Orbital angular momentum from marginals of quadrature distributions

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(Received 7 June 2013; published 25 November 2013)

We set forth a method to analyze the orbital angular momentum of a light field. Instead of using the canonical formalism for the conjugate pair angle-angular momentum, we model this latter variable by the superposition of two independent harmonic oscillators along two orthogonal axes. By describing each oscillator by a standard Wigner function, we derive, via a consistent change of variables, a comprehensive picture of the orbital angular momentum. We compare this with previous approaches and show how this method works in some relevant examples.

DOI: 10.1103/PhysRevA.88.053839 PACS number(s): 42.25.Bs, 03.30.+p, 78.67.Pt, 78.20.Ci

I. INTRODUCTION

The term vortex is commonly used to designate a region of concentrated rotation in a flow, such as an eddy, a whirlpool, or the depression at the center of a whirling body of air or water. Naturally occurring vortices include hurricanes, tornadoes, waterspouts, and dust devils [1]. Yet vortices can also be created in many different media: they manifest in plasmas [2], superfluids [3], ferromagnets [4], acoustical waves [5], quantum Hall fluids [6], Bose-Einstein condensates [7], and electron wave packets [8], to cite only a few relevant examples. This points to the ubiquity of this phenomenon and reveals a growing interest in these singularities.

The case of optical vortices deserves a special mention [9]. An optical vortex is a beam of light exhibiting a pure screw phase dislocation along the propagation axis, i.e., an azimuthal phase dependence $\exp(i\ell\varphi)$. The number ℓ plays the role of a topological charge: the phase changes its value in ℓ cycles of 2π in any closed circuit about the axis, while the amplitude is zero there.

One of the most interesting properties of vortices is that they carry orbital angular momentum (OAM): the integer ℓ can be seen as the eigenvalue of the OAM operator and its sign defines the helicity or direction of rotation. Indeed, the OAM of such a field can be easily manipulated and transferred, which opens many experimental perspectives, such as optical tweezers and spanners [10], as well as potential astronomical [11] and communication applications [12].

The fact that individual photons also carry OAM presents the most exciting possibilities for using this variable in the quantum domain, and a number of uses has already been demonstrated [13–17].

In quantum theory, the operator representing the OAM has an unbounded spectrum that includes positive and negative integers. Accordingly, its conjugate variable, the azimuthal angle, might be expected to be represented by a bona fide operator. Periodicity, however, brings out subtleties that have triggered long and heated discussions [18–20].

Here, we look at this issue from a phase-space perspective. Such an approach was introduced in the very early days of quantum theory to avoid some of the troubles arising in the abstract Hilbert-space formulation. The pioneering works of Weyl [21], Wigner [22], and Moyal [23] paved the way to formally picturing the quantum world as a statistical theory on phase space [24–26].

In a few words, the key idea is to look for a mapping that relates operators (in Hilbert space) to functions (in phase space). For the conjugate pair angle-OAM, the phase space is the discrete cylinder $\mathscr{S} \times \mathbb{Z}$ ($\mathscr{S}$ denotes the unit circle associated with the angle, while $\mathbb{Z}$ are the integers labeling OAM). It seems natural to work out a Wigner function (or any other quasiprobability) therein. A pioneer attempt in that direction was made by Mukunda [27]; his work was subsequently elaborated and developed in a variety of directions by other authors [28–40].

However, one might properly argue that in such a (correct) way of proceeding one is overlooking significant information about the transverse distribution. This means, for example, that using cylindrical coordinates all the states $\Psi_\ell(r,\varphi) = A_\ell(r)\exp(i\ell\varphi)$ represent eigenstates of the angular momentum, irrespective of the form of the amplitudes $A_\ell(r)$. A similar problem arises in the description of spinlike systems over the Bloch sphere: one disregards in this way fluctuations in the number of particles, because a sphere of fixed radius cannot accommodate those fluctuations. To bypass this drawback one needs to include the whole Bloch space that can be envisioned as foliated in a set of nested spheres with radii proportional to the different number of particules that contribute to the state.

Below, we propose an alternative road and derive phase-space distributions via suitable marginals of distributions for field quadratures, once we remove the degrees of freedom irrelevant for the specification of the problem. The same ideas have been used also to study quantum polarization properties [41,42]. Perhaps this provides the most down-to-earth approach to the problem at hand, since the quadrature distributions can be determined by very simple experimental procedures [43]. This widespread measurability does not hold for the Wigner functions on the cylinder: the proposals for their practical reconstruction are rather cumbersome [44] and lack the simple and intuitive picture provided by schemes measuring quadrature distributions.

The plan of this paper is as follows. In Sec. II we concisely sketch the phase-space fundamentals for a single harmonic oscillator. In Sec. III we start from two kinematical independent orthogonal oscillators and express the resulting Wigner function in cylindrical coordinates. By eliminating an inessential variable (the radial momentum), we get a well-behaved distribution that gives complete information not only on the pair angle-OAM but also on the radial distribution. We apply the resulting Wigner function to some relevant states in Sec. IV, and conclude that it constitutes a most suitable tool to deal with this problem.

II. PHASE-SPACE PICTURE OF A ONE-DIMENSIONAL HARMONIC OSCILLATOR

To keep the discussion as self-contained as possible, we first boil down the rudiments of the phase-space formalism for a harmonic oscillator that we shall need later on.

The relevant dynamical observables are the conjugate coordinate and momentum operators $\hat{x}$ and $\hat{p}$, with canonical commutation relation (with $\hbar = 1$ throughout)

$$[\hat{x}, \hat{p}] = i\, \hat{\mathbb{1}}\,, \tag{2.1}$$

so that they are the generators of the Heisenberg-Weyl algebra [45]. Ubiquitous and profound, this algebra has become the hallmark of noncommutativity in quantum theory. The classical phase space is here the plane $\mathbb{R}^2$.

Sometimes, it is advantageous to use instead complex amplitudes represented by the annihilation and creation operators

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})\,, \quad \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})\,, \tag{2.2}$$

in terms of which the commutation relation (2.1) turns out to be $[\hat{a},\hat{a}^\dagger] = \hat{\mathbb{1}}$.

A pivotal role will be played in what follows by the unitary

$$\hat{D}(x,p) = \exp[i(p\hat{x} - x\hat{p})]\,, \tag{2.3}$$

which is called the displacement operator for it displaces a localized state by $(x,p) \in \mathbb{R}^2$. The Fourier transform of $\hat{D}(x,p)$,

$$\hat{w}(x,p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \exp[-i(px' - xp')]\, \hat{D}(x',p')\, dx'dp'\,, \tag{2.4}$$

is an instance of a Stratonovich-Weyl quantizer [46]. One can check that the operators $\hat{w}(x,p)$ are a complete trace-orthonormal set that transforms properly under displacements

$$\hat{w}(x,p) = \hat{D}(x,p)\, \hat{w}(0,0)\, \hat{D}^\dagger(x,p)\,, \tag{2.5}$$

where $\hat{w}(0,0) = \int_{\mathbb{R}^2} \hat{D}(x,p)\, dxdp = 2\hat{P}$, and

$$\hat{P} = \int_{\mathbb{R}} |x\rangle\langle -x|\, dx = \int_{\mathbb{R}} |p\rangle\langle -p|\, dp = (-1)^{\hat{a}^\dagger \hat{a}} \tag{2.6}$$

is the parity operator.

Let $\hat{A}$ be an arbitrary operator acting on the Hilbert space of the system. Using the Stratonovich-Weyl quantizer we can associate to $\hat{A}$ a function $a(x,p)$ representing the action of the corresponding dynamical variable in phase space. In fact, this is known as the Wigner-Weyl map and is given by [47]

$$a(x,p) = \mathrm{Tr}[\hat{A}\, \hat{w}(x,p)]\,. \tag{2.7}$$

The function $a(x,p)$ is the symbol of the operator $\hat{A}$. Conversely, we can reconstruct the operator from its symbol through

$$\hat{A} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} a(x,p)\, \hat{w}(x,p)\, dxdp\,. \tag{2.8}$$

In this context, the Wigner function is nothing but the symbol of the density matrix $\hat{\rho}$, and therefore

$$\begin{aligned} W_{\hat{\rho}}(x,p) &= \mathrm{Tr}[\hat{\rho}\, \hat{w}(x,p)]\,, \\ \hat{\rho} &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{w}(x,p) W_{\hat{\rho}}(x,p)\, dxdp\,. \end{aligned} \tag{2.9}$$

For a pure state $|\Psi\rangle$, it simplifies

$$W_{|\Psi\rangle}(x,p) = \frac{1}{4\pi} \int_{\mathbb{R}} \Psi^*(x - x')\, \Psi(x + x')\, \exp(i2px')\, dx'\,, \tag{2.10}$$

which is, perhaps, the most convenient form for actual calculations.

The Wigner function defined in Eq. (2.9) fulfills all the basic properties required for any good probabilistic description. First, due to the Hermiticity of $\hat{w}(x,p)$, it is real for Hermitian operators. Second, the probability distributions for the canonical variables can be obtained as the marginals

$$\int_{\mathbb{R}} W_{\hat{\rho}}(x,p)\, dp = \langle x|\hat{\rho}|x\rangle\,, \quad \int_{\mathbb{R}} W_{\hat{\rho}}(x,p)\, dx = \langle p|\hat{\rho}|p\rangle\,. \tag{2.11}$$

Third, $W_{\hat{\rho}}(x,p)$ is translationally covariant, which means that for the displaced state $\hat{\rho}' = \hat{D}(x',p')\, \hat{\rho}\, \hat{D}^\dagger(x',p')$, one has

$$W_{\hat{\rho}'}(x,p) = W_{\hat{\rho}}(x - x', p - p')\,, \tag{2.12}$$

so that it follows displacements rigidly without changing its form, reflecting the fact that physics should not depend on a certain choice of the origin.

Finally, the overlap of two density operators is proportional to the integral of the associated Wigner functions:

$$\mathrm{Tr}(\hat{\rho}\, \hat{\rho}') \propto \int_{\mathbb{R}^2} W_{\hat{\rho}}(x,p) W_{\hat{\rho}'}(x,p)\, dxdp\,. \tag{2.13}$$

This property (known as traciality) offers practical advantages, since it allows one to predict the statistics of any outcome, once the Wigner function of the measured state is known.

The displacements constitute also a basic ingredient in the concept of coherent states. If we choose a fixed normalized reference state $|\Psi_0\rangle$, we have [48]

$$|x,p\rangle = \hat{D}(x,p)\, |\Psi_0\rangle\,, \tag{2.14}$$

so they are parametrized by phase-space points. These states have a number of remarkable properties inherited from those of $\hat{D}(x,p)$. In particular, $\hat{D}(x,p)$ transforms any coherent state in another coherent state:

$$\hat{D}(x',p')\, |x,p\rangle = \exp[i(x'p - p'x)/2]\, |x + x', p + p'\rangle\,. \tag{2.15}$$

The standard choice for the fiducial vector $|\Psi_0\rangle$ is the vacuum $|0\rangle$, which has quite a number of relevant properties.

III. PHASE-SPACE PICTURE OF A TWO-DIMENSIONAL HARMONIC OSCILLATOR

Next, we analyze the superposition of two oscillators in orthogonal directions, say x and y, with momenta $\hat{p}_x$ and $\hat{p}_y$, respectively. The corresponding complex amplitudes $\hat{a}_x$ and $\hat{a}_y$ fulfill $[\hat{a}_j,\hat{a}_k^\dagger] = \delta_{jk}\hat{\mathbb{1}}$ ($j,k \in \{x,y\}$). Since these oscillators are kinematically independent (i.e., they play the role of modes for the problem), the total system is represented by the product of the corresponding kernels

$$\hat{w}(x,p_x;y,p_y) = \hat{w}(x,p_x)\,\hat{w}(y,p_y). \tag{3.1}$$

The information is thus encoded in the four real variables (x,p_x) and (y,p_y). The resulting Wigner function $W(x,p_x;y,p_y)$ is informationally complete, but it is hard to grasp any physical flavor from it. In particular, it cannot be plotted (which is always a major advantage when depicting complex phenomena) and one must content oneself with sections of $W(x,p_x;y,p_y)$, which illustrate only partial aspects [49].

Because we are interested in elaborating on the behavior of OAM, which mostly appears when cylindrical symmetry is present, we make the change from Cartesian (x,y) to polar (r,φ) coordinates:

$$r = \sqrt{x^2+y^2}, \quad \varphi = \arctan(y/x). \tag{3.2}$$

Simultaneously, we change from (p_x,p_y) to

$$p_r = \frac{1}{r}(xp_x + yp_y), \quad \ell = xp_y - yp_x, \tag{3.3}$$

where p_r is the radial momentum and ℓ is the OAM. This transition from Cartesian to polar coordinates is not smooth at the origin and needs qualification because it takes from a contractible space to one which is not contractible. This lies at the root of the problems appearing when dealing with angle variables [50–57]. In quantum optics there are, however, a number of ways to bypass this drawback [58–65]. In the same vein, the radial momentum p_r is singular at the origin, which reflects a classical symptom of quantum illness [66], for such an operator is not self-adjoint (nor has self-adjoint extensions) [67–69]. Precisely, the use of Wigner-Weyl kernels alleviates these problems arising in a direct quantization. However, we brush aside these mathematical subtleties and move on to find a suitable solution for our problem.

Using the explicit form (2.5) for each orthogonal oscillator and after disentangling the exponentials, we can rewrite (3.1) in the equivalent way,

$$\begin{aligned}\hat{w}(r,p_r;\varphi,\ell) = 4(-1)^{\hat{N}} &\exp[-2\cos\varphi(\alpha_r\hat{a}_x^\dagger - \alpha_r^*\hat{a}_x)]\\ &\times \exp[2i\lambda_r\sin\varphi(\hat{a}_x^\dagger + \hat{a}_x)]\\ &\times \exp[-2\sin\varphi(\alpha_r\hat{a}_y^\dagger - \alpha_r^*\hat{a}_y)]\\ &\times \exp[-2i\lambda_r\cos\varphi(\hat{a}_y^\dagger + \hat{a}_y)],\end{aligned} \tag{3.4}$$

where

$$\hat{N} = \hat{a}_x^\dagger\hat{a}_x + \hat{a}_y^\dagger\hat{a}_y \tag{3.5}$$

is the total number of excitations and we have denoted $\alpha_r = (r+ip_r)/\sqrt{2}$ and $\lambda_r = \ell/(\sqrt{2}r)$.

The structure of this kernel suggests the use of the rotated operators

$$\hat{a}_+ = \frac{1}{\sqrt{2}}(\hat{a}_x - i\hat{a}_y), \quad \hat{a}_- = \frac{1}{\sqrt{2}}(\hat{a}_x + i\hat{a}_y), \tag{3.6}$$

in terms of which the OAM operator reads as

$$\hat{L} = \hat{a}_+^\dagger\hat{a}_+ - \hat{a}_-^\dagger\hat{a}_-. \tag{3.7}$$

In this way, we can interpret ℓ as the difference of quanta with opposite chirality. Note that the form of $\hat{N}$ and $\hat{L}$ suggests that the boson operators $\hat{a}_x$ and $\hat{a}_y$ furnish a Jordan-Schwinger representation for the problem at hand [much in the same way as the original oscillator construction for SU(2)], which can be justified on very general grounds [70]. On the other hand, such a representation should not come as a surprise, for it is well known that any three-dimensional Lie algebra (as the one we are dealing with here) can be realized in terms of creation and annihilation operators of two orthogonal oscillators [71].

By noticing that $e^{-i\varphi\hat{L}}\,\hat{a}_\pm\,e^{i\varphi\hat{L}} = \hat{a}_\pm e^{\pm i\varphi}$, we can recast the Wigner kernel (3.4) as the displaced version

$$\hat{w}(r,p_r;\varphi,\ell) = e^{-i\varphi\hat{L}}\,\hat{w}(r,p_r;\ell)\,e^{i\varphi\hat{L}}, \tag{3.8}$$

with

$$\begin{aligned}\hat{w}(r,p_r,\ell) = 4(-1)^{\hat{N}} &\exp[2i\lambda_r(\hat{p}_+ - \hat{p}_-)]\\ &\times \exp[-\sqrt{2}ip_r(\hat{x}_+ + \hat{x}_-)]\\ &\times \exp[\sqrt{2}ir(\hat{p}_+ + \hat{p}_-)]e^{-2ip_r r},\end{aligned} \tag{3.9}$$

and we have introduced the corresponding quadratures for the rotated amplitudes

$$\hat{x}_\pm = \frac{1}{\sqrt{2}}(\hat{a}_\pm + \hat{a}_\pm^\dagger), \quad \hat{p}_\pm = \frac{1}{\sqrt{2}i}(\hat{a}_\pm - \hat{a}_\pm^\dagger). \tag{3.10}$$

The radial momentum p_r plays no relevant role in the dynamics, so it seems entirely reasonable to integrate over this variable. To evaluate the resulting kernel we use an entangled state basis $|\xi\rangle$ (the properties of these states are briefly reviewed in Appendix A), such that

$$(\hat{a}_+ + \hat{a}_-^\dagger)|\xi\rangle = \xi|\xi\rangle, \quad (\hat{a}_+^\dagger + \hat{a}_-)|\xi\rangle = \xi^*|\xi\rangle. \tag{3.11}$$

The calculations are lengthy and the details are sketched in Appendix B. The final result for the Wigner function for a pure state $|\Psi\rangle$ turns out to be remarkably simple:

$$\begin{aligned}W_{|\Psi\rangle}(r,\varphi,\ell) = 4\int_{\mathbb{R}} &\Psi^*(r-ir',\varphi)\,\Psi(r+ir',\varphi)\\ &\times \exp(i2\ell r'/r)\,dr',\end{aligned} \tag{3.12}$$

which is the central result of this work. Here $\Psi(r,\varphi)$ denotes the wave function of $|\Psi\rangle$ in the entangled representation; i.e.,

$$\Psi(r,\varphi) = \langle\xi|e^{i\varphi\hat{L}}|\Psi\rangle = \langle\xi e^{-i\varphi}|\Psi\rangle. \tag{3.13}$$

Notice that the similarity with the single-mode Wigner function (2.10) is manifest. Obviously, the marginal over the radial variable

$$W(\varphi,\ell) = \int_0^\infty W(r,\varphi,\ell)\,dr \tag{3.14}$$

contains complete information about the pair angle-OAM and can be constructed from first principles [39].

IV. EXAMPLES

To gain further insight into this formalism, we work out Eq. (3.12) for several states of interest. First, we look for the case of a simultaneous eigenstate of both the total number of particles and the orbital angular momenta $|N,\ell_0\rangle$, viz.,

$$\hat{N}|N,\ell_0\rangle = N|N,\ell_0\rangle\,, \quad \hat{L}|N,\ell_0\rangle = \ell_0|N,\ell_0\rangle\,. \qquad (4.1)$$

Using the entangled representation, it is easy to check that

$$\Psi_{|N,\ell_0\rangle}(r,\varphi) = \frac{e^{-|\xi|^2/2}}{\sqrt{\left(\frac{N-\ell_0}{2}\right)!\left(\frac{N+\ell_0}{2}\right)!}} \times H_{\frac{N-\ell_0}{2},\frac{N+\ell_0}{2}}(\xi e^{-i\varphi},\xi^* e^{i\varphi}), \qquad (4.2)$$

where $H_{m,n}(\lambda,\lambda^*)$ stands for the two-variable Hermite polynomial. In terms of the generalized Laguerre polynomials $L_p^{\ell}(x)$, this reduces to

$$\Psi_{|N,\ell_0\rangle}(r,\varphi) = C_{N,\ell_0} e^{-\frac{1}{2}r^2} r^{|\ell_0|} L^{|\ell_0|}_{\frac{N-|\ell_0|}{2}}(r^2) e^{-i\ell_0\varphi}\,, \qquad (4.3)$$

where C_{N,ℓ_0} is a normalization constant. This wave function is very reminiscent of the standard Laguerre-Gauss modes employed in classical optics. The associated Wigner function is

$$W_{|N,\ell_0\rangle}(r,\varphi,\ell) = 4|C_{N,\ell_0}|^2 \int_{\mathbb{R}} (r+ir')^{2|\ell_0|}\left[L^{\ell_0}_{\frac{N-|\ell_0|}{2}}(r^2+r'^2)\right]^2 \times \exp[-(r^2+r'^2+2i\ell r'/r)]\,dr'\,. \qquad (4.4)$$

This integral can be computed in a closed way, although the expression is involved enough to be of practical use. If we sum over N, we get the state

$$|\ell_0\rangle = \sum_N \frac{1}{\sqrt{N+1}}|N,\ell_0\rangle\,. \qquad (4.5)$$

In Fig. 1 we have plotted an isocontour surface corresponding to $W_{|\ell_0\rangle}(r,\ell,\phi) = \text{constant}$, for $\ell_0 = 0$. We clearly appreciate quite a rich radial structure. At the top of the surface, we also include a density plot of a section by the plane $\ell = 0$, displaying the characteristic rings of the Laguerre modes. We recall that the standard Wigner function for the pair angle-OAM simplifies in this case to

$$W_{|\ell_0\rangle}(\ell,\varphi) = \frac{1}{2\pi}\delta_{\ell,\ell_0}\,, \qquad (4.6)$$

which is flat in φ and the integral over the whole phase space gives the unity, reflecting the normalization of $|\ell_0\rangle$. We can recognize the amount of information lost in this approach when compared with $W(r,\varphi,\ell)$. A similar procedure can be used for the case of the eigenstates of the angle $|\varphi_0\rangle$.

As our second example, we address the superposition

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\ell_1\rangle + e^{i\phi_0}|\ell_2\rangle) \qquad (4.7)$$

of two angular-momentum eigenstates with a relative phase $e^{i\phi_0}$. The resulting features are nicely illustrated in Fig. 2. The state $|\Psi\rangle$ is plotted for $\ell_2 = -3$ and $\ell_1 = 3$. Changing the relative phase ϕ_0 results in a global rotation of the cylinder.

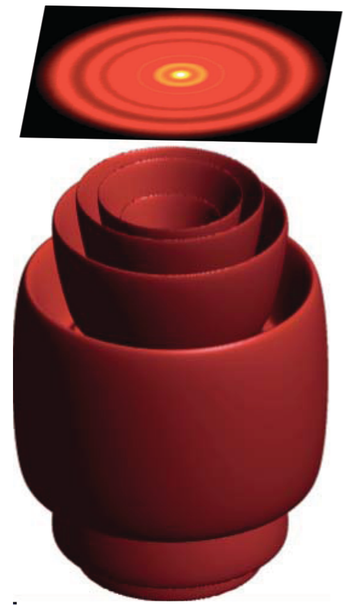

FIG. 1. (Color online) Isocontour surface of the level $1/e$ from the maximum of the Wigner function $W(r,\varphi,\ell)$ for an eigenstate of the OAM $|\ell_0\rangle$. At the top, we show a density plot of a section of that surface by the plane $\ell = 0$.

Again a rich radial structure can be appreciated. The “holes” in the isosurface correspond to points for which the Wigner function takes on negative values [72], as can be appreciated in the inset, where we draft the corresponding Wigner function $W(\ell,\varphi)$ for this state.

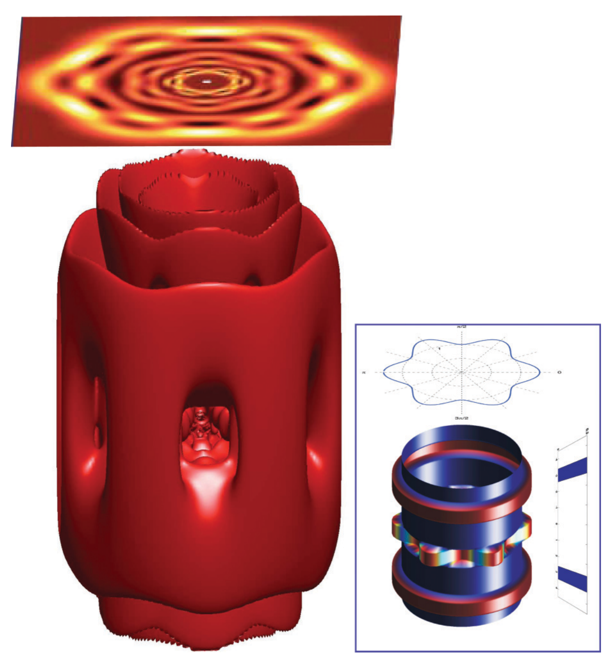

FIG. 2. (Color online) Isocontour surface of the level $1/e$ from the maximum of the Wigner function $W(r,\varphi,\ell)$ for the superposition state in Eq. (4.7), with $\ell_2 = -3$ and $\ell_1 = 3$. At the top, we show a density plot of a section of that surface by the plane $\ell = 0$. In the inset we show the standard Wigner function $W(\ell,\varphi)$ for this state, as well as the associated marginals.

V. CONCLUDING REMARKS

In summary, we have shown how to extend in a consistent way all the techniques developed for a continuous-variable phase space to the case of angle and angular momentum, including significant information about the radial variable. While we have not left aside the mathematical details, our main emphasis has been on presenting a simple and useful toolkit that any practitioner in the field should master. In our view, far from being an academic curiosity, the ideas expressed here have a wide range of potential applications in numerous hot topics in which OAM plays a key role.

ACKNOWLEDGMENTS

The inspiring ideas in this paper originated after many discussions with Prof. W. Schleich. Over the years, they have been further developed and completed with questions, suggestions, criticism, and advice from many colleagues. Particular thanks for help in various ways go to A. G. Barriuso, B.-G. Englert, J. C. Gallego, H. de Guise, and H. Kastrup. The work was supported by the EU FP7 (Grant Q-ESSENCE), the Spanish DGI (Grant FIS2011-26786), the Mexican CONACyT (Grant 106525), the Czech Ministry of Education (Project MSM6198959213), and the Czech Ministry of Industry and Trade (Project FR-TI1/364).

APPENDIX A: ENTANGLED-STATE REPRESENTATION

For the two modes $\pm$ defined in Eq. (3.6), the Fock space is spanned by

$$|n_+,n_-\rangle = \frac{(\hat{a}_+^\dagger)^{n_+}(\hat{a}_-^\dagger)^{n_-}}{\sqrt{n_+!n_-!}}|0,0\rangle\,, \tag{A1}$$

where $|0,0\rangle$ is the two-mode vacuum. Then one can immediately check that the vectors [73,74]

$$\begin{aligned}|\xi\rangle &= \exp\left[-\frac{1}{2}|\xi|^2 + \xi\hat{a}_+^\dagger + \xi^*\hat{a}_-^\dagger - \hat{a}_+^\dagger\hat{a}_-^\dagger\right]|0,0\rangle\,,\\ |\eta\rangle &= \exp\left[-\frac{1}{2}|\eta|^2 + \eta\hat{a}_+^\dagger - \eta^*\hat{a}_-^\dagger + \hat{a}_+^\dagger\hat{a}_-^\dagger\right]|0,0\rangle\end{aligned} \tag{A2}$$

are indeed eigenstates of the following operators:

$$\begin{aligned}&\hat{x}_+ - \hat{x}_-|\eta\rangle = \sqrt{2}\,\mathrm{Re}(\eta)|\eta\rangle\,, \quad \hat{p}_+ + \hat{p}_-|\eta\rangle = \sqrt{2}\,\mathrm{Im}(\eta)|\eta\rangle\,,\\ &\hat{x}_+ + \hat{x}_-|\xi\rangle = \sqrt{2}\,\mathrm{Re}(\xi)|\xi\rangle\,, \quad \hat{p}_+ - \hat{p}_-|\xi\rangle = \sqrt{2}\,\mathrm{Im}(\xi)|\xi\rangle\,,\end{aligned} \tag{A3}$$

where $\hat{x}_\pm$ and $\hat{p}_\pm$ are the quadrature operators associated to the modes $\pm$. This shows that these states are the continuous-variable versions of the original Einstein-Podolsky-Rosen states [75].

Using the technique of integration within an ordered product of operators [76], we can prove the orthogonal property and completeness relation

$$\langle\eta'|\eta\rangle = \pi\delta^{(2)}(\eta-\eta')\,, \quad \frac{1}{\pi}\int d^2\eta\,|\eta\rangle\langle\eta| = \hat{\mathbb{1}}\,, \tag{A4}$$

and an analogous one for ξ. In fact, one can also check that

$$\langle\xi|\eta\rangle = \tfrac{1}{2}\exp[(\xi\eta^* - \xi^*\eta)/2]\,. \tag{A5}$$

We observe also that if we use the Shapiro-Wagner angle operator [60]

$$\hat{E} = \sqrt{\frac{\hat{a}_+ + \hat{a}_-^\dagger}{\hat{a}_+^\dagger + \hat{a}_-}}\,, \tag{A6}$$

then

$$\hat{E}|\xi\rangle = \sqrt{\frac{\hat{a}_+ + \hat{a}_-^\dagger}{\hat{a}_+^\dagger + \hat{a}_-}}|\xi\rangle = \sqrt{\frac{\xi}{\xi^*}}|\xi\rangle = e^{i\varphi}|\xi\rangle\,, \tag{A7}$$

so these states have a well-defined angle.

If we recall that the two-variable Hermite polynomials, defined as [77]

$$H_{m,n}(\lambda,\lambda^*) = \sum_{\ell=0}^{\min(m,n)} \frac{m!n!}{\ell!(m-\ell)!(n-\ell)!}(-1)^\ell\lambda^{m-\ell}\lambda^{*n-\ell}\,, \tag{A8}$$

have the generating function

$$\sum_{m,n}^{\infty}\frac{t^m t'^n}{m!n!}H_{m,n}(\lambda,\lambda^*) = \exp(-tt' + t\lambda + t'\lambda^*)\,, \tag{A9}$$

by simple inspection we note that

$$\begin{aligned}|\eta\rangle &= \exp(-|\eta|^2/2)\sum_{n_+,n_-}\frac{(-1)^{n_-}}{\sqrt{n_+!n_-!}}H_{n_+,n_-}(\eta,\eta^*)|n_+,n_-\rangle\,,\\ |\xi\rangle &= \exp(-|\xi|^2/2)\sum_{n_+,n_-}\frac{1}{\sqrt{n_+!n_-!}}H_{n_+,n_-}(\xi,\xi^*)|n_+,n_-\rangle\,,\end{aligned} \tag{A10}$$

which constitute a compact expression of these entangled vectors in the Fock basis.

APPENDIX B: EVALUATING THE WIGNER-WEYL KERNEL

Our task here is to evaluate the kernel [Eq. (3.8)] and then integrate over the variable p_r. Using the properties of the entangled states in the previous appendix, we can write

$$\begin{aligned}\hat{w}(r,\ell) &= \int dp_r\;\hat{w}(r,p_r,\ell)\\ &= \frac{1}{\pi}(-1)^{\hat{N}}\int d^2\eta d^2\xi\;|\eta\rangle\langle\xi|\exp[(\xi\eta^* - \xi^*\eta)/2]\\ &\quad\times\exp[\sqrt{2}\lambda_r(\xi-\xi^*)]\exp[r(\eta-\eta^*)]\delta[r-\mathrm{Re}(\xi)]\,.\end{aligned} \tag{B1}$$

To simplify as much as possible what follows, we assume pure states, for which

$$W(r,\varphi,\ell) = \langle\Psi|\hat{w}(r,\varphi,\ell)|\Psi\rangle = \langle\Psi|e^{-i\varphi\hat{L}}\hat{w}(r,\ell)e^{i\varphi\hat{L}}|\Psi\rangle\,. \tag{B2}$$

This is in fact a marginal of the Wigner function of the problem. Next, we choose to expand $e^{i\varphi\hat{L}}|\Psi\rangle$ in the $|\xi\rangle$ basis. Taking into account the properties of these states, we have

$$\Psi(\xi,\varphi) = \langle \xi | e^{i\varphi \hat{L}} | \Psi \rangle = \langle \xi e^{-i\varphi} | \Psi \rangle . \tag{B3}$$

Therefore, we get

$$W(r,\varphi,\ell) = \frac{1}{\pi^2} \int d^2\xi' d^2\eta d^2\xi \, \langle \xi' | (-1)^{\hat{N}} | \eta \rangle \, \delta[r - \mathrm{Re}(\xi)] \times \Psi^*(\xi',\varphi) \Psi(\xi,\varphi) \exp[(\xi \eta^* - \xi^* \eta)/2] \times \exp[\sqrt{2}\lambda_r(\xi - \xi^*)] \exp[r(\eta - \eta^*)] . \tag{B4}$$

If we use the decomposition of these entangled states in terms of double-variable Hermite polynomials in Eq. (A10), and we recall that $H_{m,n}(\xi,\xi^*) = H^*_{n,m}(\xi,\xi^*)$, then it is easy to check that $\langle \xi' | (-1)^{\hat{N}} | \eta \rangle = \langle \eta | \xi' \rangle$. Consequently, we have

$$\int d^2\eta \langle \eta | \xi' \rangle \, \exp[(\xi \eta^* - \xi^* \eta)/2 + r(\eta - \eta^*)] = 4\pi^2 \delta^{(2)}(\xi + \xi' - 2r) . \tag{B5}$$

Finally, if we perform the integral over ξ' using this result we get

$$W(r,\varphi,\ell) = 4 \int d^2\xi \, \Psi^*(2r - \xi,\varphi) \Psi(\xi,\varphi) \times \exp[\sqrt{2}\lambda_r(\xi - \xi^*)] \, \delta[r - \mathrm{Re}(\xi)] . \tag{B6}$$

By separating the differential $d^2\xi$ in real and imaginary parts, after integrating over the real part $\mathrm{Re}(\xi)$ we get the result (3.12).

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