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Sub-Heisenberg phase uncertainties

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Phase shift estimation with uncertainty below the Heisenberg limit, $\Delta\phi_{\rm HL} \propto 1/\bar{N}_T$, where $\bar{N}_T$ is the total average number of particles employed, is a mirage of linear quantum interferometry. Recently, Rivas and Luis, [New J. Phys. **14**, 093052 (2012)] proposed a scheme to achieve a phase uncertainty $\Delta\phi \propto 1/\bar{N}_T^k$, with k an arbitrary exponent. This sparked an intense debate in the literature which, ultimately, does not exclude the possibility to overcome $\Delta\phi_{\rm HL}$ at specific phase values. Our numerical analysis of the Rivas and Luis proposal shows that sub-Heisenberg uncertainties are obtained *only* when the estimator is strongly biased. No violation of the Heisenberg limit is found after bias correction or when using a bias-free Bayesian analysis.

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Introduction. How well can one estimate an unknown phase shift using a linear interferometer and finite energy resources? Despite the fundamental interest, the technological relevance, and the recent theoretical advancements [1], this long-standing problem is still highly debated [2–13]. Since the early seminal works on quantum metrology [14], the fundamental phase uncertainty bound—the so-called Heisenberg limit (HL) [15]—is believed to be

$$\Delta\phi_{\rm HL} = \frac{\kappa}{\bar{N}_T}, \tag{1}$$

where κ is a unit constant [16] and $\bar{N}_T$ is the total average number of particles used in the phase estimation. In general, $\bar{N}_T = \bar{n} \times m$, where $\bar{n}$ is the average number of particles in the probe state and m is the number of repeated measurements done with identical copies of the probe. Notice that $\Delta\phi = 1/\bar{n}\sqrt{m} = \sqrt{m}/\bar{N}_T$, taken by some authors as ultimate phase uncertainty bound, is not fundamental [3–5,17].

Overcoming Eq. (1) is a highly desired goal in quantum metrology. A first attempt to beat Eq. (1), by Shapiro and coworkers, dates back to the late 1980s [18]. While this proposal was soon criticized [19,20], it stimulated the optimization of phase uncertainties over measurement strategies, probe states, and estimators [21–29]. Recently, Rivas and Luis (RL) [2] challenged again the validity of Eq. (1). They proposed an estimation strategy achieving $\Delta\phi \propto 1/\bar{N}_T^k$ with $k > 1$ an arbitrary exponent. Their proposal is particularly appealing and revived the interest in sub-Heisenberg uncertainties [6–10]. Indeed, it makes use of familiar probe states, noncovariant measurements, and well-understood statistical tools, such as the Cramer-Rao bound and the maximum likelihood estimator, which have been applied in recent phase estimation experiments [34–37]. The current proofs of Eq. (1) [5–10] do not exclude the possibility to overcome the HL at specific, but unknown, phase values (the so-called "sweet spots" [9]) and in the presence of number coherence in the probe state and output measurement, as in Ref. [2].

The aim of this Rapid Communication is to set a standard in the numerical analysis of sweet spots phase estimation. Such an analysis is necessary to corroborate claims of sub-Heisenberg phase uncertainties. Specifically, we show that the violation of the HL, achieved in [2] with a maximum likelihood (ML) analysis, is not a consequence of special features of the probe state but rather due to the bias properties of the estimator. For a small number of measurements the ML estimator is strongly biased and no sub-Heisenberg uncertainty is found after bias correction. A bias-free Bayesian analysis shows that a sufficiently large number of measurements is needed to reduce the long tails which characterize the probability phase distributions. These strongly affect the resource counting, preventing sub-Heisenberg uncertainties from being achieved.

Locally biased phase estimation. We briefly review here the general theory of phase inference with special emphasis on locally biased estimators. Let $\hat{\rho}$ be the probe state of $\bar{n}$ average particles. It is transformed according to $\hat{\rho}(\phi) = e^{-i\phi\hat{H}}\hat{\rho}e^{+i\phi\hat{H}}$, where ϕ is the unknown value of the phase shift and $\hat{H}$ an arbitrary Hermitian operator. The goal is to estimate ϕ with the smallest possible uncertainty, given finite $\bar{N}_T$.

In a frequentist setting, ϕ is estimated as $\Phi_{\rm est}(\{x\}_m)$ from the results $\{x\}_m \equiv x_1,\ldots,x_m$ of m independent measurements, obtained with probability $P(\{x\}_m|\phi) = \prod_{i=1}^m {\rm Tr}[\hat{E}(x_i)\hat{\rho}(\phi)]$. Here $\{\hat{E}(x)\}$ is a set of non-negative Hermitian operators with $\int dx\, \hat{E}(x) = \mathbb{1}$, which forms a positive operator-valued measure (POVM). The estimator $\Phi_{\rm est}$ is said to be locally unbiased at ϕ if $\langle\Phi_{\rm est}\rangle_\phi = \phi$ *and* $\frac{d\langle\Phi_{\rm est}\rangle_\phi}{d\phi} = 1$. We indicate as $\langle\ldots\rangle_\phi$ the statistical average at phase ϕ. Unbiased estimators are rare. One of the most important estimators, the maximum likelihood, $\Phi_{\rm ML} = \arg[\max_\psi P(\{x\}_m|\psi)]$, is known to be unbiased, in general, only in the large-m limit. The variance $(\Delta\Phi_{\rm est})^2_\phi = \langle\Phi^2_{\rm est}\rangle_\phi - \langle\Phi_{\rm est}\rangle^2_\phi$ of locally biased estimators can be arbitrarily small. No bound exists, in this case, in terms of energy resources [38]. Let us therefore consider $\langle\Phi_{\rm est}\rangle_\phi$ to be (locally, around a reference phase ϕ_0) a linear function of ϕ,

$$\langle\Phi_{\rm est}\rangle_\phi = \langle\Phi_{\rm est}\rangle_{\phi_0} + b_{\phi_0}(\phi - \phi_0), \tag{2}$$

where

$$b_{\phi_0} = \left.\frac{d\langle\Phi_{\rm est}\rangle_\phi}{d\phi}\right|_{\phi_0}. \tag{3}$$

We assume that $\langle\Phi_{\rm est}\rangle_{\phi_0}$ and b_{ϕ_0} are known and that Eq. (2) holds sufficiently close to ϕ_0. In this situation one can easily correct the bias by introducing the new estimator

$$\tilde{\Phi}_{\rm est} = \frac{\Phi_{\rm est} - \langle\Phi_{\rm est}\rangle_{\phi_0}}{b_{\phi_0}} + \phi_0, \tag{4}$$

which is unbiased for all the phase values ϕ where Eq. (2) holds. The variance of $\tilde{\Phi}_{\rm est}$ is related to $(\Delta\Phi_{\rm est})^2_\phi$ as

$$(\Delta\tilde{\Phi}_{\rm est})^2_\phi = \frac{(\Delta\Phi_{\rm est})^2_\phi}{b^2_{\phi_0}}. \tag{5}$$

Taking, without loss of generality, $\phi_0 = 0$ and $\langle\Phi_{\rm est}\rangle_0 = 0$, Eq. (5) reduces to the usual figure of merit of phase estimation [23,39]. Under mild assumptions on $P(x|\phi)$, Eq. (5) fulfills the Cramer-Rao (CR) theorem, $(\Delta\tilde{\Phi}_{\rm est})^2_\phi \geqslant (\Delta\Phi_{\rm CR})^2_\phi$, where

$$(\Delta\Phi_{\rm CR})^2_\phi = \frac{1}{mF(\phi)} \tag{6}$$

is the CR bound and

$$F(\phi) = \int dx\, \frac{1}{P(x|\phi)}\left(\frac{dP(x|\phi)}{d\phi}\right)^2 \tag{7}$$

is the Fisher information (FI). The FI, maximized over all possible POVMs, defines the quantum Fisher information, $F \leqslant F_{\rm Q}$. For pure states one finds $F_{\rm Q} = 4(\Delta\hat{H})^2$, where $(\Delta\hat{H})^2 = {\rm Tr}[\hat{\rho}\hat{H}^2] - {\rm Tr}[\hat{\rho}\hat{H}]^2$ [23].

A different (nonfrequentist) approach to phase estimation is based on the Bayes theorem,

$$P(\psi|\{x\}_m) = P(\{x\}_m|\psi)P(\psi)/P(\{x\}_m), \tag{8}$$

where $P(\psi)$ is the prior knowledge about the phase shift, $P(\{x\}_m)$ provides the normalization of the *a posteriori* probability distribution $P(\psi|\{x\}_m)$, and ψ is a phase variable. In the following we will take a constant prior $P(\psi)$. Equation (8) defines the Bayesian phase probability distribution, $P(\psi|\{x\}_m) \propto \prod_{i=1}^m P(\psi|x_i)$, conditioned by the m independent measurement results obtained at fixed ϕ. In the limit $m \to \infty$, $P(\psi|\{x\}_m)$ becomes a Gaussian, centered at ϕ and of variance given by $1/mF(\phi)$. This result is a consequence of the Laplace-Bernstein-von Mises theorem [40]. In the Bayesian setting, the concept of bias is lost: Given the m measurement results, one can choose a phase estimate $\Phi_{\rm est}$ [e.g., the mean or the maximum of $P(\psi|\{x\}_m)$] and calculate the corresponding confidence interval. For the 68.27% confidence, for instance, this can be taken as $\Delta\Phi_{\rm B} = (\Delta\Phi_{\rm max} + \Delta\Phi_{\rm min})/2$ such that

$$0.6827 = \int_{\Phi_{\rm est}-\Delta\Phi_{\rm min}}^{\Phi_{\rm est}+\Delta\Phi_{\rm max}} d\psi\, P(\psi|\{x\}_m), \tag{9}$$

where the phase intervals $[\Phi_{\rm est}, \Phi_{\rm est} + \Delta\Phi_{\rm max}]$ and $[\Phi_{\rm est} - \Delta\Phi_{\rm min}, \Phi_{\rm est}]$ contain the same probability [41]. The smaller the confidence interval around $\Phi_{\rm est}$, the likelier it is to find $\Phi_{\rm est}$ close to ϕ. For further use, we indicate with $\mathcal{P}(\Delta\Phi_{\rm B})$ the statistical distribution of Bayesian confidence intervals.

The proposal of Rivas and Luis. In the remainder of this Rapid Communication we will consider the probe state [2]

$$|\psi\rangle = \mu|{\rm vac}\rangle + \nu|\alpha,\xi\rangle, \tag{10}$$

where $|{\rm vac}\rangle$ is the vacuum and $|\alpha,\xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|{\rm vac}\rangle$ is the quadrature squeezed state, defined in terms of the displacement, $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, and squeezing, $\hat{S}(\xi) = \exp(\frac{\xi^*}{2}\hat{a}^2 - \frac{\xi}{2}\hat{a}^{\dagger 2})$, operators [42]. Here $\hat{a}$ ($\hat{a}^\dagger$) is the mode annihilator (creation) operator, $\alpha = |\alpha|e^{i\varphi_\alpha}$ and $\xi = re^{i\varphi_\xi}$ ($r \geqslant 0$). The average number of particles is $\bar{n} = \langle\psi|\hat{n}|\psi\rangle = |\nu|^2(|\alpha|^2 + \sinh^2 r)$, where $\hat{n} = \hat{a}^\dagger\hat{a}$. The phase shift is generated by the number operator, so that $F_{\rm Q} = 4(\Delta\hat{n})^2$. A straightforward calculation gives

$$F_{\rm Q} = 2|\nu|^2(\sinh^2 2r + 2|\alpha|^2e^{-2r}\cos^2\theta + 2|\alpha|^2e^{2r}\sin^2\theta), \tag{11}$$

where $\theta \equiv \varphi_\alpha - \varphi_\xi/2$. In the following we set $\varphi_\alpha = \pi/2$, $\varphi_\xi = 0$ [thus $\theta = \pi/2$] and $|\alpha|^2 = \sinh^2 r$, maximizing Eq. (11) when $r \gg 1$. We find

$$F_{\rm Q} = 6\bar{n}^2/|\nu|^2, \tag{12}$$

showing that, by increasing the weight of the vacuum ($\nu \to 0$) in Eq. (10) while keeping $\bar{n}$ constant, one can arbitrarily increase $F_{\rm Q}$.

In the RL proposal, the phase is estimated from homodyne measurements on $e^{-i\phi\hat{n}}|\psi\rangle$, where $\hat{n} = \hat{a}^\dagger\hat{a}$. The quadrature operators are $\hat{X} = \hat{a}^\dagger + \hat{a}$ and $\hat{Y} = i(\hat{a}^\dagger - \hat{a})$ with $[\hat{X},\hat{Y}] = 2i$ and we indicate with x ($|x\rangle$) the eigenvalues (eigenstates) of $\hat{X}$. The conditional probability is $P(x|\phi) = |\mu\langle x|{\rm vac}\rangle + \nu\langle x|\alpha,\xi,\phi\rangle|^2$, where we used the notation $|\alpha,\xi,\phi\rangle \equiv e^{-i\phi\hat{a}^\dagger\hat{a}}|\alpha,\xi\rangle$. Without any approximation [43], $P(x|\phi)$ can be calculated by using

$$\langle x|\alpha,\xi,\phi\rangle = \frac{e^{i\bar{Y}x/2}e^{-[(x-\bar{X})^2(1-i\sin 2\phi\sinh 2r)]/4(\Delta X)^2}e^{-i\Gamma/2}e^{-i(|\alpha|^2\sin 2\phi)/2}}{[2\pi(\Delta X)^2]^{1/4}}, \tag{13}$$

where $\bar{X} = 2|\alpha|\sin\phi$, $\bar{Y} = 2|\alpha|\cos\phi$, $(\Delta X)^2 = e^{2r}\sin^2\phi + e^{-2r}\cos^2\phi$, and $\Gamma = \arcsin[\frac{\sin 2\phi\,\sinh 2r}{2\Delta X\cosh r}]$. The FI, Eq. (7), can be calculated from the conditional probability. Taking real coefficients ν and μ, at $\phi = 0$ one obtains

$$F(0) \approx 4\bar{n}^2/\nu^2 \tag{14}$$

[see Fig. 1(a)] [44]. Although the homodyne measurement is not optimal [the FI is smaller than Eq. (12) by a factor 2/3], Eq. (14) leads to an interesting result: If we fix $\bar{n}$ and take $\nu \propto 1/\bar{n}^{k-1}$, where k is an arbitrary integer, then Eq. (14) predicts $F(0) \propto \bar{n}^{2k}$. This suggests that, *if* there exists an unbiased estimator which saturates the CR bound for a fixed value of m, independently of $\bar{n}$ and ν, then the phase shift $\phi = 0$ can be estimated with uncertainty $\Delta\Phi_{\rm est} \propto 1/\bar{N}_T^k$. Following this idea, Rivas and Luis [2] considered $\nu = 0.05/\bar{n}$ and performed a maximum likelihood analysis focusing on the case $m = 1$ and $\phi = 0$. Even though their scheme does not saturate the CR bound ($\Delta\Phi_{\rm CR} = 0.025/\bar{N}_T^2$, in this case), they claimed a phase uncertainty $\Delta\Phi_{\rm ML} = 0.0354/\bar{N}_T^{3/2}$ [45], overcoming Eq. (1).

This result is astonishing: According to [2], it is possible to beat Eq. (1) by standard detection and a probe state simply obtained by amplifying the vacuum component and boosting

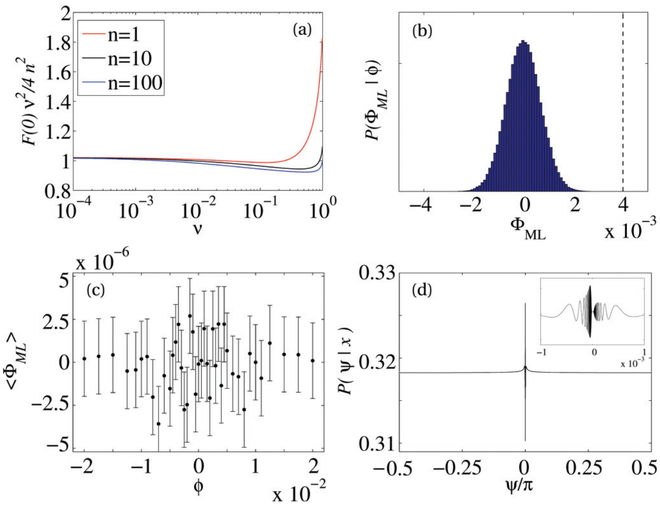

FIG. 1. (Color online) Panel (a) shows the Fisher information $F(0)$ (divided by $4\bar{n}^2/\nu^2$) as a function of ν and for different values of $\bar{n}$. (b) ML histogram obtained for $m = 1$ and $\phi = 0.004$ (highlighted by the vertical dotted line). (c) $\langle \phi_{\rm ML}\rangle_\phi$ for $m = 1$, as a function of ϕ. Error bars are $(\Delta\Phi_{\rm ML})_\phi/\sqrt{N_s}$, where N_s is the sample size [$N_s = 10^6$ in panels (b) and (c)]. (d) Typical Bayesian probability phase distribution (here $x = -0.2$) obtained for $m = 1$. The inset is a zoom around $\psi \approx 0$. In panels (b)–(d) $\bar{N}_T = 10$ and $\nu = 0.05$.

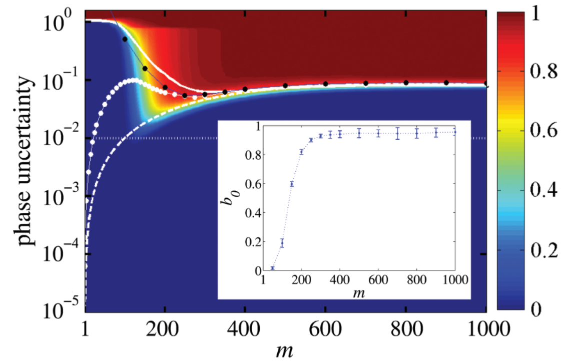

FIG. 2. (Color online) Results of a ML and a Bayesian analysis at $\phi = 0$ and $\bar{N}_T = 50$. The main panel shows $\Delta\Phi_{\rm ML}$ (white dots—guide to the eye), $\Delta\tilde{\Phi}_{\rm ML}$ (black dots—guide to the eye), and $\langle\Delta\Phi_{\rm B}\rangle$ (solid white line). The dashed line is $(\Delta\Phi_{\rm CR})_0$ and the horizontal dotted line is $1/2\bar{N}_T$. The inset shows b_0 as a function of m. It is extracted by a linear fit of $\langle\Phi_{\rm ML}\rangle_\phi$ around $\phi = 0$. The color scale shows the cumulative Bayesian phase uncertainty distribution $\mathcal{D}(\Delta\Phi_{\rm B})$. All results are obtained from large statistical samples (typically of 10^4 realizations).

the energy of the coherent squeezed state. This stimulated us to repeat the ML analysis of Ref. [2] and to extend it to different values of the parameters. The typical ML distribution for $m = 1$ and $\phi \approx 0$ is shown in Fig. 1(b). It is approximately Gaussian, centered at $\Phi_{\rm ML} = 0$ and of width $\Delta\Phi_{\rm ML} = 0.0221/\bar{N}_T^{3/2}$, much narrower than $1/\bar{N}_T$. However, the ML distribution barely changes with ϕ and it is easy to find a situation where the true value of the phase shift lies well outside the bell of the corresponding ML distribution [see Fig. 1(b), where ϕ is located at the vertical dotted line]. Figure 1(c) plots $\langle\Phi_{\rm ML}\rangle_\phi$ as a function of ϕ. It shows that in this case ($m = 1$, $\phi \approx 0$), the ML estimator is strongly biased: Even though $\langle\Phi_{\rm ML}\rangle_0 \approx 0$, $\langle\Phi_{\rm ML}\rangle_\phi$ barely changes with ϕ [$d\langle\Phi_{\rm ML}\rangle_\phi/d\phi|_{\phi=0} \ll 1$]. The typical Bayesian distribution $P(\psi|x)$ for $m = 1$ is shown in Fig. 1(d): It is almost constant with low visibility oscillations around $\psi \approx 0$. Clearly, $\Delta\Phi_{\rm B}$ is of the order of π, in this case.

As the case $m = 1$ is strongly biased, we have performed a systematic ML and Bayesian analysis for larger values of m. The results for $\bar{N}_T = 50$ are shown in Fig. 2. Taking $\nu = 0.05/\bar{n}$, as in [2], the number of measurements is limited in $1 \leqslant m \leqslant \bar{N}_T/0.05$ where small values of m correspond to small values of ν ($\nu = 0.001$ for $m = 1$, $m = 1000$ for $\nu = 1$). For relatively large values of m, $\Delta\Phi_{\rm ML}$ (white dots), $\Delta\tilde{\Phi}_{\rm ML}$ (black dots), and the statistically averaged $\langle\Delta\Phi_{\rm B}\rangle$ (solid white line) converge, as expected, to the CR bound $(\Delta\Phi_{\rm CR})_0$ (dashed white line). In this regime, we have $b_0 \approx 1$, as shown in the inset of Fig. 2. By decreasing m, b_0 rapidly tends to zero: The ML distribution becomes biased. For small m, $\Delta\Phi_{\rm ML}$ decreases below the HL (see Fig. 2) reaching its minimum value for $m = 1$, $\Delta\tilde{\Phi}_{\rm ML}$ tends to diverge as $b_0 \to 0$ [46], and $\langle\Delta\Phi_{\rm B}\rangle$ converge to 0.34π, which is the 68% confidence interval of a flat distribution centered in $\psi = 0$. The underlying color scale in Fig. 2 shows the cumulative probability of $\Delta\Phi_{\rm B}$, $\mathcal{D}(\Delta\Phi_{\rm B}) = \int_0^{\Delta\Phi_{\rm B}} d\delta\,\mathcal{P}(\delta)$. For large m, $\mathcal{P}(\Delta\Phi_{\rm B})$ concentrates around the CR bound. At the transition region, around $m = 200$ for the parameters of Fig. 2, the probability $\mathcal{P}(\Delta\Phi_{\rm B})$ broadens between the CR bound and 0.34π: It is statistically possible to obtain both narrow and wide Bayesian distributions, depending on the outcomes $\{x\}_m$. In this regime, the median of $\mathcal{P}(\Delta\Phi_{\rm B})$ [i.e., the value $\Delta\Phi_{\rm B}^{\rm med}$ such that $\mathcal{D}(\Delta\Phi_{\rm B}^{\rm med}) = 0.5$] is more representative than the mean and follows the CR to lower values of m. Interestingly, below a certain m, the probability to have $\Delta\Phi_{\rm B} \approx \Delta\Phi_{\rm CR}$ drops suddenly. At small ν (and m) the Bayesian distributions are characterized by long tails, which are not reduced by repeating the measurements. Notice that the probability to find $\Delta\Phi_{\rm B} \leqslant 1/2\bar{N}_T$ is negligible.

The above analysis points out the difficulties of the RL proposal: Small values of ν increase the FI but are associated with the presence of biases (and long tails of the phase distribution) which increase the phase uncertainty. Increasing the number of measurements m reduces the biases and tails but affects the resource counting. In order to decide whether the RL proposal beats the HL, or not, we need to evaluate the central limit, i.e., the minimum number of measurements, $m_{\rm cl}(\nu,\bar{N}_T)$, for which the CR bound is saturated [47]. One has

$$\Delta\Phi \approx \Delta\Phi_{\rm CR} \approx \frac{\nu\sqrt{m}}{2\bar{N}_T} \quad \text{for} \quad m \gtrsim m_{\rm cl}(\nu,\bar{N}_T), \tag{15}$$

where we have used Eq. (14). If $\nu^2 m_{\rm cl} \propto \bar{N}_T^{-k}$ (with $k > 0$), one can beat Eq. (1). To find $m_{\rm cl}(\nu,\bar{N}_T)$, we have repeated the numerical analysis outlined above, for different parameters $\bar{N}_T$, m, and ν, releasing the relation $\nu = 0.05/\bar{n}$ considered so far [2]. The results are shown in Fig. 3 for $\bar{N}_T = 50$ (left panels) and $\bar{N}_T = 500$ (right panels). We plot $\Delta\tilde{\Phi}_{\rm ML}/\Delta\Phi_{\rm CR}$ [(a),(b)], b_0 [(c),(d)], and $\Delta\Phi_{\rm B}^{\rm med}/\Delta\Phi_{\rm CR}$ [(e),(f)]. For $\nu \ll 1$ the different colored lines in all panels follow the general trend $m \propto 1/\nu^2$. We thus conclude that $\nu^2 m_{\rm cl}(\nu,\bar{N}_T) = {\rm const}$. For instance, a fit of $\Delta\tilde{\Phi}_{\rm ML}/\Delta\Phi_{\rm CR} = 2$ [dashed white line in panels (a) and (b)], gives $\nu^2 m_{\rm cl}(\nu,\bar{N}_T) \approx 12$ for both $\bar{N}_T = 50$

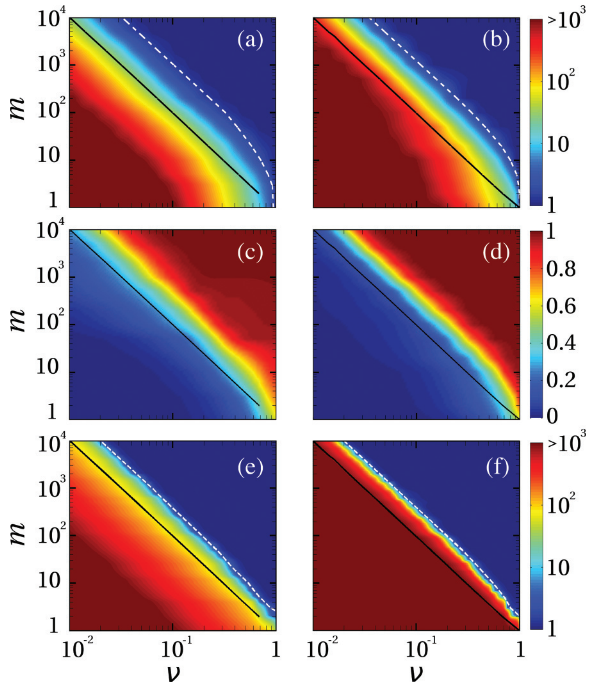

FIG. 3. (Color online) Normalized phase uncertainty $\Delta\tilde{\Phi}_{\rm ML}/\Delta\Phi_{\rm CR}$ [(a),(b)], b_0 [(c),(d)], and $\Delta\Phi_{\rm B}^{\rm med}/\Delta\Phi_{\rm CR}$ [(e),(f)] as a function of ν and m and for $\bar{N}_T = 50$ [left panels] and $\bar{N}_T = 500$ [right panels]. We recall here that $\Delta\Phi_{\rm B}^{\rm med}$ is the median of the Bayesian phase uncertainty distribution $\mathcal{P}(\Delta\Phi_{\rm B})$. The solid black lines in all panels are $\Delta\Phi_{\rm CR} = 1/2\bar{N}_T$: Phase uncertainties below the HL are predicted by the CR for small ν and m values on the left side of this line. The dashed white line is $\Delta\tilde{\Phi}_{\rm ML}/\Delta\Phi_{\rm CR} = 2$ in panels (a) and (b), and $\Delta\Phi_{\rm B}^{\rm med}/\Delta\Phi_{\rm CR} = 2$ in panels (e) and (f).

and $\bar{N}_T = 500$, for $\nu \ll 1$. We thus observe no violation of the HL. Analogous conclusion is found from the Bayesian analysis. In this case, a fit of $\Delta\Phi_{\rm B}^{\rm med}/\Delta\Phi_{\rm CR} = 2$ [dashed white line in panels (e) and (f)] gives $\nu^2 m_{\rm cl}(\nu,\bar{N}_T) \approx 4$ for both $\bar{N}_T = 50$ and $\bar{N}_T = 500$, for $\nu \ll 1$. One can also see that $\Delta\tilde{\Phi}_{\rm ML} = \Delta\Phi_{\rm B}^{\rm med} = \Delta\Phi_{\rm CR}$ and $b_0 = 1$ for large values of m and ν, in particular where the CR bound predicts a sensitivity above the HL (right side of the black dotted line all panels). Around the values of m and ν for which the CR uncertainty is below the HL, we find that b_0 rapidly decreases and tends to zero and thus $\Delta\tilde{\Phi}_{\rm ML}/\Delta\Phi_{\rm CR} \gg 1$. In the same limit, $\Delta\Phi_{\rm B}^{\rm med}$ saturates to 0.34π, which explains the values $\Delta\Phi_{\rm B}^{\rm med}/\Delta\Phi_{\rm CR} \gg 1$ observed for small ν and m.

Conclusions. The quantum Fisher information, proportional to $(\Delta\hat{n})^2$, may achieve arbitrary large values when keeping $\bar{n}$ fixed. This effect is the crucial ingredient of a number of proposals, appearing in the recent [2–4] and old [18] literature, claiming the possibility to beat Eq. (1) with linear interferometers. Thoughtful bounds are obtained only after evaluating the central limit, i.e., the minimum number of measurements for which biases of point estimators and/or the long tails of the Bayesian probability distribution are reduced and the phase uncertainty approaches the Cramer-Rao bound. This generally requires a careful numerical analysis, such as the one performed in this Rapid Communication. Specifically, we have focused on the proposal of Ref. [2]. Our numerical results show that the responsibility for the sub-HL uncertainties discussed in [2] is the strong bias of the estimator, rather than special properties of the probe state (10). Our methods should be understood as a complement of the current proofs of the HL [6,8–11], for point inference, where such proofs do not hold. In conclusion, no evidence of sub-Heisenberg uncertainties is found in the literature, except for biased estimators.

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[41] Care must be taken when $\Phi_{\rm est}$ is close to the boundaries $\phi_{\rm min}$ and $\phi_{\rm max}$ of the phase domain. If, for instance, $\int_{\Phi_{\rm est}}^{\phi_{\rm max}} d\psi\, P(\psi|\{x\}_m) \leqslant 0.3413$, then we take $\Delta\Phi_{\rm max} = \phi_{\rm max} - \Phi_{\rm est}$ and $\Delta\Phi_{\rm min}$ from the integral $\int_{\Phi_{\rm est}-\Delta\Phi_{\rm min}}^{\phi_{\rm max}} d\psi\, P(\psi|\{x\}_m) = 0.6827$. In this case the intervals $[\Phi_{\rm est}, \Phi_{\rm est} + \Delta\Phi_{\rm max}]$ and $[\Phi_{\rm est} - \Delta\Phi_{\rm min}, \Phi_{\rm est}]$ do not contain the same probability. Analogous considerations if $\Phi_{\rm est}$ is sufficiently close to $\phi_{\rm min}$.

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[45] The prefactor 0.0354 ± 0.0003 is an artifact of the approximations used in [2]. We rather find $\Delta\Phi_{\rm ML} = (0.0221 \pm 0.0002)/\bar{N}_T^{3/2}$.

[46] The divergence of $\Delta\tilde{\Phi}_{\rm ML}$, thus reaching values much larger than 2π, is an artifact of Eq. (5) in the presence of very strong biases, i.e., $b_\phi \to 0$ while $\Delta\Phi_{\rm ML}$ remains finite, as in the case considered here.

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