

Local exclusion principle for identical particles obeying intermediate and fractional statistics

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A local exclusion principle is observed for identical particles obeying intermediate and fractional exchange statistics in one and two dimensions, leading to bounds for the kinetic energy in terms of the density. This has implications for models of Lieb-Liniger and Calogero-Sutherland type and implies a nontrivial lower bound for the energy of the anyon gas whenever the statistics parameter is an odd numerator fraction. We discuss whether this is actually a necessary requirement.

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I. INTRODUCTION

The majority of interesting phenomena in many-body quantum mechanics are in some way associated with the fundamental concepts of identical particles and statistics. Elementary identical particles in three spatial dimensions are either bosons, obeying Bose-Einstein statistics, or fermions, obeying Fermi-Dirac statistics. The former are usually represented using wave functions which are symmetric under particle permutations, while the latter implement Pauli's exclusion principle by exhibiting total antisymmetry under particle interchange. On the other hand, for point particles living in one and two dimensions, there are logical possibilities different from bosons and fermions, so-called intermediate or fractional statistics [1–3]. Although first regarded as of purely academic interest—filling the loopholes in the arguments leading to the two standard permutation symmetries—these have recently become a reality in the laboratory, with the advent of trapped bosonic gases [4] and quantum Hall physics [5], and thus the discoveries of effective models of particles (or quasiparticles) that seem to obey these generalized rules for identical particles and statistics. We refer to Refs. [6–8] for extensive reviews on these topics.

Although noninteracting bosons and fermions are well understood in terms of single-particle Hilbert spaces and operators, the same cannot be said about particles obeying these generalized interchange statistics. Namely, despite some effort in this direction [9,10], many-particle quantum states for intermediate and fractional exchange statistics have in general not admitted a simple description in terms of single-particle states restricted by some exclusion principle. The reason for this difficulty is that the general symmetry of the wave function under particle interchange is naturally modeled using pairwise or many-body interactions, hence leaving the much simpler realm of single-particle Hamiltonians (and also introducing other mathematical difficulties as well, already at the formulation of these models).

As a different approach, we stress in the following that the effects of exclusion are also encoded in inequalities for many-particle energy forms, such as the Lieb-Thirring inequality [11]. For the case of identical spinless fermions in an external potential V in d -dimensional space, it states that there is a *uniform* bound for the energy of a normalized N -particle state ψ :

$$\langle \psi, \hat{H} \psi \rangle \geq - \sum_{k=0}^{N-1} |\lambda_k| \geq -C_d \int |V_-(x)|^{1+d/2} d^d x, \quad (1)$$

with the N -particle Hamiltonian operator

$$\hat{H} = \hat{T}_0 + \hat{V} = \sum_{j=1}^N \left(-\frac{1}{2} \nabla_j^2 + V(\mathbf{x}_j) \right),$$

the conventions $\hbar = m = 1$, $V_{\pm} := (V \pm |V|)/2$, and a positive constant C_d . The inequality (1) incorporates Pauli's exclusion principle via the intermediate sum over the negative energy levels λ_k of the one-particle Hamiltonian $\hat{h} = -\frac{1}{2} \nabla^2 + V(x)$. It furthermore incorporates the uncertainty principle and is in fact equivalent to the kinetic energy inequality

$$\langle \psi, \hat{T}_0 \psi \rangle \geq \frac{d(2/C_d)^{2/d}}{(d+2)^{1+2/d}} \int \rho(x)^{1+2/d} d^d x, \quad (2)$$

involving the one-particle density ρ of ψ , normalized $\int \rho(x) d^d x = N$. In dimension $d = 3$, the expression on the right-hand side of relation (2) may be recognized as the kinetic energy approximation from Thomas-Fermi theory. It is in this case conjectured [11] that relation (2) holds with exactly the Thomas-Fermi expression on the right. The best-known result is, however, smaller by a factor of $(3/\pi^2)^{1/3}$ [12].

The bounds (1) and (2) need to be weakened in the case of weaker exclusion. In the case that each single-particle state can be filled q times (e.g., in models with q spin states or cf. Gentile intermediate statistics [13]) the right-hand side of the inequalities (1) [respectively (2)] are to be multiplied by q (respectively $q^{-2/d}$). Bosons can then be accommodated by $q = N$, yielding trivial bounds as $N \rightarrow \infty$.

In this work we report on a new set of Lieb-Thirring-type inequalities for intermediate and fractional statistics, which follow from a corresponding *local* version of the exclusion principle, applicable to such interacting systems. Our approach is very much inspired by the work [14] of Dyson and Lenard (see also Ref. [15]), who used only such a local form of the Pauli principle to rigorously prove the stability of ordinary fermionic matter in the bulk [the inequalities (1) and (2) were subsequently invented by Lieb and Thirring to simplify their proof]. Although the numerical constants resulting from our method are comparatively weak, we believe the forms of our bounds to be conceptually very useful, and as a result we also learn something nontrivial about the elusive anyon gas.

Starting by recalling the models for intermediate and fractional statistics which we are concerned with here, we proceed by showing how a local form of the exclusion principle can be established for such statistics, leading to bounds for the

kinetic energy in terms of the one-particle density $\rho(\mathbf{x})$. For clarity, we leave out some of the technical details, referring to the mathematical papers [16,17], and instead focus on general aspects of the procedure. With these preparations, we consider the problem of determining the ground-state energy for a large number N of anyons in a harmonic oscillator potential and can conclude that the energy grows like $N^{3/2}$ under the assumption that the anyonic statistics phase is an odd numerator rational multiple of π . In the final section we discuss a structural difference between such odd and even numerator fractions using a class of trial states which are related to the Read-Rezayi states for the fractional quantum Hall effect.

II. IDENTICAL PARTICLES IN ONE AND TWO DIMENSIONS

We recall three well-established models for intermediate and fractional exchange statistics for scalar nonrelativistic quantum mechanical particles in one or two spatial dimensions. As mentioned in the introduction, there are by now a number of standard references for their background and derivations, which we accordingly skip here. We mainly follow the notation in Ref. [6], with technical details addressed in Ref. [17].

Identical particles in 2D, anyons, have the possibility to pick up an arbitrary but fixed phase $e^{i\alpha\pi}$ upon continuous simple interchange of two particles [2,3]. A standard way to model such (Abelian) anyons, in the so-called magnetic gauge, is by means of bosons in \mathbb{R}^2 together with a statistical magnetic interaction given by the vector potential

$$\mathbf{A}_j = \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)I}{|\mathbf{x}_j - \mathbf{x}_k|^2}, \quad \alpha \in \mathbb{R} \pmod{2},$$

where $\mathbf{x}I$ denotes a 90° counterclockwise rotation of the vector \mathbf{x} . This attaches to every particle an Aharonov-Bohm point flux of strength $2\pi\alpha$, felt by all the other particles. The kinetic energy for N such particles is thus given by $T_A := \langle \psi, \hat{T}_A \psi \rangle$,

$$\hat{T}_A := \frac{1}{2} \sum_{j=1}^N D_j^2, \quad (3)$$

where $D_j = -i\nabla_j + \mathbf{A}_j$, and the wave function ψ is represented as a completely symmetric square-integrable function on $(\mathbb{R}^2)^N$. The case $\alpha = 0$ then corresponds to bosons, and $\alpha = 1$ corresponds to fermions.

The case of identical particles confined to move in only one spatial dimension is special and in some sense degenerate, since particles cannot be interchanged continuously without colliding. In quantum mechanics this necessitates some choice of boundary conditions for the wave function at the collision points. It turns out that, depending on which approach one takes to quantization [2,6,18], identical particles in one dimension (1D) can again be modeled as bosons, i.e., wave functions symmetric under the flip $r \mapsto -r$ of any two relative particle coordinates $r := x_j - x_k$, together with a local interaction potential, singular at $r = 0$, and either of the form $\delta(r)$ or $1/r^2$. We write

$$V_{LL}(r) := 2\eta\delta(r), \quad V_{CS}(r) := \frac{\alpha(\alpha-1)}{r^2}, \quad (4)$$

with statistics parameters $\eta, \alpha \in \mathbb{R}$, for the respective models resulting from a Schrödinger (respectively, Heisenberg) approach to quantization. These statistics potentials correspond to the choices of boundary conditions for the wave function ψ at the boundary $r = 0$ of the configuration space

$$\frac{\partial \psi}{\partial r} = \eta \psi, \quad \text{at } r = 0^+, \quad \text{resp. } \psi(r) \sim r^\alpha, \quad \text{as } r \rightarrow 0^+.$$

Here $\eta = 0$ (respectively $\alpha = 0$) represent bosons (Neumann boundary conditions) while $\eta = +\infty$ (respectively $\alpha = 1$) represent fermions (Dirichlet or analytically vanishing boundary conditions; see Ref. [17]). As suggested by such pairwise boundary conditions, one may define [19] the total kinetic energy for a normalized completely symmetric wave function ψ describing N identical particles on the full real line \mathbb{R} to be $T_{LL/CS} := \langle \psi, \hat{T}_{LL/CS} \psi \rangle$ where

$$\hat{T}_{LL/CS} := -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} V_{LL/CS}(x_j - x_k). \quad (5)$$

In other words, the LL case in our notation is nothing but the Lieb-Liniger model for one-dimensional bosons with pairwise Dirac δ interactions [20], while the CS case corresponds to the homogeneous part of the Calogero-Sutherland model with inverse-square interactions [21]. It is well known that these models can describe a continuous interpolation between the properties of bosons and fermions for certain ranges of the statistics parameters. We restrict the following results to $\eta \geq 0$ (Lieb-Liniger-type intermediate statistics) and $\alpha \geq 1$ (Calogero-Sutherland-type superfermions) for which the statistics potentials (4) are non-negative, i.e., repulsive.

In all of the above cases, the one-particle density $\rho(\mathbf{x})$ is defined such that the expected number of particles on a local region Q of space (typically a d -dimensional cube in the following) equals

$$\int_Q \rho(\mathbf{x}) d\mathbf{x} = \sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\psi|^2 \chi_Q(\mathbf{x}_j) d\mathbf{x},$$

where $\chi_Q \equiv 1$ on Q and $\chi_Q \equiv 0$ on the complement Q^c . In particular, $\int_{\mathbb{R}^d} \rho = N$. Similarly, it is useful to be able to speak about the expected kinetic energy of a wave function on a local region. For fermions or bosons we naturally define this quantity to be

$$T_0^Q := \frac{1}{2} \sum_{j=1}^N \int_{\mathbb{R}^{dN}} |\nabla_j \psi|^2 \chi_Q(\mathbf{x}_j) d\mathbf{x}.$$

Analogously for anyons,

$$T_A^Q := \frac{1}{2} \sum_{j=1}^N \int_{\mathbb{R}^{2N}} |D_j \psi|^2 \chi_Q(\mathbf{x}_j) d\mathbf{x},$$

and for 1D intermediate statistics, $T_{LL/CS}^Q :=$

$$\frac{1}{2} \sum_{j=1}^N \int_{\mathbb{R}^N} \left(|\partial_j \psi|^2 + \sum_{k \neq j} V_{LL/CS}(x_j - x_k) |\psi|^2 \right) \chi_Q(x_j) d\mathbf{x}.$$

Note that if the full space \mathbb{R}^d has been partitioned into a family of nonoverlapping regions $\{Q_k\}$ then the total kinetic energy

decomposes as $T_{0/LL/CS/A} = \sum_k T_{0/LL/CS/A}^{Q_k}$. We furthermore write $T_F = T_0$ to denote the free kinetic energy for the particular case of fermions in \mathbb{R}^3 , i.e., totally antisymmetric ψ .

III. LOCAL EXCLUSION

The starting point for our energy bounds will be the following local consequence of the Pauli exclusion principle for fermions, which was used by Dyson and Lenard in their proof of stability of matter [14]: Let ψ be a wave function of n spinless fermions in \mathbb{R}^3 , i.e., antisymmetric with regard to every pair of particle indices, and let Q be a cube of side length l . Then, for the contribution to the free kinetic energy with all particles in Q ,

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |\nabla_j \psi|^2 dx \geq (n-1) \frac{\xi_F^2}{l^2} \int_{Q^n} |\psi|^2 dx, \quad (6)$$

where $\xi_F = \pi/\sqrt{2}$. In other words, due to the Pauli principle, the energy is nonzero for $n \geq 2$ and grows at least linearly with n (indeed linearity proves to be sufficient; cf. also [15]). In Ref. [14], Q was replaced by a ball of radius l and $\sqrt{2}\xi_F$ by the smallest positive root of the equation $(d^2/d\xi^2)(\sin \xi/\xi) = 0$. The inequality (6) follows by expanding ψ in the eigenfunctions of the Neumann Laplacian on Q , or by the pairwise method below at the cost of a slightly weaker constant ξ_F .

Now, for the 1D case we introduce $\xi_{LL}(\eta l)$ [respectively $\xi_{CS}(\alpha)$] to be the smallest positive solutions of $\xi \tan \xi = \eta l$ [respectively $(d/d\xi)[\xi^{1/2}J(\xi)] = 0$], where J is the Bessel function of order $\alpha - 1/2$. These $\xi_{LL/CS}$ arise as quantization conditions for the wave function upon considering the Neumann problems

$$[-\partial_r^2 + V_{LL/CS}(r)]\psi = \lambda\psi, \quad \partial_r \psi|_{r=\pm l} = 0, \quad (7)$$

in the pairwise relative coordinate r on an interval $[-l, l]$, yielding a lowest bound for the energy $\lambda = \xi_{LL/CS}^2/l^2$. A good numerical approximation to ξ_{LL} is given by $\xi_{LL}(t) \approx \arctan \sqrt{t + 4t^2/\pi^2}$ for all $t \geq 0$ (see Fig. 1), while we have $\xi_{CS}(1) = \pi/2$ and, asymptotically, $\xi_{CS}(\alpha) \sim \alpha$ as $\alpha \rightarrow \infty$ (see Fig. 2).

In the case of anyons we define the expression

$$\xi_A(\alpha, n) := \min_{p \in \{0, 1, \dots, n-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|, \quad (8)$$

which measures the fractionality of the parameter α and arises in a bound for a local pairwise magnetic operator, which is the 2D analog to relation (7) (and defined on an annulus instead of an interval [16]). The absolute quantity which is being minimized in the expression (8) can be understood as the magnetic gauge phase $(2p+1)\alpha\pi$ arising from a pairwise interchange of two anyons, with the odd integer $2p+1$ depending on the number p of other anyons that can appear inside such a two-anyon interchange loop and the additional $+1$ stemming from the statistics flux of the interchanging pair itself. This is taken modulo the pairwise orbital angular momentum of the wave function which is an even integer $-2q$ due to the underlying bosonic symmetry. Note that for bosons $\xi_A(\alpha = 0, n) \equiv 0$ while for fermions we have $\xi_A(\alpha = 1, n) \equiv 1$ for all n .

We call the following observation a *local exclusion principle* for generalized exchange statistics since it implies that the

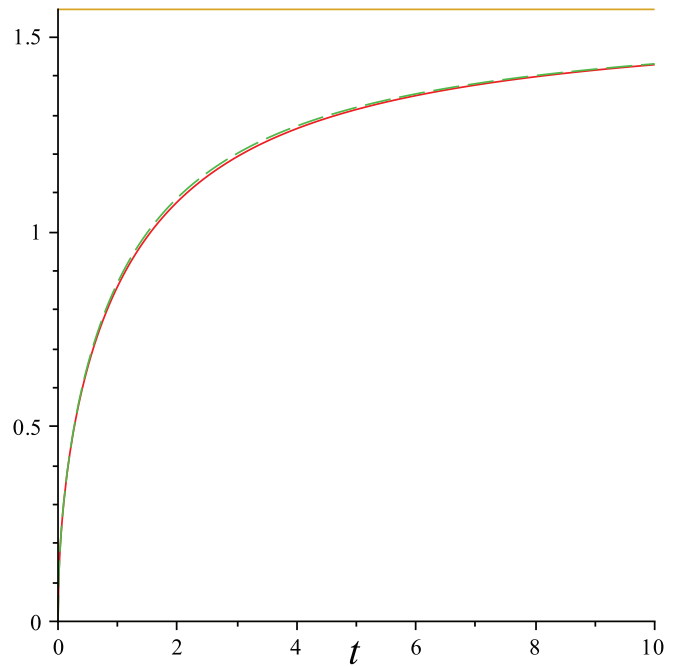


FIG. 1. (Color online) Plot of $\xi_{LL}(t)$ (solid) and $\arctan \sqrt{t + 4t^2/\pi^2}$ (dashed) as a function of $t \geq 0$.

local kinetic energy is nonzero whenever we have more than one particle, and hence that the particles cannot occupy the same single-particle state (which on a local region would be the zero-energy ground state).

Lemma 1: Local exclusion principle. Given any finite interval $Q \subset \mathbb{R}$ of length $|Q|$, we have for $\eta \geq 0$

$$\int_{Q^n} \bar{\psi} \hat{T}_{LL} \psi dx \geq (n-1) \frac{\xi_{LL}(\eta|Q|)^2}{|Q|^2} \int_{Q^n} |\psi|^2 dx, \quad (9)$$

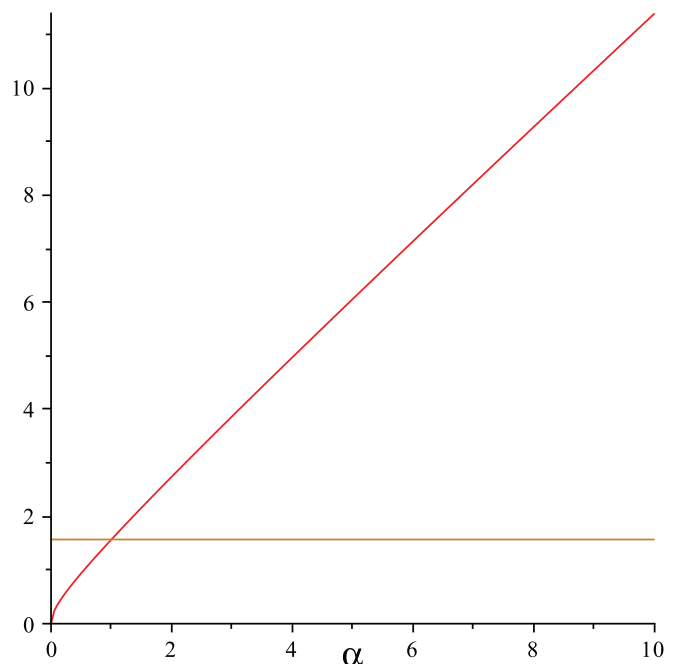


FIG. 2. (Color online) Plot of $\xi_{CS}(\alpha)$ as a function of $\alpha \geq 0$.

and for $\alpha \geq 1$

$$\int_{Q^n} \bar{\psi} \hat{T}_{CS} \psi dx \geq (n-1) \frac{\xi_{CS}(\alpha)^2}{|Q|^2} \int_{Q^n} |\psi|^2 dx, \quad (10)$$

while for a square $Q \subset \mathbb{R}^2$ with area $|Q|$ and any $\alpha \in \mathbb{R}$

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |D_j \psi|^2 dx \geq (n-1) \frac{c \xi_A(\alpha, n)^2}{|Q|} \int_{Q^n} |\psi|^2 dx, \quad (11)$$

with $c = 0.056$. It then follows for the expected kinetic energy on a d -dimensional cube Q with volume $|Q|$ that

$$T_{LL/CS/A/F}^Q \geq \frac{\xi_{LL/CS/A/F}^2}{|Q|^{2/d}} \left(\int_Q \rho(x) dx - 1 \right)_+, \quad (12)$$

where $\xi_{LL/CS/A/F}$ here stands for $\xi_{LL}(\eta|Q|)$, $\xi_{CS}(\alpha)$, $\sqrt{c} \xi_A(\alpha, N)$ (respectively ξ_F), with corresponding dimension $d = 1, 1, 2, 3$.

Let us consider the proof for the one-dimensional (1D) Calogero-Sutherland case. Using the separation of the center-of-mass $n \sum_j \partial_j^2 = (\sum_j \partial_j)^2 + \sum_{j < k} (\partial_j - \partial_k)^2$, the

(Neumann) kinetic energy for $n \geq 2$ particles on an interval $Q = [a, b]$ is

$$\begin{aligned} & \int_{Q^n} \bar{\psi} \hat{T}_{CS} \psi dx \\ & \geq \int_{Q^n} \sum_{j < k} \bar{\psi} \left(-\frac{1}{2n} (\partial_j - \partial_k)^2 + V_{CS}(x_j - x_k) \right) \psi dx \\ & \geq \frac{2}{n} \sum_{j < k} \int_{Q^{n-2}} \int_Q \int_{[-\delta(R), \delta(R)]} \bar{\psi} [-\partial_r^2 + V_{CS}(r)] \psi dr dR dx' \\ & \geq \frac{2}{n} \sum_{j < k} \int_{Q^{n-2}} \int_Q \frac{\xi_{CS}(\alpha)^2}{\delta(R)^2} \int_{[-\delta(R), \delta(R)]} |\psi|^2 dr dR dx', \end{aligned} \quad (13)$$

where for each particle pair we have introduced $R := (x_j + x_k)/2$, $r := x_j - x_k$, $x' = (x_1, \dots, x_j, \dots, x_k, \dots, x_N)$, and $\delta(R) := 2 \min\{|R - a|, |R - b|\}$. We then use relation (13) and $\delta(R)^{-2} \geq |Q|^{-2}$ to obtain relation (10). By inserting the partition of unity $1 = \sum_{A \subseteq \{1, \dots, N\}} \prod_{l \in A} \chi_Q(x_l) \prod_{l \notin A} \chi_{Q^c}(x_l)$ into the expression for T_{CS}^Q we then obtain (cf. Refs. [14–17])

$$\begin{aligned} T_{CS}^Q &= \sum_A \int_{\mathbb{R}^N} \sum_{j \in A} \frac{1}{2} \left(|\partial_j \psi|^2 + \sum_{(j \neq) k=1}^N V_{CS}(x_j - x_k) |\psi|^2 \right) \prod_{l \in A} \chi_Q(x_l) \prod_{l \notin A} \chi_{Q^c}(x_l) dx \\ &\geq \sum_A \int_{(Q^c)^{N-|A|}} \int_{Q^{|A|}} \frac{1}{2} \left(\sum_{j \in A} |\partial_j \psi|^2 + \sum_{j \neq k \in A} V_{CS}(x_j - x_k) |\psi|^2 \right) \prod_{l \in A} dx_l \prod_{l \notin A} dx_l \\ &\geq \sum_A (|A| - 1) \frac{\xi_{CS}(\alpha)^2}{|Q|^2} \int_{(Q^c)^{N-|A|}} \int_{Q^{|A|}} |\psi|^2 \prod_{l \in A} dx_l \prod_{l \notin A} dx_l \\ &= \frac{\xi_{CS}(\alpha)^2}{|Q|^2} \int_{\mathbb{R}^N} \left(\sum_{j=1}^N \chi_Q(x_j) - 1 \right) |\psi|^2 dx, \end{aligned}$$

where in the last step we again used the partition of unity. This proves relation (12) in the $\alpha \geq 1$ Calogero-Sutherland case. The Lieb-Liniger case follows similarly, while in the anyon case the application of the above-mentioned pairwise magnetic operator inequality gives rise to a local repulsive inverse-square pair potential, with its strength measured by the expression (8). We refer to Refs. [16,17] for the detailed proofs.

The constants $\xi_{LL/CS/A/F}^2$ of proportionality in relation (12) appear as lower bounds on the strength of local exclusion, and could, e.g., be compared with the global constant of proportionality in Haldane's generalized exclusion statistics [9]. For the case of anyons, the constant $\xi_A \propto \xi_A(\alpha, N)$ is actually N dependent, and it is clear from the definition (8) that this constant can become identically zero for sufficiently large N if α is an even numerator (reduced) fraction. However, we have shown in Ref. [16] that for $\alpha = \mu/\nu$ an *odd numerator fraction*, the limiting constant is nonzero and equal to $\lim_{N \rightarrow \infty} \xi_A(\alpha, N) = 1/\nu$ (see Fig. 3). It hence also becomes weaker with a bigger denominator ν in the statistics parameter.

For irrational α the constant is nonzero for all finite N , but the limit is again zero. We return to a discussion on the true dependence on α for the exclusion and statistics of anyons below.

IV. LIEB-THIRRING-TYPE INEQUALITIES

The inequalities (1) and (2) for fermions combine the Pauli exclusion principle with the uncertainty principle to produce nontrivial and useful bounds for the energy as the number of particles N becomes large. We complement the local form of the exclusion principle above with the following *local form of the uncertainty principle* on a d -dimensional cube Q , valid for the free kinetic energy of any bosonic wave function ψ , and hence applicable in our cases of intermediate statistics after discarding the positive statistics potentials or, in the case of anyons, using the diamagnetic inequality $|D_j \psi| \geq |\nabla_j \psi|$:

$$T_{0/LL/CS/A}^Q \geq c_1 \frac{\int_Q \rho^{1+2/d} dx}{(\int_Q \rho dx)^{2/d}} - c_2 \frac{\int_Q \rho dx}{|Q|^{2/d}}. \quad (14)$$

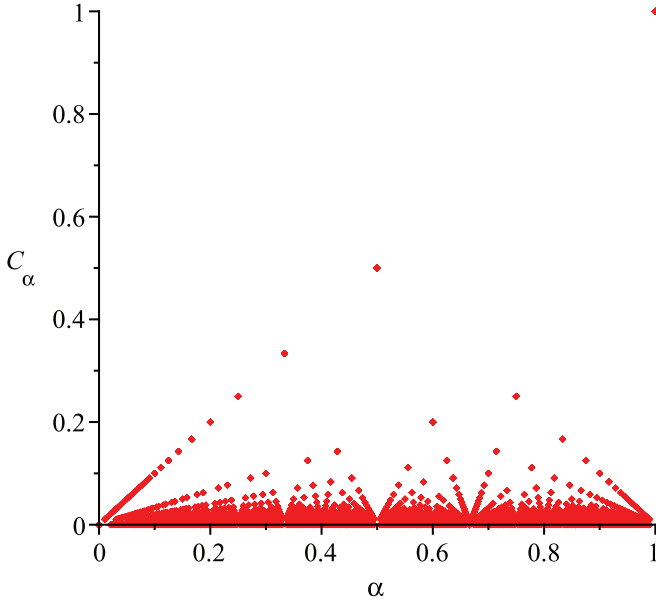


FIG. 3. (Color online) A sketch of $C_\alpha = \lim_{N \rightarrow \infty} \xi_A(\alpha, N)$ as a function of α .

The constants $c_2 > c_1 > 0$ only depend on d . Mathematically, relation (14) is a form of Poincaré-Sobolev inequality, and we refer to refs. [16,17] for details and proofs. Note that the right-hand side is bigger for less constant density but scales with the number of particles only as N (in contrast to the Lieb-Thirring inequality).

Combining local uncertainty with local exclusion, and cleverly splitting the space into smaller cubes depending on the density [the bound (12) is strongest for cubes Q such that the expected number of particles to be found on Q is $\int_Q \rho \approx 2$], one can then prove the following energy bounds:

Theorem 2: L-T inequalities for anyons. For any $\alpha \in \mathbb{R}$, the free kinetic energy for N anyons satisfies the bound

$$T_A \geq C_A \xi_A(\alpha, N)^2 \int_{\mathbb{R}^2} \rho(x)^2 dx, \quad (15)$$

for some constant $10^{-4} \leq C_A \leq \pi$. It follows that if $\alpha = \mu/\nu$ is a reduced fraction with odd numerator μ and the density ρ is supported on an area L^2 then

$$T_A/L^2 \geq C_A \frac{\bar{\rho}^2}{\nu^2}, \quad \bar{\rho} := N/L^2. \quad (16)$$

Theorem 3: L-T inequalities for 1D Lieb-Liniger. For $\eta \geq 0$

$$T_{LL} \geq C_{LL} \int_{\mathbb{R}} \xi_{LL}[2\eta/\rho(x)]^2 \rho(x)^3 dx, \quad (17)$$

for some constant $10^{-5} \leq C_{LL} \leq 2/3$. In particular, if ρ is homogeneous, e.g., $\rho \leq \gamma \bar{\rho}$ for some $\gamma > 0$, then

$$T_{LL} \geq C_{LL} \xi_{LL}[2\eta/(\gamma \bar{\rho})]^2 \int_{\mathbb{R}} \rho(x)^3 dx, \quad (18)$$

and if ρ is supported on an interval of length L

$$T_{LL}/L \geq C_{LL} \xi_{LL}[2\eta/(\gamma \bar{\rho})]^2 \bar{\rho}^3, \quad \bar{\rho} := N/L. \quad (19)$$

It is illustrative to compare with the results of Lieb and Liniger [20], where for a free system in the thermodynamic

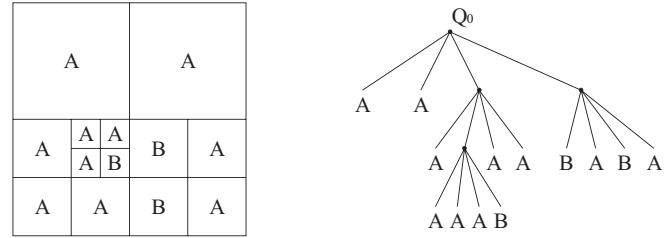


FIG. 4. Example of a splitting of Q_0 and a corresponding tree \mathbb{T} of subsquares. For the B square at level 3 in the tree, the set $\mathcal{A}(Q)$ of all associated A squares (cf. $\mathcal{A}_1(Q)$ in the text) consists of eight elements, whereas for the two B squares at level 2, $\mathcal{A}(Q)$ coincides and has four elements.

limit $N, L \rightarrow \infty$ with fixed density $\bar{\rho}$, $T_{LL}/L \rightarrow \frac{1}{2} e(2\eta/\bar{\rho}) \bar{\rho}^3$ with $e(t) \sim t, t \ll 1, e(t) \rightarrow \frac{\pi^2}{3}, t \rightarrow \infty$ (see also Ref. [22]).

Theorem 4: L-T inequalities for 1D C.-S. For $\alpha \geq 1$ and arbitrary intervals Q such that the expected number of particles $\int_Q \rho \geq 2$

$$T_{CS}^Q \geq C_{CS} \xi_{CS}(\alpha)^2 \frac{(\int_Q \rho(x) dx)^3}{|Q|^2}, \quad (20)$$

with a constant $1/32 \leq C_{CS} \leq 2/3$. It follows in particular that if ρ is confined to a length L and $N \geq 2$ then

$$T_{CS}/L \geq C_{CS} \xi_{CS}(\alpha)^2 \bar{\rho}^3, \quad \bar{\rho} := N/L. \quad (21)$$

Compare with Calogero and Sutherland [21], where one finds $T_{CS}/L \rightarrow \frac{\pi^2}{6} \alpha^2 \bar{\rho}^3$ in the thermodynamic limit $N, L \rightarrow \infty$.

The reason for the more technical forms (17) and (20) as compared to the bounds (2) and (15) is the local dependence of the strength of exclusion in the Lieb-Liniger case [respectively, the possibility for arbitrarily strong exclusion ($\alpha \rightarrow \infty$) in the Calogero-Sutherland case]. We sketch a proof below only for the simpler anyonic case and refer readers to Refs. [16,17,23] for further details. For an application of the same method to fermions in three dimensions (3D) and the generalization to q spin states, we refer to Ref. [24], where a model for point interactions was considered.

Let us for simplicity assume ρ to be supported on some square Q_0 in the plane which we proceed to split into four smaller squares iteratively, organizing the resulting subsquares Q in a tree \mathbb{T} (see Fig. 4). The procedure can be arranged so that Q_0 is finally covered by a collection $Q \in \mathbb{T}_B$ of nonoverlapping squares marked B such that $2 \leq \int_Q \rho < 8$, and $Q \in \mathbb{T}_A$ marked A such that $0 \leq \int_Q \rho < 2$, and such that at least one B square is at the topmost level of every branch of the tree \mathbb{T} . On the B squares we use relation (12) together with relation (14) to obtain (with $c'_k > 0$ some numerical constants)

$$T_A^Q \geq \xi_A(\alpha, N)^2 \left(c'_1 \int_Q \rho^2 + \frac{c'_2}{|Q|} \right), \quad Q \in \mathbb{T}_B. \quad (22)$$

The A squares are further grouped into a subclass A_2 on which the density is sufficiently nonconstant, $\int_Q \rho^2 > \frac{2c_2}{c_1} (\int_Q \rho)^2 / |Q|$ for $Q \in \mathbb{T}_{A_2} \subseteq \mathbb{T}_A$, so that by

relation (14)

$$T_A^Q > \frac{c_1}{4} \int_Q \rho^2, \quad Q \in \mathbb{T}_{A_2}, \quad (23)$$

and a subclass A_1 on which $\int_Q \rho^2 \leq \frac{2c_2}{c_1} (\int_Q \rho)^2 / |Q|$. One can then use the structure of the splitting of squares to prove that, for the set $\mathcal{A}_1(Q_B)$ of all such A_1 squares which can be found by stepping backwards in the tree \mathbb{T} from a fixed B-square Q_B and then one step forward,

$$\sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \leq \sum_{k=0}^{\infty} 3 \frac{2c_2}{c_1} \frac{2^k}{4^k |Q_B|} = \frac{32c_2}{c_1} \frac{1}{|Q_B|}. \quad (24)$$

In other words the energy on all subsquares with constant low density is dominated by that from exclusion on the B squares. We therefore find from relation (22) that

$$T_A^{Q_B} \geq \xi_A(\alpha, N)^2 \left(c'_1 \int_{Q_B} \rho^2 + c'_3 \sum_{Q \in \mathcal{A}_1(Q_B)} \int_Q \rho^2 \right),$$

and hence, since all A_1 squares are covered in this way, $T_A = \sum_{Q \in \mathbb{T}} T_A^Q \geq \sum_{Q \in \mathbb{T}_B \cup \mathbb{T}_{A_2}} T_A^Q \geq C_A \xi_A(\alpha, N)^2 \int_{Q_0} \rho^2$ for some numerical constant $C_A > 0$.

For relations (19) and (16) we use $\int_{Q_0} \rho^p dx \geq N^p |Q_0|^{1-p}$, and for relation (16) we use the fact that $\lim_{N \rightarrow \infty} \xi_A(\alpha, N) = 1/\nu$ for such odd numerator fractions and zero otherwise.

V. AN APPLICATION TO HARMONIC OSCILLATOR CONFINEMENT

Consider N anyons with statistics parameter α confined in an external one-body harmonic oscillator potential $V(\mathbf{x}) = \frac{\omega^2}{2} |\mathbf{x}|^2$. Using the bound (15) for the kinetic energy we obtain as a lower bound for the total energy the following functional of the density:

$$T_A + \langle \hat{V} \rangle_\psi \geq \int_{\mathbb{R}^2} \left(C_A \xi_A(\alpha, N)^2 \rho(\mathbf{x})^2 + \frac{\omega^2}{2} |\mathbf{x}|^2 \rho(\mathbf{x}) \right) dx. \quad (25)$$

It is straightforward [17] to extremize this functional with regard to ρ under the constraint $\int_{\mathbb{R}^2} \rho = N$ to obtain the minimizer

$$\rho(\mathbf{x}) = \frac{(\omega \xi_A(\alpha, N) \sqrt{2C_A N / \pi} - \omega^2 |\mathbf{x}|^2 / 2)_+}{2C_A \xi_A(\alpha, N)^2},$$

and therefore the (rigorous) lower bound for the ground-state energy E_0 :

$$T_A + \langle \hat{V} \rangle_\psi \geq E_0 \geq \frac{1}{3} \sqrt{\frac{8C_A}{\pi}} \xi_A(\alpha, N) \omega N^{3/2}. \quad (26)$$

In the case of odd numerator rational α this improves the bound given in Ref. [25] (which is also valid for arbitrary α):

$$E_0 \geq \omega \left(N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right), \quad (27)$$

where L denotes the total angular momentum of the state ψ . Note that if $L = -\alpha \binom{N}{2}$ (which could occur for certain N and rational α as long as the right-hand side is an even integer)

then this bound reduces to the bosonic bound for the energy, which is always valid as a trivial lower bound.

It was argued in Ref. [25] using perturbation theory that the behavior for the exact ground-state energy as $N \rightarrow \infty$ is approximately $E_0 \sim \sqrt{\alpha} \omega N^{3/2}$ for $\alpha \sim 0$ and $E_0 \sim \frac{1}{3} \sqrt{8} \omega N^{3/2}$ for $\alpha \sim 1$, requiring $L = -\alpha \binom{N}{2} + O(N^{3/2})$ by relation (27). In fact, we can show that the ground-state energy necessarily always satisfies the upper bound $E_0 \lesssim \omega N^{3/2}$, up to a constant independent of α . Namely, given a (possibly nonsymmetric) N -particle wave function ϕ such that all particles are supported on disjoint sets, we can form its symmetrization

$$\psi(x) := \frac{1}{\sqrt{N!}} \sum_{\sigma \in S_N} \phi(x_{\sigma(1)}, \dots, x_{\sigma(N)}) \quad (28)$$

and conclude by the properties of the supports that $\|\psi\| = \|\phi\|$ and $\int \sum_j |D_j \psi|^2 dx = \int \sum_j |D_j \phi|^2 dx$. Now take, e.g.,

$$\phi(x) := \prod_{j=1}^N \varphi(x_j - \mathbf{y}_j) \prod_{k<l} e^{-i\alpha \phi_{kl}}, \quad (29)$$

where \mathbf{y}_j are fixed points in the plane separated by a minimal length r ; the function φ localizes each particle in a ball of radius $r/2$, and ϕ_{kl} is the angle between particle k and l relative to a fixed axis. Note that this angle is well defined and smooth and that the resulting phase factor (gauge) cancels the magnetic potentials A_j in $D_j \phi$. Then, by choosing the points \mathbf{y}_j to fill a disk of radius $R \sim \sqrt{N}r$, we conclude that the energy E of ψ is bounded by $E \lesssim N/r^2 + \omega^2 N R^2 \sim N^2/R^2 + \omega^2 N R^2$, and hence, choosing r such that $R^2 \sim \sqrt{N}/\omega$, we have $E \lesssim \omega N^{3/2}$. It therefore follows that, for odd numerator α , relation (26) yields the correct dependence in N up to the value of the constant. In a similar way one can also prove that the ground-state energy per unit area for the free anyon gas is uniformly bounded by a constant times $\bar{\rho}^2$.

For comparison, we can also consider the 1D Calogero-Sutherland case together with an external potential, for which some exact results are available [21]. After splitting the real line into intervals big enough to contain a sufficient number of particles, the local bound (20) can be applied on each such interval with the addition of an external potential, thereby obtaining an energy functional and lower estimates for the ground-state energy (along with estimates for the corresponding ground-state density). Depending on the potential, the lower bound can be optimized to the better by choosing the splitting suitably. As an example, we considered in Ref. [17] the external one-body potential $V(x) = a^\mu |x|^\mu$ and obtained a lower bound for the total ground-state energy

$$T_{CS} + \langle \hat{V} \rangle_\psi \geq E_0 \geq C(\mu) [\xi_{CS}(\alpha) a]^{2\mu} N^{\frac{3\mu+2}{\mu+2}}, \quad (30)$$

in the limit $N \rightarrow \infty$ and with an explicit constant $C(\mu)$. In the harmonic oscillator case $\mu = 2$, $a = \omega/\sqrt{2}$, one obtains $E_0 \geq \frac{\sqrt{3}}{8\pi} \xi_{CS}(\alpha) \omega N^2$, which can be compared to the exact ground-state energy for the Calogero-Sutherland model, $E_0 = \frac{1}{2} \omega N [1 + \alpha(N-1)]$. We can also compare these rigorous bounds with the approximate Thomas-Fermi theory [26] and collective field theory [27].

VI. DISCUSSION

The bound (16) provides a nontrivial lower bound for the energy per unit area for an ideal gas of anyons with odd-fractional statistics parameter α . The numerical constant $C_A \geq 10^{-4}$ in this bound has in Ref. [17] been improved to ≥ 0.021 , which is still quite far from the exact semiclassical constant π for the two-dimensional spinless fermion gas. In any case, these nontrivial bounds (16) and (26) raise the very interesting question of whether such Lieb-Thirring inequalities are in fact not valid for even numerator and irrational α . We give some reasons for why this could be the case by considering the following observations.

In these bounds the expression $\xi_A(\alpha, N)$ appears as a measure of exclusion. Its complicated behavior in α and N is related to the fact that only for even numerator fractions do the anyons appear to have the possibility to completely cancel out the statistical phase which is responsible for a local repulsive force between them, by assuming certain configurations. Consider, e.g., a pair of $\alpha = 2/3$ anyons which symmetrically encircle a third one with relative angular momentum -2 , leading to a local cancellation of the interchange phase with the orbital phase, and $\xi_A(\alpha, 3) = 0$. A similar complete cancellation would never be possible for odd numerator α , and indeed $\xi_A(\alpha, N)$ remains strictly bounded away from zero for all N .

Let us again consider the model of N anyons in a harmonic oscillator potential. It is well known that energy levels and degeneracies in this model depend very nontrivially on both N and α , and we can point out certain similarities in the limiting graph of $\xi_A(\alpha, N)$ (see Fig. 3) with known features in spectra for $N = 2, 3, 4$ and corresponding extrapolations to large N [2, 7, 25, 28]. It is intriguing to compare this graph—which can be obtained by cutting out a wedge of slope ν from the upper half-plane at every even numerator rational point μ/ν on the horizontal axis—with the general structure indicated in Fig. 1 in Ref. [25].

The question remains whether for particular α (even numerator rational and/or irrational) the energy could be of lower order than $O(N^{3/2})$ for some special states with $L \sim -\alpha \binom{N}{2}$. With the above considerations, and motivated by the Laughlin states in the fractional quantum Hall effect [29], we could for particular $N = \nu K$ consider trial wave functions of the form $\psi = \Phi \psi_\alpha$, with

$$\psi_\alpha := \prod_{j < k} |z_{jk}|^{-\alpha} \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k) \in \mathcal{E}_q} (\bar{z}_{jk})^\mu \prod_{l \in \mathcal{V}_q} \varphi_0(\mathbf{x}_l) \right] \quad (31)$$

for even numerator fractions $\alpha = \mu/\nu \in [0, 1]$, and

$$\psi_\alpha := \prod_{j < k} |z_{jk}|^{-\alpha} \mathcal{S} \left[\prod_{q=1}^{\nu} \prod_{(j,k) \in \mathcal{E}_q} (\bar{z}_{jk})^\mu \bigwedge_{k=0}^{K-1} \varphi_k(\mathbf{x}_{l \in \mathcal{V}_q}) \right] \quad (32)$$

for odd numerators μ , where the role of the factor Φ is to regularize the short-scale behavior [necessary due to the singular Jastrow factor in expressions (31) and (32)]. We have written $z_{jk} := z_j - z_k$ for the complex relative particle coordinates; φ_k denotes the eigenstates of the one-particle Hamiltonian $\hat{h} = -\frac{1}{2}\nabla^2 + V$ and of which we may form a Slater determinant $\bigwedge_k \varphi_k$, while \mathcal{E}_q and \mathcal{V}_q are sets of edges and vertices of ν disjoint complete graphs involving K particles

each, and \mathcal{S} denotes the operation of symmetrization [compare expression (28)]. Two possible choices of regularizing symmetric functions Φ , giving rise to the expected pairwise short-scale behavior $\sim |z_{jk}|^\alpha$ in ψ , could be

$$\Phi_{r_0} = \prod_{j < k} |z_{jk}|^{2\alpha} (r_0^2 + |z_{jk}|^2)^{-\alpha}, \quad (33)$$

with a parameter $r_0 > 0$, or the parameter-free (but less smooth)

$$\Phi = \prod_{j=1}^N \prod_{k=1}^{\nu-1} |z_{jk(j)}|^\alpha, \quad (34)$$

with $k(j)$ denoting the k th nearest neighbor of particle j . These states ψ have $L = -\alpha \binom{N}{2} + \alpha \frac{\nu-1}{2} N$ [for expression (31) and for certain magic numbers K in expression (32)] and the property that only up to ν particles can be selected in each term without involving a repulsive factor $(\bar{z}_{jk})^\mu$ from an edge in \mathcal{E}_q for some q , allowing for the formation of groups of ν anyons with integer statistics flux μ . Namely, while the Jastrow factor acts to attract all particles, this attraction is on large scales exactly balanced whenever a group of ν nonrepelling anyons has formed, since an anyon \mathbf{x}_j far outside the group, seeing the total attractive factor $\sim (r^{-\alpha})^\nu = r^{-\mu}$ where r is the distance from the group, is also repelled by at least one anyon \mathbf{x}_k in the group, with a factor $|\bar{z}_{jk}|^\mu \sim r^\mu$ from that corresponding edge in \mathcal{E}_q . This balance could act to distribute the anyons, on the average, in such groups of ν . Furthermore, the total contribution from such a group to the statistics potential A_j seen by the distant particle \mathbf{x}_j would be $\sim \nu \alpha r I / r^2 = \mu r I / r^2$, while the particle also has an orbital angular momentum $-\mu$ around the group [due to that same edge to \mathbf{x}_k and phase of $(\bar{z}_{jk})^\mu$] with velocity $\sim -\mu r I / r^2$, again leading to a cancellation of terms in the kinetic energy $D_j \psi$.

The forms (31) and (32) bring out a structural difference between even and odd numerators μ . The limit $\alpha = 1$ of expression (32) is the fermionic ground state in the bosonic representation, and also generalizes to the correct gauge copies for arbitrary integer α , whereas the states with $\nu > 1$ in expression (31) are (modulo the Jastrow factor) actually found to be exactly the Read-Rezayi states for fractional quantum Hall liquids in their bosonic form [30]. The state (31) is an exact but singular (requiring the regularization by Φ) eigenstate of the Hamiltonian with energy $E = \omega(N + \deg \psi_\alpha)$, where $\deg \psi_\alpha = -\alpha \frac{\nu-1}{2} N$ is the total degree of the non-Gaussian part of the wave function (cf. Ref. [31]). In all known exact eigenstates there is this simple correspondence between the degree and the energy. It is an interesting fact that adding the degree of Φ in the nearest-neighbor form (34) produces $\omega(1 + \alpha \frac{\nu-1}{2})N$, i.e., exactly the right-hand side of relation (27) for the above value of L , speaking for a low energy for even numerator fractions. On the other hand, the degree of the odd numerator states (32) necessarily grows with K as $\sim K^{3/2}$ due to the Slater determinant. While the resulting energy $E = \omega(N + \deg \psi_\alpha) \sim \omega \nu (N/\nu)^{3/2}$ satisfies but does not match the bound (26) exactly with regard to α , a corresponding picture of ideal anyons forming essentially free ν -anyon groups with fermionic-type statistics would actually match the form of the bound (16), involving the reduced density $\bar{\rho}/\nu = K/L^2$.

We finally remark that there are also many interesting connections between the forms of the fractions appearing here and those of fractionally charged quantum Hall quasiparticles [32]. Another question concerns possible relations with q -commutation relations, with $q = e^{i\alpha\pi}$ [33].

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