r_x,r_y,0)^\top. \quad (44)$$

For an initial quantum state with $\mathbf{r}_i := (r_{ix},r_{iy},r_{iz})^\top$, the solution of the master equation (44) is given by

$$\mathbf{r}(t) = (e^{-2\hat{\gamma}t}r_{ix},e^{-2\hat{\gamma}t}r_{iy},r_{iz})^\top. \quad (45)$$

The locus of the fixed points for this model is given by the z axis, i.e., it is the set of points with $\mathbf{r}_{\rm fp} = (0,0,\bar{r}_{\rm fp})^\top$ and any $\bar{r}_{\rm fp} \in [0,1]$, while the speed of purity change is

$$v_{\rm PD}(r,\theta) = -2\hat{\gamma}r^2\sin^2\theta. \quad (46)$$

From Eq. (45) and the trace distance condition (8), we then find that the relaxation time from r_i to the fixed point in the absence of quantum control is

$$T_{\rm free}^{\rm PD}(\mathbf{r}_i;\epsilon) = \frac{1}{2\hat{\gamma}}\ln\left[\frac{\sqrt{r_{ix}^2 + r_{iy}^2}}{\epsilon}\right]. \quad (47)$$

In this case, since the locus of the fixed points is the whole z axis, the task of speeding up the relaxation is ambiguous. Given an arbitrary initial state $\mathbf{r}_i = (r_{ix},r_{iy},r_{iz})^\top$, the *natural* fixed point of the channel would be $\mathbf{r}_f = (0,0,r_{iz})^\top$. Quantum control can then be used to achieve two different tasks: speeding up the relaxation towards an arbitrary fixed point along the z axis or, alternatively, towards the *natural* fixed point associated with the initial state. The first task is trivial since it can be achieved instantaneously via a unitary rotation to the z axis. On the other hand, the second task is nontrivial and the optimal control strategy is analogous to the one used for the amplitude-damping channel: one should first rotate the state to a position where the absolute value of the speed of purity change is maximum (i.e., to the equator), let the phase-damping channel act and, once the desired purity is reached, perform a final rotation to the *natural* fixed point. In this case, the corresponding optimal relaxation time is given (for $r_i > |r_{iz}| + \epsilon$) by

$$T_{\rm fast}^{\rm PD}(\mathbf{r}_i;\epsilon) = \frac{1}{2\hat{\gamma}}\ln\left[\frac{r_i}{|r_{iz}| + \epsilon}\right]. \quad (48)$$

Comparing Eq. (47) with (48), one can see that quantum control speeds up the relaxation for all initial states with $r_{iz} \neq 0$. However, if we use, as done for the previous channels, the figure of merit based on the worst-case scenario, this advantage is lost. Indeed, it is easy to check that the maximum over $\mathbf{r}_i$ of the free evolution time (47) and of the optimal relaxation time (48) is, in both cases, equal to $|\ln\epsilon|/(2\hat{\gamma})$.

Furthermore, also for the phase-damping channel, similarly to the case of the depolarizing channel, quantum control can keep the qubit near its initial state for indefinite time only if the initial state happens to be a fixed point along the z axis.

VI. DISCUSSION

We have studied how the rate of relaxation of a qubit in the presence of some paradigmatic Markovian quantum channels (generalized amplitude damping, depolarization, and phase damping) can be sped up or slowed down using optimal control. We analytically discussed the situation in which a generic initial state should reach the fixed point of the dynamics up to an arbitrarily small distance. Our results suggest that optimal control can not speed up the natural cooling rate of a thermal qubit in the presence of a cold bath. However, it is possible to heat the qubit from an initial thermal state to its fixed point (another thermal state with lower purity) in finite time in the presence of a quantum control of large strength. We have also analyzed the relaxation of a qubit in the presence of a generic control Hamiltonian with infinitesimal strength m. Here, the optimized relaxation time decreases linearly with m, with the slope depending on the explicit form of the Hamiltonian. We have also presented numerical data supporting our analytical results. Finally, we have given a measure of the performance of the quantum control in the worst-case scenario by maximizing the time duration of the evolutions with respect to the possible initial states of the qubit. Quantum control enhances this performance with respect to the uncontrolled decoherence in the cases of the generalized amplitude damping and depolarizing channels. Time-optimal control of a two-level dissipative quantum system has also been studied elsewhere [12–20] using the Pontryagin maximum principle and geometrical methods [38]. In our simplified approach, we further addressed the case of the time-optimal relaxation of a qubit towards the fixed point of a depolarizing channel. Moreover, the inverse problem of slowing down the relaxation from an arbitrary initial quantum state of the qubit was not considered in [12–20]. Note that this situation can be also thought as a "storage" procedure for certain special states. We also found analytical expressions for the optimal-time durations, which was possible in the geometric approach only for the saturation problem in NMR subject to longitudinal and transverse relaxation [17]. Finally, we considered the broader situation in which the final target of the quantum motion need not be reached exactly, but up to an arbitrarily small trace distance. The next step would be to consider time-optimal quantum control with fixed target fidelity for open systems in higher dimensions (e.g., dissipative channels with interacting qubits, exploiting some recent results on optimal coherent control [39]) and to consider the case of time-dependent Lindblad operators.

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APPENDIX: SPEED OF CHANGE FOR THE PURITY

When we are only concerned about the quantum motion of the qubit along the radial coordinate, in other words when we are only interested in the speed of change of the purity of our quantum system, we have to study the quantity $v = dP/dt$. Using the relation $P = (1 + r^2)/2$ and the master equation (5) in spherical coordinates, a simple algebra shows that the speed of change of the purity can be explicitly written in general as

$$\begin{aligned}\frac{v(r,\theta,\varphi)}{r} &= -(a_+ - a_-)\cos\theta + 2\,\mathrm{Re}[(d_+ - d_-^*)e^{i\varphi}]\sin\theta \\ &\quad + \frac{r}{2}\{-(b + a_+ + a_-) + \mathrm{Re}(ce^{2i\varphi}) \\ &\quad + [b - a_+ - a_- - \mathrm{Re}(ce^{2i\varphi})]\cos 2\theta \\ &\quad + 2\,\mathrm{Re}[(d_+ + d_-^*)e^{i\varphi}]\sin 2\theta\},\end{aligned} \tag{A1}$$

where the coefficients $a_\pm, b, c, d_\pm$ depend upon the Lindblad operators in the following manner:

$$\begin{aligned} a_\pm &:= \sum_a \gamma_a |l_{a\pm}|^2, \quad b := \sum_a \gamma_a (1 + |l_{az}|^2), \\ c &:= \sum_a \gamma_a l_{a+}^* l_{a-}, \quad d_\pm := \sum_a \gamma_a l_{a\pm}^* l_{az}, \end{aligned} \tag{A2}$$

and we have defined $l_{a\pm} := l_{ax} \pm i l_{ay}$.

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