# Transport theory for a dilute Bose-Einstein condensate

L. E. Reichl* and Erich D. Gust
*The Center for Complex Quantum Systems, The University of Texas at Austin, Austin, Texas 78712, USA*
(Received 5 June 2013; published 6 November 2013)

We obtain microscopic expressions for the six hydrodynamic modes of a dilute Bose-Einstein condensate: two transverse (shear) modes and four longitudinal modes corresponding to the first sound (elastic waves) and second sound (temperature waves). Our microscopic expressions include both the speed of the two types of sound and the rate of relaxation of the sound waves. We obtain numerical values for the shear viscosity of a dilute BEC composed of bosons which interact via a contact potential. Our values for the shear viscosity are obtained using the eigenvalues and eigenvectors of the three types of collision operators that govern the relaxation of the condensate.

DOI: 10.1103/PhysRevA.88.053603 PACS number(s): 67.85.−d, 47.37.+q, 03.75.Kk

## I. INTRODUCTION

In 1995, Bose-Einstein condensation of dilute Bose gases was first observed in a gas of rubidium $^{87}$Rb atoms [1] and then in a gas of sodium $^{23}$Na atoms [2]. Since those first experiments, there have been a number of experiments investigating properties of dilute Bose-Einstein condensates (BECs), and it has been shown that the mean-field theory of dilute BECs, first proposed by Bogoliubov [3], shows excellent agreement with experiment for temperatures below about 60% of the critical temperature [4–6]. An excellent review of the mean field theory for dilute BECs can be found in [7].

Since the early work of Bogoliubov, there have also been a number of works that have incorporated mean-field theory into a kinetic theory that can describe the relaxation of BECs to thermodynamic equilibrium. One of the earliest efforts was due to Hohenberg and Martin [8], who related two fluid hydrodynamics to a microscopic mean-field theory using a Green's function approach. This led to the later work of Kirkpatrick and Dorfman [9], who derived expressions for transport coefficients in a BEC using mean-field theory and a Green's function method. However, Kirkpatrick and Dorfman do not give a unified microscopic theory for all the hydrodynamic modes in the BEC and they obtain an incomplete set of collision operators [10]. Contemporary to Hohenberg and Martin was the work of Peletminksii and Yatsenko [11,12], who derived a more traditional kinetic equation that could incorporate a mean-field description of relaxation processes in the superfluids. This approach to superfluid kinetic theory has been used by a number of authors to describe relaxation processes in superfluids [13–16], although none of these references specifically deal with dilute BECs. Gust and Reichl [10,17] have applied this method to a spatially uniform nonequilibrium BEC and have found collision operators describing processes not previously included in BEC kinetic equations. For a noncondensed dilute monatomic gas of bosons with temperatures higher than the critical temperature $T_c$ for Bose-Einstein condensation, five slowly varying hydrodynamic variables govern the relaxation to equilibrium. These five variables correspond to the five quantities conserved during elastic collisions between the particles: the particle number, momentum (three components), and kinetic energy of the particles. Above $T_c$, relaxation is governed by three transport coefficients: shear viscosity, thermal conductivity, and bulk viscosity (which is 0 for the dilute gas [18]). Below the critical temperature $T_c$, the boson gas has six hydrodynamic modes, but the microscopic collision processes occur between Bogoliubov excitations (bogolons) and only four quantities are conserved: bogolon momentum and energy. The additional hydrodynamic modes are due to particle number conservation and broken gauge symmetry.

Once a kinetic equation is derived there are two traditional approaches to computing transport coefficients: the method outlined by Chapman and Enskog [19] and a method due to Resibois [20], which directly uses the microscopic hydrodynamic modes of the system. In subsequent sections, we use the Peletminksii and Yatsenko method to derive the kinetic equation of a dilute BEC and the Resibois method to derive the microscopic hydrodynamic modes of the BEC (microscopic modes for a superconductor have been obtained [16] using this method). We obtain microscopic expressions for the six hydrodynamic modes of a dilute Bose-Einstein condensate: two transverse (shear) modes and four longitudinal modes corresponding to the first sound (density waves) and second sound (temperature waves). Our microscopic expressions include both the speed of the two types of sound and the rate of relaxation of the sound waves. We also obtain a microscopic expression for the shear viscosity of the BEC, and we compute the shear viscosity of a dilute Bose-Einstein condensed gas composed of sodium atoms for a variety of temperatures and densities.

We begin in Sec. II, with an outline of the derivation of the kinetic equation for a dilute gas of neutral particles that interact via a contact potential and have undergone Bose-Einstein condensation. Our derived kinetic equations contain the macroscopic phase of the condensate and the superfluid velocity. In Sec. III, we specialize the kinetic equations to the hydrodynamic regime where all quantities are slowly varying in space, and in Sec. IV, we linearize the kinetic equations about absolute equilibrium. In Sec. V, we show that our derived kinetic equations conserve the total particle number. In Sec. VI, we transform the particle kinetic equations to kinetic equations describing the dynamics of Bogoliubov excitations (bogolons) in the presence of the condensate and we determine the closure relation that connects the macroscopic phase to the bogolon

*reichl@mail.utexas.edu

 

dynamics. The microscopic hydrodynamic modes are derived in Sec, VII and the speeds of the first and second sound are plotted. By matching the frequency of the microscopic and macroscopic transverse hydrodynamic modes we can obtain a microscopic expression for the shear viscosity. This is done in Sec. VIII, where we also plot the shear viscosity of the dilute BEC gas for temperatures $0 < T < 0.6T_c$. Finally, in Sec. IX we make some concluding remarks.

## II. KINETIC THEORY

The derivation of the kinetic equation for a spatially uniform, nonequilibrium BEC was discussed in [21]. We outline the key steps here for a spatially nonuniform gas. The Hamiltonian for $N$ bosons of mass $m$, in a cubic box with very large volume $V$, can be written in the form

$$\hat{H} = \int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2\right)\hat{\Psi}(\mathbf{r}) + \frac{1}{2}\int\int d\mathbf{r}_1 d\mathbf{r}_2 \times V(|\mathbf{r}_1-\mathbf{r}_2|)\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)\hat{\Psi}(\mathbf{r}_2)\hat{\Psi}(\mathbf{r}_1), \quad (1)$$

where $\hat{\Psi}^\dagger(\mathbf{r})$ [$\hat{\Psi}(\mathbf{r})$] creates (removes) a particle at point $\mathbf{r}$, and the integration is over the entire volume $V$. These operators satisfy the boson commutation relations $[\hat{\Psi}(\mathbf{r}_1),\hat{\Psi}^\dagger(\mathbf{r}_2)] = \delta(\mathbf{r}_1-\mathbf{r}_2)$. We assume that the interaction between particles is given by the contact potential $V(|\mathbf{r}_1-\mathbf{r}_2|) = g\delta(\mathbf{r}_1-\mathbf{r}_2)$. The probability density operator $\hat{\rho}$ for this system satisfies the Liouville equation

$$\frac{\partial\hat{\rho}(t)}{\partial t} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)]. \quad (2)$$

Equations (1) and (2) give the exact behavior of the BEC gas.

Below the critical temperature for Bose-Einstein condensation, the gauge symmetry of the fluid is broken. To accurately describe the behavior of the BEC this broken symmetry needs to be incorporated into the dynamics. We introduce the one-body reduced density operator

$$\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2) = \begin{pmatrix}\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_2) & \hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)\\ \hat{\Psi}(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_2) & \hat{\Psi}(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)\end{pmatrix}. \quad (3)$$

We also introduce the one-body reduced density matrix

$$\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t) = \mathrm{Tr}\,[\hat{\rho}(t)\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)] = \begin{pmatrix}\mathcal{F}_{1,1}(\mathbf{r}_1,\mathbf{r}_2,t) & \mathcal{F}_{1,2}(\mathbf{r}_1,\mathbf{r}_2,t)\\ \mathcal{F}_{2,1}(\mathbf{r}_1,\mathbf{r}_2,t) & \mathcal{F}_{2,2}(\mathbf{r}_1,\mathbf{r}_2,t)\end{pmatrix} = \begin{pmatrix}\langle\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_2)\rangle & \langle\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)\rangle\\ \langle\hat{\Psi}(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_2)\rangle & \langle\hat{\Psi}(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_2)\rangle\end{pmatrix}. \quad (4)$$

The one-body reduced density matrix evolves in time as

$$-i\hbar\frac{\partial\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t)}{\partial t} = \mathrm{Tr}\{\hat{\rho}(t)[\hat{H},\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)]\}. \quad (5)$$

According to the Bogoliubov assumption, after a very short time $t$ the density operator $\hat{\rho}(t)$ will be a functional of the single-particle reduced density operator $\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t)$ [11,12]. The density operator then can be written

$$\hat{\rho}(t) = \hat{\rho}(\tilde{\mathcal{F}}(t)), \quad (6)$$

where $\tilde{\mathcal{F}}(t)$ denotes a vector containing $\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t)$ for all values of $(\mathbf{r}_1,\mathbf{r}_2)$. The average $\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t)$ is defined self-consistently so that

$$\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t) = \mathrm{Tr}\,[\hat{\rho}(\tilde{\mathcal{F}}(t))\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)]. \quad (7)$$

We make the existence of the broken symmetry explicit by introducing a mean-field Hamiltonian. We rewrite the total Hamiltonian in the form $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = \int d\mathbf{r}\hat{\Psi}^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m}\nabla_{\mathbf{r}}^2 - \mu\right)\hat{\Psi}(\mathbf{r}) + \hat{U}, \quad (8)$$

$$\hat{H}_1 = \frac{1}{2}\int\int d\mathbf{r}_1 d\mathbf{r}_2 \mathrm{V}(|\mathbf{r}_1-\mathbf{r}_2|)\hat{\Psi}^\dagger(\mathbf{r}_1) \times \hat{\Psi}^\dagger(\mathbf{r}_2)\hat{\Psi}(\mathbf{r}_2)\hat{\Psi}(\mathbf{r}_1) - \hat{U}, \quad (9)$$

$$\hat{U} = \frac{1}{2}\int d\mathbf{r}_1[\nu(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1) + \nu(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_1)] + \frac{1}{2}\int d\mathbf{r}_1\Delta^\dagger(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1) + \frac{1}{2}\int d\mathbf{r}_1\Delta(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_1), \quad (10)$$

where $\nu(\mathbf{r}_1) = 2g\langle\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1)\rangle$, $\Delta(\mathbf{r}_1) = g\langle\hat{\Psi}(\mathbf{r}_1)\hat{\Psi}(\mathbf{r}_1)\rangle$, $\Delta^\dagger(\mathbf{r}_1) = g\langle\hat{\Psi}^\dagger(\mathbf{r}_1)\hat{\Psi}^\dagger(\mathbf{r}_1)\rangle$, and $\mu$ is the equilibrium chemical potential.

### *Kinetic equation*

As discussed in [21], the kinetic equation describing the dynamic evolution of the one-body density matrix is given by

$$-i\hbar\frac{\partial\bar{\mathcal{F}}(\mathbf{r}_1,\mathbf{r}_2,t)}{\partial t} = \mathrm{Tr}\,\{\hat{\rho}(\tilde{\mathcal{F}})[\hat{H}_0,\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)]\} + \mathrm{Tr}\,\{\hat{\rho}(\tilde{\mathcal{F}})[\hat{H}_1,\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)]\} + \frac{i}{\hbar}\int_{-\infty}^{0} ds\mathrm{Tr}\,\{\hat{\rho}(\tilde{\mathcal{F}}(t)[\hat{H}_1,\hat{U}^{0,\dagger}(0,s) \times [\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2),\hat{H}_1]\hat{U}^0(0,s)\}, \quad (11)$$

where

$$\hat{U}^0(s_1,s_2) = e^{-\hat{H}_0(s_1-s_2)/\hbar}. \quad (12)$$

The form of the mean-field Hamiltonian $\hat{H}_0$, defined in Eq. (8), has been chosen so that

$$\mathrm{Tr}\,\{\hat{\rho}(\tilde{\mathcal{F}})[\hat{H}_1,\hat{\bar{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)]\} = 0. \quad (13)$$

This removes secular terms from the kinetic equation (at least to lowest order in the coupling constant). The kinetic equation takes the form of four coupled equations,

$$-i\hbar\frac{\partial\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2\rangle}{\partial t} = (\hat{E}_1-\hat{E}_2)\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2\rangle - \Delta_2\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2^\dagger\rangle + \Delta_1^\dagger\langle\hat{\Psi}_1\hat{\Psi}_2\rangle + \mathcal{I}_{11},$$

$$-i\hbar\frac{\partial\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2^\dagger\rangle}{\partial t} = (\hat{E}_1+\hat{E}_2)\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2^\dagger\rangle + \Delta_2^\dagger\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2\rangle + \Delta_1^\dagger\langle\hat{\Psi}_1\hat{\Psi}_2^\dagger\rangle + \mathcal{I}_{12},$$

$$-i\hbar\frac{\partial\langle\hat{\Psi}_1\hat{\Psi}_2\rangle}{\partial t} = (-\hat{E}_1-\hat{E}_2)\langle\hat{\Psi}_1\hat{\Psi}_2\rangle - \Delta_2\langle\hat{\Psi}_1\hat{\Psi}_2^\dagger\rangle - \Delta_1\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2\rangle + \mathcal{I}_{21},$$

$$-i\hbar\frac{\partial\langle\hat{\Psi}_1\hat{\Psi}_2^\dagger\rangle}{\partial t} = (-\hat{E}_1+\hat{E}_2)\langle\hat{\Psi}_1\hat{\Psi}_2^\dagger\rangle + \Delta_2^\dagger\langle\hat{\Psi}_1\hat{\Psi}_2\rangle - \Delta_1\langle\hat{\Psi}_1^\dagger\hat{\Psi}_2^\dagger\rangle + \mathcal{I}_{22}, \quad (14)$$

where $\hat{\Psi}_1=\hat{\Psi}(\mathbf{r}_1)$, $\hat{\Psi}_1^\dagger=\hat{\Psi}^\dagger(\mathbf{r}_1)$, $\hat{E}_i=-\frac{\hbar^2}{2m}\nabla_i^2+\nu_i-\mu$ for $i=1,2$, and the quantities

$$\begin{pmatrix}\mathcal{I}_{1,1} & \mathcal{I}_{1,2}\\ \mathcal{I}_{2,1} & \mathcal{I}_{2,2}\end{pmatrix} = \frac{i}{\hbar}\int_{-\infty}^{0} ds\,\mathrm{Tr}\,\{\hat{\rho}(\tilde{\mathcal{F}}(t)[\hat{H}_1,\hat{U}^{0,\dagger}(0,s)] \times[\hat{\hat{\Theta}}(\mathbf{r}_1,\mathbf{r}_2),\hat{H}_1]\hat{U}^0(0,s)]\} \quad (15)$$

are the collision integrals which govern the relaxation processes in the BEC gas. The next step is to transform these equations to the superfluid rest frame.

### Superfluid velocity

The field operators $\hat{\Psi}^\dagger(\mathbf{r})$ and $\hat{\Psi}(\mathbf{r})$ describe motion in the laboratory frame. We introduce the unitary transformation to the reference frame moving with the superfluid (superfluid rest frame) [9,22]. It is given by

$$\hat{U}(t)=\exp\left[-i\int d\mathbf{r}\phi(\mathbf{r},t)\hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})\right], \quad (16)$$

where $\phi(\mathbf{r},t)$ is the macroscopic phase of the condensate wave function. We let $\hat{\psi}^\dagger(\mathbf{r})$ and $\hat{\psi}(\mathbf{r})$ denote particle creation and annihilation operators in the superfluid rest frame. Then

$$\hat{U}^\dagger(t)\hat{\Psi}(\mathbf{r})\hat{U}(t)=e^{-i\phi(\mathbf{r},t)}\hat{\Psi}(\mathbf{r})=\hat{\psi}(\mathbf{r}), \quad (17)$$

where $\hat{\psi}(\mathbf{r})$ $[\hat{\psi}^\dagger(\mathbf{r})]$ removes (creates) a particle in the superfluid rest frame. Thus we find

$$\hat{\Psi}(\mathbf{r})=e^{i\phi(\mathbf{r},t)}\hat{\psi}(\mathbf{r}) \quad\text{and}\quad \hat{\Psi}^\dagger(\mathbf{r})=e^{-i\phi(\mathbf{r},t)}\hat{\psi}^\dagger(\mathbf{r}). \quad (18)$$

The one-body reduced density matrix now takes the form

$$\hat{\hat{\Theta}}(\mathbf{r}_1,\mathbf{r}_2)=\hat{T}(\mathbf{r}_1)\cdot\hat{\hat{\theta}}(\mathbf{r}_1,\mathbf{r}_2)\cdot\hat{T}^*(\mathbf{r}_2), \quad (19)$$

where

$$\hat{\hat{\theta}}(\mathbf{r}_1,\mathbf{r}_2)=\begin{pmatrix}\hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2) & \hat{\psi}^\dagger(\mathbf{r}_1)\hat{\psi}^\dagger(\mathbf{r}_2)\\ \hat{\psi}(\mathbf{r}_1)\hat{\psi}(\mathbf{r}_2) & \hat{\psi}(\mathbf{r}_1)\hat{\psi}^\dagger(\mathbf{r}_2)\end{pmatrix}$$
$$\text{and}\quad \hat{\hat{T}}(\mathbf{r}_1)=\begin{pmatrix}e^{i\phi_1} & 0\\ 0 & e^{-i\phi_1}\end{pmatrix}. \quad (20)$$

$\hat{\hat{\theta}}(\mathbf{r}_1,\mathbf{r}_2)$ is the one-body density operator in the superfluid rest frame and $\phi_i=\phi(\mathbf{r}_i,t)$.

The kinetic equations can now be written

$$-i\hbar\frac{\partial}{\partial t}\langle\hat{\psi}_1^\dagger\hat{\psi}_2\rangle=(\hat{e}_1^{(+)}-\hat{e}_2^{(-)})\langle\hat{\psi}_1^\dagger\hat{\psi}_2\rangle-\Delta_2\langle\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\rangle+\Delta_1^\dagger\langle\hat{\psi}_1\hat{\psi}_2\rangle+\mathcal{I}_{11},$$
$$-i\hbar\frac{\partial}{\partial t}\langle\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\rangle=(\hat{e}_2^{(+)}+\hat{e}_1^{(+)})\langle\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\rangle+\Delta_2^\dagger\langle\hat{\psi}_1^\dagger\hat{\psi}_2\rangle+\Delta_1^\dagger\langle\hat{\psi}_1\hat{\psi}_2^\dagger\rangle+\mathcal{I}_{12},$$
$$-i\hbar\frac{\partial}{\partial t}\langle\hat{\psi}_1\hat{\psi}_2\rangle=-(\hat{e}_2^{(-)}+\hat{e}_1^{(-)})\langle\hat{\psi}_1\hat{\psi}_2\rangle-\Delta_1\langle\hat{\psi}_1^\dagger\hat{\psi}_2\rangle-\Delta_2\langle\hat{\psi}_1\hat{\psi}_2^\dagger\rangle+\mathcal{I}_{21},$$
$$-i\hbar\frac{\partial}{\partial t}\langle\hat{\psi}_1\hat{\psi}_2^\dagger\rangle=(\hat{e}_2^{(+)}-\hat{e}_1^{(-)})\langle\hat{\psi}_1\hat{\psi}_2^\dagger\rangle-\Delta_1\langle\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\rangle+\Delta_2^\dagger\langle\hat{\psi}_1\hat{\psi}_2\rangle+\mathcal{I}_{22}, \quad (21)$$

where

$$\hat{e}_j^{(\pm)}=\epsilon_j\pm i\frac{\hbar}{2}[\nabla_j\cdot\mathbf{v}_s(j)]+\frac{m}{2}v_s^2(j)\pm i\hbar\mathbf{v}_s(j)\cdot\nabla_j+\hbar\frac{\partial\phi_j}{\partial t}, \quad (22)$$

with

$$\epsilon_j=-\frac{\hbar^2}{2m}\nabla_j^2+\nu_j-\mu,\quad \mathbf{v}_s(j)=\frac{\hbar}{m}\nabla_j\phi,$$
$$\nu_j=2g\langle\hat{\psi}_j^\dagger\hat{\psi}_j\rangle,\quad \Delta_j=g\langle\hat{\psi}_j\hat{\psi}_j\rangle,\quad \Delta_j^\dagger=g\langle\hat{\psi}_j^\dagger\hat{\psi}_j^\dagger\rangle. \quad (23)$$

The coupled kinetic equations contain the full quantum dynamics of the BEC gas (to second order in the coupling $\hat{H}_1$). We now can specialize these equations to the hydrodynamic regime where all macroscopic quantities are slowly varying in space and time. This can best be achieved by writing the kinetic equations in terms of Wigner functions.

## III. KINETIC EQUATIONS IN TERMS OF WIGNER FUNCTIONS

Wigner functions [23] provide a means to describe the behavior of a quantum system in phase space and are defined in Appendix A. The one-body density matrix, in terms of Wigner functions, can be written

$$\begin{pmatrix}F_{11}(\mathbf{K},\mathbf{R}) & F_{12}(\mathbf{K},\mathbf{R})\\ F_{21}(\mathbf{K},\mathbf{R}) & F_{22}(\mathbf{K},\mathbf{R})\end{pmatrix} = \frac{1}{V}\int_{\mathbf{r}} d\mathbf{r}e^{+i\mathbf{K}\cdot\mathbf{r}}\begin{pmatrix}\langle\hat{\psi}_1^\dagger\hat{\psi}_2\rangle & \langle\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\rangle\\ \langle\hat{\psi}_1\hat{\psi}_2\rangle & \langle\hat{\psi}_1\hat{\psi}_2^\dagger\rangle\end{pmatrix}, \quad (24)$$

where $\hbar\mathbf{K}$ is the momentum of particles in the gas and $\mathbf{R}$ is their position. We now specialize the kinetic equations to the hydrodynamic regime, where all macroscopic quantities are slowly varying in space. We therefore keep only the lowest order derivatives with respect to $\mathbf{R}$ in the kinetic equations. The superfluid velocities, $\mathbf{v}_s(\mathbf{r}_1)=\mathbf{v}_s(\mathbf{R}+\frac{1}{2}\mathbf{r})$ and $\mathbf{v}_s(\mathbf{r}_2)=\mathbf{v}_s(\mathbf{R}-\frac{1}{2}\mathbf{r})$ can be written

$$\mathbf{v}_s\left(\mathbf{R}\pm\tfrac{1}{2}\mathbf{r}\right)=\mathbf{v}_s(\mathbf{R})\pm\tfrac{1}{2}\mathbf{r}\cdot\nabla_{\mathbf{R}}\mathbf{v}_s(\mathbf{R})+\tfrac{1}{8}(\mathbf{r}\cdot\nabla_{\mathbf{R}})^2\mathbf{v}_s(\mathbf{R})+\cdots. \quad (25)$$

We note that $\delta\nu(\mathbf{r}_j)$, $\Delta^\dagger(\mathbf{r}_j)$, and $\Delta(\mathbf{r}_j)$ are also slowly varying functions of $\mathbf{R}$ in the hydrodynamic limit. Thus, we again neglect higher order derivatives with respect to $\mathbf{R}$, write

$$\delta\nu\left(\mathbf{R}\pm\tfrac{1}{2}\mathbf{r}\right)=\delta\nu(\mathbf{R})\pm\tfrac{1}{2}\mathbf{r}\cdot\nabla_{\mathbf{R}}\delta\nu(\mathbf{R})+\cdots, \quad (26)$$

and write similar expansions for $\Delta\left(\mathbf{R}\pm\frac{1}{2}\mathbf{r}\right)$ and $\Delta^\dagger\left(\mathbf{R}\pm\frac{1}{2}\mathbf{r}\right)$. We also note that

$$\frac{1}{V}\int d\mathbf{r}e^{i\mathbf{K}\cdot\mathbf{r}}\mathbf{r}F_{\alpha,\beta}(\mathbf{R},\mathbf{r})=-i\nabla_{\mathbf{K}}F_{\alpha,\beta}(\mathbf{R},\mathbf{K}). \quad (27)$$

At this point, it is useful to separate $\nu(\mathbf{r}_i)$ into its equilibrium part $\nu_0$ and a deviation from equilibrium $\delta\nu(\mathbf{r}_i)$, so that $\nu(\mathbf{r}_i)=\nu_0+\delta\nu(\mathbf{r}_i)$. Then define

$$\tilde{\epsilon}_{\mathbf{R},\mathbf{r}}^{(\pm)}=-\frac{\hbar^2}{2m}\left(\frac{1}{2}\nabla_{\mathbf{R}}\pm\nabla_{\mathbf{r}}\right)\cdot\left(\frac{1}{2}\nabla_{\mathbf{R}}\pm\nabla_{\mathbf{r}}\right)+\nu^0-\mu. \quad (28)$$

This decomposition into an equilibrium component and a deviation from equilibrium assumes that we have decomposed the density operator $\hat{\rho}(\tilde{\mathcal{F}}(t))$ in terms of a density operator describing the equilibrium state of the gas and a correction describing deviations from equilibrium.

The kinetic equations now take the form

$$\begin{aligned}-i\hbar\frac{\partial F_{11}(\mathbf{K},\mathbf{R})}{\partial t}&=(\tilde{\epsilon}^{(+)}_{\mathbf{K},\mathbf{R}}-\tilde{\epsilon}^{(-)}_{\mathbf{R},\mathbf{K}})F_{11}(\mathbf{K},\mathbf{R})+\hat{\mathcal{V}}_{\mathbf{R},\mathbf{K}}F_{11}(\mathbf{K},\mathbf{R})\\&\quad-i\,[\nabla_{\mathbf{R}}\Phi(\mathbf{R})]\cdot\nabla_{\mathbf{K}}F_{11}(\mathbf{K},\mathbf{R})-\hat{\Delta}_{(+)}(\mathbf{K},\mathbf{R})F_{12}(\mathbf{K},\mathbf{R})\\&\quad+\hat{\Delta}^{\dagger}_{(+)}(\mathbf{K},\mathbf{R})F_{21}(\mathbf{K},\mathbf{R})+\mathcal{I}_{11}(\mathbf{K},\mathbf{R}),\end{aligned}\qquad(29)$$

$$\begin{aligned}-i\hbar\frac{\partial F_{12}(\mathbf{K},\mathbf{R})}{\partial t}&=(\tilde{\epsilon}^{(+)}_{\mathbf{K},\mathbf{R}}+\tilde{\epsilon}^{(-)}_{\mathbf{K},\mathbf{R}})F_{12}(\mathbf{K},\mathbf{R})+\hat{\mathcal{V}}_{\mathbf{R},\mathbf{K}}F_{12}(\mathbf{K},\mathbf{R})\\&\quad+2\Phi(\mathbf{R})F_{12}(\mathbf{K},\mathbf{R})+\hat{\Delta}^{\dagger}_{(-)}(\mathbf{K},\mathbf{R})F_{11}(\mathbf{K},\mathbf{R})\\&\quad+\hat{\Delta}^{\dagger}_{(+)}(\mathbf{K},\mathbf{R})F_{22}(\mathbf{K},\mathbf{R})+\mathcal{I}_{12}(\mathbf{K},\mathbf{R}),\end{aligned}\qquad(30)$$

$$\begin{aligned}-i\hbar\frac{\partial F_{21}(\mathbf{K},\mathbf{R})}{\partial t}&=-(\tilde{\epsilon}^{(+)}_{\mathbf{K},\mathbf{R}}+\tilde{\epsilon}^{(-)}_{\mathbf{K},\mathbf{R}})F_{21}(\mathbf{K},\mathbf{R})+\hat{\mathcal{V}}_{\mathbf{R},\mathbf{K}}F_{21}(\mathbf{K},\mathbf{R})\\&\quad-2\Phi(\mathbf{R})F_{21}(\mathbf{K},\mathbf{R})-\hat{\Delta}_{(-)}(\mathbf{K},\mathbf{R})F_{11}(\mathbf{K},\mathbf{R})\\&\quad-\hat{\Delta}_{(+)}(\mathbf{K},\mathbf{R})F_{22}(\mathbf{K},\mathbf{R})+\mathcal{I}_{21}(\mathbf{K},\mathbf{R}),\end{aligned}\qquad(31)$$

$$\begin{aligned}-i\hbar\frac{\partial F_{22}(\mathbf{K},\mathbf{R})}{\partial t}&=-(\tilde{\epsilon}^{(+)}_{\mathbf{K},\mathbf{R}}-\tilde{\epsilon}^{(-)}_{\mathbf{R},\mathbf{K}})F_{22}(\mathbf{K},\mathbf{R})+\hat{\mathcal{V}}_{\mathbf{R},\mathbf{K}}F_{22}(\mathbf{K},\mathbf{R})\\&\quad+i[\nabla_{\mathbf{R}}\Phi(\mathbf{R})]\cdot\nabla_{\mathbf{K}}F_{22}(\mathbf{K},\mathbf{R})\\&\quad-\hat{\Delta}_{(-)}(\mathbf{K},\mathbf{R})F_{12}(\mathbf{K},\mathbf{R})\\&\quad+\hat{\Delta}^{\dagger}_{(-)}(\mathbf{K},\mathbf{R})F_{21}(\mathbf{K},\mathbf{R})+\mathcal{I}_{11}(\mathbf{K},\mathbf{R}),\end{aligned}\qquad(32)$$

where

$$\begin{aligned}\hat{\mathcal{V}}_{\mathbf{R},\mathbf{K}}&=+i\hbar\nabla_{\mathbf{R}}\cdot\mathbf{v}_s(\mathbf{R})+i\hbar\mathbf{v}_s(\mathbf{R})\cdot\nabla_{\mathbf{R}}\\&\quad-i\hbar[\nabla_{\mathbf{R}}(\mathbf{K}\cdot\mathbf{v}_s(\mathbf{R})]\cdot\nabla_{\mathbf{K}},\end{aligned}\qquad(33)$$

$$\hat{\Delta}_{(\pm)}(\mathbf{K},\mathbf{R})=\Delta(\mathbf{R})\pm\frac{i}{2}[\nabla_{\mathbf{R}}\Delta(\mathbf{R})]\cdot\nabla_{\mathbf{K}},\qquad(34)$$

$$\Phi(\mathbf{R})=\hbar\frac{\partial\phi(\mathbf{R})}{\partial t}+\delta\nu(\mathbf{R}),\qquad(35)$$

$$\tilde{\epsilon}^{(\pm)}_{\mathbf{R},\mathbf{K}}=-\frac{\hbar^2}{2m}\left(\frac{1}{2}\nabla_{\mathbf{R}}\mp i\mathbf{K}\right)\cdot\left(\frac{1}{2}\nabla_{\mathbf{R}}\mp i\mathbf{K}\right)+\nu^0-\mu.\qquad(36)$$

It is easy to show that the Wigner functions satisfy the conditions $F_{11}(\mathbf{K},\mathbf{R})=F^*_{22}(-\mathbf{K},\mathbf{R})$ and $F_{12}(\mathbf{K},\mathbf{R})=F^*_{21}(-\mathbf{K},\mathbf{R})$.

Expressions for transport coefficients can be determined from kinetic equations that are linearized about absolute equilibrium. Therefore, in the next section we describe the process of linearizing these kinetic equations.

## IV. LINEARIZED KINETIC EQUATIONS

We now write the hydrodynamic variables in terms of their equilibrium values plus small deviations from their equilibrium values,

$$\begin{aligned}F_{i,j}(\mathbf{K},\mathbf{R})&=F^{\mathrm{eq}}_{i,j}(\mathbf{K})+\delta F_{i,j}(\mathbf{K},\mathbf{R}),\quad\mathbf{v}_s(\mathbf{R})=\mathbf{v}^0_s+\delta\mathbf{v}_s(\mathbf{R}),\\\Delta(\mathbf{R})&=\Delta+\delta\Delta(\mathbf{R}),\quad\Delta^{\dagger}(\mathbf{R})=\Delta+\delta\Delta^{\dagger}(\mathbf{R}),\end{aligned}\qquad(37)$$

where $F^{\mathrm{eq}}_{i,j}(\mathbf{K})$, $\mathbf{v}^0_s$, and $\Delta$ denote the equilibrium values of the various quantities. Furthermore, we evaluate the transport properties of the BEC at temperatures below about 60% of the BEC critical temperature $T_c$. For these low temperatures, the "Popov" approximation to equilibrium quantities has been shown to give good agreement with experiment [4–6]. In this approximation $F^{\mathrm{eq}}_{11}(\mathbf{0})\approx F^{\mathrm{eq}}_{12}(\mathbf{0})\approx F^{\mathrm{eq}}_{21}(\mathbf{0})\approx F^{\mathrm{eq}}_{22}(\mathbf{0})\approx \mathrm{N}^{\mathrm{eq}}_{\mathbf{0}}$, where $\mathrm{N}^{\mathrm{eq}}_{\mathbf{0}}$ is the number density of particles in the condensate at equilibrium.

We next Fourier transform the space dependence of the kinetic equations and write the linearized kinetic equations in terms of the component of the spatial variation with wave vector $\mathbf{q}$ (see Appendix A):

$$F_{ij}(\mathbf{K},\mathbf{R})=\frac{1}{V}\sum_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{R}}F_{ij}(\mathbf{K},\mathbf{q}).\qquad(38)$$

Since the kinetic equations are linearized, each wave-vector component evolves independently. We also introduce the notation

$$e^{(\pm)}_{\mathbf{K},\mathbf{q}}=\frac{\hbar^2}{2m}\left|\mathbf{K}\pm\frac{1}{2}\mathbf{q}\right|^2+\nu^0-\mu.\qquad(39)$$

The resulting linearized kinetic equations can be written in matrix form,

$$\begin{aligned}-i\hbar\frac{\partial\delta\bar{F}}{\partial t}&=\{\bar{\epsilon}^{(+)}_{K,q}\delta\bar{F}-\delta\bar{F}\bar{\epsilon}^{(-)}_{K,q}\}+\hbar\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q})\bar{F}^{\mathrm{eq}}\\&\quad-\hbar\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q})\mathbf{q}\cdot\nabla_{\mathbf{K}}\bar{F}^{\mathrm{eq}}+\{\bar{B}\bar{F}^{\mathrm{eq}}-\bar{F}^{\mathrm{eq}}\bar{B}'\}\\&\quad+\mathbf{q}\cdot\nabla_{\mathbf{K}}\{\bar{D}\bar{F}^{\mathrm{eq}}-\bar{F}^{\mathrm{eq}}\bar{D}'\}+\delta\bar{\mathcal{I}},\end{aligned}\qquad(40)$$

where

$$\delta\bar{F}=\begin{pmatrix}\delta F_{11}(\mathbf{K},\mathbf{q},t)&\delta F_{12}(\mathbf{K},\mathbf{q},t)\\\delta F_{21}(\mathbf{K},\mathbf{q},t)&\delta F_{22}(\mathbf{K},\mathbf{q},t)\end{pmatrix},$$

$$\bar{F}^{\mathrm{eq}}=\begin{pmatrix}F^{\mathrm{eq}}_{11}(\mathbf{K})&F^{\mathrm{eq}}_{12}(\mathbf{K})\\F^{\mathrm{eq}}_{21}(\mathbf{K})&F^{\mathrm{eq}}_{22}(\mathbf{K})\end{pmatrix},\qquad(41)$$

$$\bar{\epsilon}^{(+)}_{K,q}=\begin{pmatrix}e^{(+)}_{\mathbf{K},\mathbf{q}}&\Delta\\-\Delta&-e^{(+)}_{\mathbf{K},\mathbf{q}}\end{pmatrix},\quad\bar{\epsilon}^{(-)}_{K,q}=\begin{pmatrix}e^{(-)}_{\mathbf{K},\mathbf{q}}&-\Delta\\\Delta&-e^{(-)}_{\mathbf{K},\mathbf{q}}\end{pmatrix},\qquad(42)$$

$$\bar{B}=\begin{pmatrix}\Phi(\mathbf{q})&\delta\Delta^{\dagger}(\mathbf{q})\\-\delta\Delta(\mathbf{q})&-\Phi(\mathbf{q})\end{pmatrix},\quad\bar{B}'=\begin{pmatrix}\Phi(\mathbf{q})&-\delta\Delta^{\dagger}(\mathbf{q})\\\delta\Delta(\mathbf{q})&-\Phi(\mathbf{q})\end{pmatrix},\qquad(43)$$

$$\begin{aligned}\bar{D}&=\begin{pmatrix}-\frac{1}{2}\Phi(\mathbf{q})&-\frac{1}{2}\delta\Delta^{\dagger}(\mathbf{q})\\\frac{1}{2}\delta\Delta(\mathbf{q})&\frac{1}{2}\Phi(\mathbf{q})\end{pmatrix},\\\bar{D}'&=\begin{pmatrix}\frac{1}{2}\Phi(\mathbf{q})&-\frac{1}{2}\delta\Delta^{\dagger}(\mathbf{q})\\\frac{1}{2}\delta\Delta(\mathbf{q})&-\frac{1}{2}\Phi(\mathbf{q})\end{pmatrix},\end{aligned}\qquad(44)$$

$$\delta\bar{\mathcal{I}}=\begin{pmatrix}\delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q},t)&\delta\mathcal{I}_{12}(\mathbf{K},\mathbf{q},t)\\\delta\mathcal{I}_{21}(\mathbf{K},\mathbf{q},t)&\delta\mathcal{I}_{22}(\mathbf{K},\mathbf{q},t)\end{pmatrix}.\qquad(45)$$

In these matrices $\Phi(\mathbf{q})=\hbar\frac{\partial\phi(\mathbf{q})}{\partial t}+\delta\nu(\mathbf{q})$ and $\delta\mathcal{I}_{ij}(\mathbf{K},\mathbf{q},t)$ are the linearized collision integrals for the particle kinetic equations. The collision integrals are discussed in [21] and $\delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q},t)$ given in Appendix B. [For simplicity and without loss of generality, we have set $v^0_s=0$ (superfluid velocity at equilibrium), although the space and time derivatives of $v^0_s$ are not 0.]

The particle kinetic equations are now in a form that allows us to transform them into kinetic equations for Bogoliubov excitations (bogolons) in the gas. But first we demonstrate that the linearized particle kinetic equations conserve the particle number.

## V. PARTICLE NUMBER CONSERVATION

The total particle number density in the interval $\mathbf{q} \to \mathbf{q} + d\mathbf{q}$ at time $t$ is

$$\delta \mathrm{N}(\mathbf{q},t) = \frac{1}{V}\sum_{\mathbf{K}} \delta F_{11}(\mathbf{K},\mathbf{q},t). \tag{46}$$

From Eq. (40) we can write

$$\begin{aligned} -i\hbar \frac{\partial \delta F_{11}(\mathbf{K},\mathbf{q},t)}{\partial t} &= (\tilde{\epsilon}^{(+)}_{\mathbf{K},\mathbf{q}} - \tilde{\epsilon}^{(-)}_{\mathbf{K},\mathbf{q}})\delta F_{11}(\mathbf{K},\mathbf{q},t) \\ &+ \hbar \mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathrm{N}^{\mathrm{eq}}_{\mathbf{K}} + \hbar\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathbf{q}\cdot\nabla_{\mathbf{K}}\mathrm{N}^{\mathrm{eq}}_{\mathbf{K}} \\ &- \Delta\delta F_{12}(\mathbf{K},\mathbf{q},t) - \delta\Delta(\mathbf{q},t)F^{\mathrm{eq}}_{12}(\mathbf{K}) + \Delta\delta F_{21}(\mathbf{K},\mathbf{q},t) \\ &+ \delta\Delta^{\dagger}(\mathbf{q},t)F^{\mathrm{eq}}_{21}(\mathbf{K}) + \delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q},t). \end{aligned} \tag{47}$$

Note that

$$\begin{aligned} \Delta &= \frac{g}{V}\sum_{\mathbf{K}} F^{\mathrm{eq}}_{12}(\mathbf{K}) = \frac{g}{V}\sum_{\mathbf{K}} F^{\mathrm{eq}}_{12}(\mathbf{K}), \\ \delta\Delta(\mathbf{q},t) &= \frac{g}{V}\sum_{\mathbf{K}}\delta F_{21}(\mathbf{K},\mathbf{q},t), \\ \delta\Delta^{\dagger}(\mathbf{q},t) &= \frac{g}{V}\sum_{\mathbf{K}}\delta F_{12}(\mathbf{K},\mathbf{q},t). \end{aligned} \tag{48}$$

Let us now sum over all momentum states in Eq. (47). The terms that depend on $\Delta$ cancel. One can also check that the third term on the right-hand side of Eq. (47) gives a negligible contribution, compared to the second term, when one integrates over $\mathbf{K}$ (see Appendix C), so we neglect this term in all subsequent calculations involving the particle conservation equation. Equation (47) reduces to

$$\begin{aligned} -i\hbar\frac{\partial\delta\mathrm{N}(\mathbf{q},t)}{\partial t} &= \frac{\hbar^2}{m}\frac{1}{V}\sum_{\mathbf{K}}\mathbf{K}\cdot\mathbf{q}\delta F_{11}(\mathbf{K},\mathbf{q},t) \\ &+ \hbar\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathrm{N}^{\mathrm{eq}} + \frac{1}{V}\sum_{\mathbf{K}}\delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q}), \end{aligned} \tag{49}$$

where $\mathrm{N}^{\mathrm{eq}}$ is the total particle number density. The linearized collision integral $\delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q})$ is given in Appendix B.

From the definitions of the collision integrals linearized about equilibrium in Appendix B, it is straightforward to show that $\sum_{\mathbf{K}\neq 0}\delta\mathcal{I}_{11}(\mathbf{K},\mathbf{q}) = -\delta\mathcal{I}_{11}(0,\mathbf{q})$. Therefore,

$$\begin{aligned} -i\hbar\frac{\partial\delta\mathrm{N}(\mathbf{q},t)}{\partial t} &= \frac{\hbar^2}{m}\frac{1}{V}\sum_{\mathbf{K}}\mathbf{K}\cdot\mathbf{q}\delta F_{11}(\mathbf{K},\mathbf{q},t) \\ &+ \hbar\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathrm{N}^{\mathrm{eq}}, \end{aligned} \tag{50}$$

and we obtain the continuity equation for the total particle number density.

## VI. BOGOLON KINETIC EQUATION

The hydrodynamic relaxation of the BEC is governed by the dynamics of the Bogoliubov excitations (bogolons). The collision operators that appear in the kinetic equation conserve bogolon momentum and energy. Therefore, in order to determine the hydrodynamic behavior of the BEC, we need to transform the particle kinetic equations to bogolon kinetic equations.

The Bogoliubov transformation from particle creation and annihilation operators, $\hat{a}^{\dagger}_1 = \hat{a}^{\dagger}_{\mathbf{k}_1}$ and $\hat{a}_1 = \hat{a}_{\mathbf{k}_1}$, respectively, to bogolon creation and annihilation operators, $\hat{b}^{\dagger}_1 = \hat{b}^{\dagger}_{\mathbf{k}_1}$ and $\hat{b}_1 = \hat{b}_{\mathbf{k}_1}$, respectively, can be written

$$\begin{pmatrix} \hat{a}^{\dagger}_1\hat{a}_2 & \hat{a}^{\dagger}_1\hat{a}^{\dagger}_{-2} \\ \hat{a}_{-1}\hat{a}_2 & \hat{a}_{-1}\hat{a}^{\dagger}_{-2}\end{pmatrix} = \bar{U}_1\cdot\begin{pmatrix} \hat{b}^{\dagger}_1\hat{b}_2 & \hat{b}^{\dagger}_1\hat{b}^{\dagger}_{-2} \\ \hat{b}_{-1}\hat{b}_2 & \hat{b}_{-1}\hat{b}^{\dagger}_{-2}\end{pmatrix}\cdot\bar{U}_2, \tag{51}$$

where

$$\bar{U}_1 = \begin{pmatrix} u_1 & -v_1 \\ -v_1 & u_1\end{pmatrix} \quad \text{and} \quad \bar{U}^{-1}_1 = \begin{pmatrix} u_1 & v_1 \\ v_1 & u_1\end{pmatrix}. \tag{52}$$

Since we are linearizing the kinetic equations about absolute equilibrium, it is sufficient to express the parameters $u_1$ and $v_1$ in terms of equilibrium quantities. When the BEC is at equilibrium, the mean-field Hamiltonian (in the superfluid rest frame) takes the form [21]

$$\begin{aligned} \hat{H}_0 &= \sum_i\left[(\epsilon_i - \Delta)\hat{a}^{\dagger}_i\hat{a}_i + \frac{\Delta}{2}(\hat{a}^{\dagger}_i\hat{a}_i + \hat{a}^{\dagger}_i\hat{a}_i)\right] \\ &= \frac{g}{2}N_0^2 + \sum_i E_i\hat{b}^{\dagger}_i\hat{b}_i, \end{aligned} \tag{53}$$

where

$$E_1 = \sqrt{e_1^2 - \Delta^2} \quad \text{with} \quad e_1 = \frac{\hbar^2k_1^2}{2m} + \nu^0 - \mu = \frac{\hbar^2k_1^2}{2m} + \Delta, \tag{54}$$

and we have used the Hugenholtz-Pines relation $\mu = \nu^0 - \Delta$ [24]. In terms of these equilibrium quantities, the Bogoliubov transformation parameters take the form

$$u_1 = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{e_1}{E_1}}, \quad v_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{e_1}{E_1} - 1}. \tag{55}$$

Note also that

$$\begin{aligned} u_1^2 - v_1^2 &= 1, \\ \Delta(u_1^2 + v_1^2) - 2e_1u_1v_1 &= 0, \\ e_1(u_1^2 + v_1^2) - 2\Delta u_1v_1 &= E_1. \end{aligned} \tag{56}$$

This transformation has the property that

$$\begin{aligned} &\bar{U}^{-1}_1\cdot\begin{pmatrix} e_1 & \Delta \\ -\Delta & -e_1\end{pmatrix}\cdot\bar{U}_1 \\ &= \bar{U}_1\cdot\begin{pmatrix} e_1 & -\Delta \\ \Delta & -e_1\end{pmatrix}\cdot\bar{U}^{-1}_1 = \begin{pmatrix} E_1 & 0 \\ 0 & -E_1\end{pmatrix}. \end{aligned} \tag{57}$$

Particles in a BEC form a condensate, but excitations (the bogolons) do not form a condensate. Therefore, we require that $\langle\hat{b}^{\dagger}_1\hat{b}^{\dagger}_{-1}\rangle = 0$ and $\langle\hat{b}_{-1}\hat{b}_1\rangle = 0$. Also, since we linearize the kinetic equations about absolute equilibrium, we can use the equilibrium expressions for the Bogoliubov transformation. Then

$$\begin{aligned} \begin{pmatrix} \mathcal{N}^{\mathrm{eq}}_K & 0 \\ 0 & \mathcal{F}^{\mathrm{eq}}_K\end{pmatrix} &= \begin{pmatrix} \langle\hat{b}^{\dagger}_{\mathbf{K}}\hat{b}_{\mathbf{K}}\rangle_{\mathrm{eq}} & 0 \\ 0 & \langle\hat{b}_{-\mathbf{K}}\hat{b}^{\dagger}_{-\mathbf{K}}\rangle_{\mathrm{eq}}\end{pmatrix} \\ &= \bar{U}^{-1}_K\cdot\bar{F}^{\mathrm{eq}}_{\mathbf{K}}\cdot\bar{U}^{-1}_K, \end{aligned} \tag{58}$$

where $\mathcal{N}_K^{\rm eq} = [\exp(\beta E_K) - 1]^{-1}$ is the Bose-Einstein distribution for bogolons and $\mathcal{F}_K^{\rm eq} = \mathcal{N}_K^{\rm eq} + 1$. We also require (because of the structure of the nonequilibrium density operator) that

$$\begin{pmatrix} \delta\mathcal{N}_{K,q} & 0 \\ 0 & \delta\mathcal{F}_{K,q} \end{pmatrix} = \begin{pmatrix} \delta\langle\hat{b}^\dagger_{\mathbf{k}_1}\hat{b}_{\mathbf{k}_2}\rangle & 0 \\ 0 & \delta\langle\hat{b}_{-\mathbf{k}_1}\hat{b}^\dagger_{-\mathbf{k}_2}\rangle \end{pmatrix} = \bar{U}^{-1}_{\mathbf{k}_1}\cdot\delta\bar{F}\cdot\bar{U}^{-1}_{\mathbf{k}_2}, \quad (59)$$

where $\mathbf{k}_1 = \mathbf{K} + \frac{1}{2}\mathbf{q}$ and $\mathbf{k}_2 = \mathbf{K} - \frac{1}{2}\mathbf{q}$.

We can now expand the particle number distribution in terms of bogolon distributions. We obtain

$$\delta F_{11}(\mathbf{K},\mathbf{q},t) = u_K^2\delta\mathcal{N}(\mathbf{K},\mathbf{q},t) + v_K^2\delta\mathcal{N}(-\mathbf{K},\mathbf{q},t) \quad (60)$$

and

$$\delta \mathrm{N}(\mathbf{q},t) = \frac{1}{V}\sum_{\mathbf{K}}(u_K^2 + v_K^2)\delta\mathcal{N}(\mathbf{K},\mathbf{q},t). \quad (61)$$

We can also expand the particle current in terms of bogolon currents

$$\sum_{\mathbf{K}}\mathbf{K}\delta F_{11}(\mathbf{K},\mathbf{q},t) = \sum_{\mathbf{K}}\mathbf{K}\left[u_K^2\delta\mathcal{N}(\mathbf{K},\mathbf{q},t) + v_K^2\delta\mathcal{N}(-\mathbf{K},\mathbf{q},t)\right] = \sum_{\mathbf{K}}\mathbf{K}\delta\mathcal{N}(\mathbf{K},\mathbf{q},t), \quad (62)$$

since $u_K^2 - v_K^2 = 1$. Thus, we obtain the very useful result that the bogolon momentum density is equal to the particle momentum density.

We can now obtain the bogolon kinetic equation. If we multiply Eq. (40) on the left by the Bogoliubov transformation matrix $\bar{U}_1^{-1} = \bar{U}_{\mathbf{k}_1}^{-1}$ and on the right by $\bar{U}_2^{-1} = \bar{U}_{\mathbf{k}_2}^{-1}$, we obtain

$$\begin{aligned} -i\hbar\frac{\partial\delta\bar{\mathcal{N}}}{\partial t} &= \left\{\bar{U}_1^{-1}\bar{\epsilon}_1\bar{U}_1\delta\bar{\mathcal{N}} - \delta\bar{\mathcal{N}}\bar{U}_2\bar{\epsilon}_2^T\bar{U}_2^{-1}\right\} \\ &\quad + \left\{\bar{U}_K^{-1}\bar{B}\bar{U}_K\bar{\mathcal{N}}^{\rm eq} - \bar{\mathcal{N}}^{\rm eq}\bar{U}_K\bar{B}'\bar{U}_K^{-1}\right\} \\ &\quad + \hbar\mathbf{q}\mathbf{v}_s(\mathbf{q})\bar{\mathcal{N}}^{\rm eq} - \hbar\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q})\mathbf{q}\cdot\left[\bar{U}_K^{-1}\nabla_{\mathbf{K}}\bar{F}^{\rm eq}\bar{U}_K^{-1}\right] \\ &\quad + \left\{\bar{U}_K^{-1}\bar{D}\bar{U}_K\mathbf{q}\cdot\left[\bar{U}_K^{-1}\nabla_{\mathbf{K}}\bar{F}^{\rm eq}\bar{U}_K^{-1}\right]\right. \\ &\quad \left. - \mathbf{q}\cdot\left[\bar{U}_K^{-1}\nabla_{\mathbf{K}}\bar{F}^{\rm eq}\bar{U}_K^{-1}\right]\bar{U}_K\bar{D}'\bar{U}_K^{-1}\right\} \\ &\quad + \bar{U}_1^{-1}\delta\bar{\mathcal{I}}\bar{U}_2^{-1}, \end{aligned} \quad (63)$$

where $\bar{\epsilon}^T$ denotes the transpose of matrix $\bar{\epsilon}$, and we have made the approximation $\bar{U}_1 \approx \bar{U}_K$ and $\bar{U}_2 \approx \bar{U}_K$ in the third to fifth terms on the right-hand side (neglect higher order terms in $q$), and

$$\bar{U}_1^{-1}\bar{\epsilon}_1\bar{U}_1 = \begin{pmatrix} E_1 & 0 \\ 0 & -E_1 \end{pmatrix}, \quad \bar{U}_2\bar{\epsilon}_2^T\bar{U}_2^{-1} = \begin{pmatrix} E_2 & 0 \\ 0 & -E_2 \end{pmatrix}. \quad (64)$$

Before simplifying Eq. (63), it is useful to write again the Hugenholtz-Pines equation [24] for the BEC. As discussed in [23], the time derivative of the macroscopic phase $\phi(\mathbf{R},t)$ is proportional to the chemical potential $\mu = -\hbar\frac{\partial\phi}{\partial t}$. Therefore, in the hydrodynamic regime, where we can assume that the system is locally in equilibrium, we can write the Hugenholtz-Pines equation in the form

$$\hbar\frac{\partial\phi(\mathbf{R},t)}{\partial t} + \delta\nu(\mathbf{R},t) - \delta\tilde{\Delta}(\mathbf{R},t) = 0, \quad (65)$$

where $\delta\mu(\mathbf{R},T) = -\hbar\frac{\partial\phi(\mathbf{R},t)}{\partial t}$ is the spatially varying chemical potential,

$$\delta\nu(\mathbf{R},t) = 2g\sum_{\mathbf{K}}\delta F_{11}(\mathbf{K},\mathbf{R}) = 2g\delta\mathrm{N}(\mathbf{R},t), \quad (66)$$

where $\delta\mathrm{N}(\mathbf{R},t)$ denotes deviations in the particle number density, and

$$\delta\tilde{\Delta}(\mathbf{R},t) = \frac{g}{2}\sum_{\mathbf{K}}(\delta F_{12}(\mathbf{K},\mathbf{R},t) + \delta F_{21}(\mathbf{K},\mathbf{R},t)) = \frac{1}{2}(\delta\Delta^\dagger(\mathbf{R},t) + \delta\Delta(\mathbf{R},t)) \quad (67)$$

is a real function of $\mathbf{R}$ and $t$.

In subsequent sections, we combine Eq. (50) for particle conservation with the Hugenholtz-Pines equation in order to obtain closure of the kinetic equations. This can be done using a "Bogoliubov-like" approximation for the nonequilibrium order parameter, $\delta\tilde{\Delta}(\mathbf{R},t) = g\delta\mathrm{N}(\mathbf{R},t)$ (this approximation is also used in [9]). As we see, this approximation only affects the longitudinal modes and allows us to write a microscopic expression for longitudinal modes that yields the correct speed of second sound at $T = 0$ K. In addition, it maintains the real nature of the nonequilibrium extension of the Hugenholtz-Pines equation. The Hugenholtz-Pines equation can now be written in the form

$$\hbar\frac{\partial\phi(\mathbf{R},t)}{\partial t} + g\delta\mathrm{N}(\mathbf{R},t) = 0. \quad (68)$$

This approximation limits us to very dilute gases, and we expect that it limits the accuracy of our results for the longitudinal modes to the temperature range $0 \leqslant T \leqslant 0.3T_c$.

Equation (68) gives a closure condition for the hydrodynamic equations. We analyze Eq. (63) in Appendix C using the closure condition in Eq. (68). Without loss of generality we can choose $\mathbf{q}$ to lie along the $z$ axis so $\mathbf{q} = q\hat{e}_z$, $\mathbf{v}_s(\mathbf{q}) = v_s(\mathbf{q})\hat{e}_z$, and $\hat{e}_z$ is a unit vector along the $z$ direction. We obtain the following expression for the bogolon kinetic equation:

$$\frac{\partial\delta\mathcal{N}(\mathbf{K},\mathbf{q},t)}{\partial t} = i\frac{\hbar}{m}K_z q\frac{(\epsilon_K + \Delta)}{E_K}\delta\mathcal{N}(\mathbf{K},\mathbf{q},t) + iqv_s(\mathbf{q},t)\Gamma_K^{\rm eq} + iqK_z\Theta_K^{\rm eq}\delta\mathrm{N}(\mathbf{q},t) + \frac{i}{\hbar}\delta\mathcal{G}_{11}(1,2), \quad (69)$$

where $\epsilon_K = \frac{\hbar^2K^2}{2m}$,

$$\Gamma_K^{\rm eq} = \mathcal{N}_K^{\rm eq} + \frac{\hbar^2}{2mk_BT}K_z^2\frac{2\epsilon_K}{E_k}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}, \quad (70)$$

and

$$\Theta_K^{\rm eq} = -g\frac{\hbar}{2m}\frac{2u_Kv_K}{E_k}(u_K - v_K)^2(u_K^2 + v_K^2)(2\mathcal{N}_K^{\rm eq}) + g\frac{\hbar}{2mk_BT}\frac{2\epsilon_K}{E_k}(u_K - v_K)^2(u_K^2 + v_K^2)\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}. \quad (71)$$

Note that our expression for $\Theta_K^{\rm eq}$ is consistent with the "Popov" approximation used to evaluate properties of the equilibrium BEC.

Let us now take the time derivative of Eq. (68) and use Eq. (50) for particle number conservation. The combined equations take the form

$$\frac{\partial^2\phi(\mathbf{q},t)}{\partial t^2} = -gqi\sum_{\mathbf{K}}\left[\frac{1}{m}K_z\delta\mathcal{N}(\mathbf{K},\mathbf{q},t)\right] - ig\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q},t)\mathrm{N}^{\mathrm{eq}}, \tag{72}$$

where $\mathrm{N}^{\mathrm{eq}}$ is the total particle number density.

In order to prepare for analysis of the sound modes, we consider one Fourier component of the kinetic equations. Since the equations are linear, each Fourier component propagates independently. Thus, we write

$$\delta\mathcal{N}(\mathbf{K},\mathbf{q},t) \sim e^{i\omega t}\delta\mathcal{N}(\mathbf{K},\mathbf{q},\omega). \tag{73}$$

Then the bogolon kinetic equation, (69), takes the form

$$\begin{aligned}\omega\delta\mathcal{N}(\mathbf{K},\mathbf{q},\omega) &= \frac{(\epsilon_K+\Delta)}{E_K}\frac{\hbar q K_z}{m}\delta\mathcal{N}(\mathbf{K},\mathbf{q},\omega) - qK_z\Theta_K^{\mathrm{eq}}\delta\mathrm{N}(\mathbf{q},\omega)\\ &\quad - i\Gamma_K^{\mathrm{eq}}\frac{\hbar}{m}q^2\phi(\mathbf{q},\omega) - \frac{i}{\hbar}\mathcal{G}_{11}(\mathbf{K},\mathbf{q},\omega),\end{aligned} \tag{74}$$

where we have used the fact that $\mathbf{v}_s(\mathbf{q},t) = -i\frac{\hbar}{m}\mathbf{q}\phi(\mathbf{q},t)$ and have retained terms to order $q^2$ on the right-hand side of these equations, and Eq. (72) takes the form

$$\omega^2\phi(\mathbf{q},\omega) = gqi\sum_{\mathbf{K}\neq 0}\left[\frac{1}{m}K_z\delta\mathcal{N}(\mathbf{K},\mathbf{q},\omega)\right] + gq^2\frac{1}{m}\phi(\mathbf{q},\omega)\mathrm{N}^{\mathrm{eq}}. \tag{75}$$

Equations (74) and (75) are the bogolon kinetic equations that describe the hydrodynamic behavior of a dilute BEC. We now investigate the consequences of these equations. The first step is to obtain microscopic expressions for the hydrodynamic oscillations in the condensate. We then connect these to macroscopic quantities.

### VII. MICROSCOPIC HYDRODYNAMIC MODES

For simplicity (and without loss of generality) let us assume that $\mathbf{q} = q\hat{e}_z$. Also, write the bogolon distribution as

$$\delta\mathcal{N}(\mathbf{K},\mathbf{q},\omega) = \mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}h(\mathbf{K},\mathbf{q},\omega), \tag{76}$$

where $h(\mathbf{K},\mathbf{q},\omega)$ is a small deviation from equilibrium. Equation (75) can be written in the form

$$\phi(\mathbf{q},\omega) = \frac{iq}{\omega^2 - \mathcal{A}q^2}\mathcal{J}\frac{1}{V}\sum_{\mathbf{K}}K_z\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}h(\mathbf{K},\mathbf{q},\omega), \tag{77}$$

where

$$\mathcal{A} = g\frac{1}{m}\mathrm{N}^{\mathrm{eq}} \quad \text{and} \quad \mathcal{J} = \frac{g}{m} = \mathcal{A}/\mathrm{N}^{\mathrm{eq}}. \tag{78}$$

If we now use Eq. (77), Eq. (74) for the bogolon distribution can be written

$$\begin{aligned}\omega\mathcal{N}_{\mathbf{K}}^{\mathrm{eq}}\mathcal{F}_{\mathbf{K}}^{\mathrm{eq}}h(\mathbf{K},\mathbf{q},\omega) &= q\mathcal{B}_K K_z\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}h(\mathbf{K},\mathbf{q},\omega)\\ &\quad + \frac{q^3}{\omega^2-\mathcal{A}q^2}\frac{\hbar}{m}\Gamma_K^{\mathrm{eq}}\mathcal{J}\frac{1}{V}\sum_{\mathbf{K}'}{}' K'_z\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}h(\mathbf{K}',\mathbf{q},\omega)\\ &\quad - qK_z\Theta_K^{\mathrm{eq}}\frac{1}{V}\sum_{\mathbf{K}'}\left(u_{K'}^2+v_{K'}^2\right)\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}h(\mathbf{K}',\mathbf{q},\omega)\\ &\quad - \frac{i}{\hbar}\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\hat{G}_{11}h(\mathbf{K},\mathbf{q},\omega),\end{aligned} \tag{79}$$

where $\hat{G}_{11}$ is the linearized collision operator and

$$\mathcal{B}_K = \frac{\hbar}{m}\frac{(e_K+\Delta)}{E_K}. \tag{80}$$

There are four microscopic hydrodynamic modes corresponding to the four conserved quantities, $\psi_1^{(0)}(\mathbf{K}) = C_1K_x$, $\psi_2^{(0)}(\mathbf{K}) = C_2K_y$, $\psi_3^{(0)}(\mathbf{K}) = C_3K_z$, and $\psi_4^{(0)}(\mathbf{K}) = C_4E_K$, where $C_i$ $(i = 1,\ldots,4)$ are normalization constants. Given the form of Eq. (79), these modes naturally separate into transverse modes $\psi_1^{(0)}$ and $\psi_2^{(0)}$ and longitudinal modes $\psi_3^{(0)}$ and $\psi_4^{(0)}$.

Below, first we find the frequencies associated with the transverse modes and then we consider the longitudinal modes. For both cases we need to expand the frequencies $\omega$ and the deviations from equilibrium in powers of $q$, which is the wave vector of the hydrodynamic modes. We let

$$\omega_n = \omega_n^{(0)} + q\omega_n^{(1)} + q^2\omega_n^{(2)} + O(q^3) \tag{81}$$

and

$$\begin{aligned}h(\mathbf{K},\mathbf{q},\omega) &= \psi_n^{(0)}(\mathbf{K},\omega) + q\psi_n^{(1)}(\mathbf{K},\omega)\\ &\quad + q^2\psi_n^{(2)}(\mathbf{K},\omega) + O(q^3).\end{aligned} \tag{82}$$

Note that, for hydrodynamic modes, $\omega_n^{(0)} = 0$, $\omega_n^{(1)}$ corresponds to the speed of the mode if it is a propagating mode, and $\omega_n^{(2)}$ gives the rate of decay of the mode. For nonhydrodynamic modes $\omega_n^{(0)} = \lambda_n$, where $\lambda_n$ is the $n$th nonzero eigenvalue of the collision operator. Also, we will normalize the states using the scaling factor $\mathcal{N}_k^{\mathrm{eq}}\mathcal{F}_k^{\mathrm{eq}}$. Therefore, we require

$$\sum_{\mathbf{K}}\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\psi_n^{(0)}(\mathbf{K})\psi_m^{(0)}(\mathbf{K}) = \delta_{m,n}. \tag{83}$$

The normalization constants $C_i$ are defined as $C_1^{-2} = \langle K_x^2\rangle$, $C_2^{-2} = \langle K_y^2\rangle$, $C_3^{-2} = \langle K_z^2\rangle$, and $C_4^{-2} = \langle E_K^2\rangle$, where

$$\langle f(\mathbf{K})\rangle = \sum_{\mathbf{K}}\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}f(\mathbf{K}). \tag{84}$$

To zeroth order in $q$, Eq. (79) reduces to

$$\omega_n^{(0)}\psi_n^{(0)}(\mathbf{K}) = -\frac{i}{\hbar}\hat{G}_{11}\psi_n^{(0)}(\mathbf{K}) = 0 \quad \text{for} \quad n = 1,2,3,4 \tag{85}$$

since the states $\psi_n^{(0)}(\mathbf{K})$ $(n = 1,\ldots,4)$ are eigenvectors of the collision operator $\hat{G}_{11}$ with eigenvalue equal to 0. Therefore, $\omega_n^{(0)} = 0$ for $n = 1,2,3,4$, and to second order in $q$, Eq. (79)

takes the form

$$\begin{aligned}&\left(q\omega_n^{(1)}+\cdots\right)\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\left(\psi_n^{(0)}(\mathbf{K})+q\psi_n^{(1)}(\mathbf{K})\right)\\&=q\mathcal{B}_K K_z\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\left(\psi_n^{(0)}(\mathbf{K})+q\psi_n^{(1)}(\mathbf{K})\right)+\frac{q^3}{\left(q\omega_n^{(1)}\right)^2-\mathcal{A}q^2}\frac{\hbar}{m}\Gamma_K^{\rm eq}\mathcal{J}\frac{1}{V}\sum_{\mathbf{K}'}{}^{'}K_z'\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}\left(\psi_n^{(0)}(K')+q\psi_n^{(1)}(\mathbf{K}')\right)\\&\quad-qK_z\Theta_K^{\rm eq}\frac{1}{V}\sum_{\mathbf{K}'}\left(u_{K'}^2+v_{K'}^2\right)\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}\left(\psi_n^{(0)}(\mathbf{K}')+q\psi_n^{(1)}(\mathbf{K}')\right)-\frac{i}{\hbar}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\hat{G}_{II}\left(q\psi_n^{(1)}(\mathbf{K})+q^2\psi_n^{(2)}(\mathbf{K})\right).\end{aligned}\tag{86}$$

We now can compute the frequencies of the hydrodynamic modes.

### A. Transverse modes

The two transverse modes correspond to $\psi_1^{(0)}(\mathbf{K})=C_1K_x$ and $\psi_2^{(0)}(\mathbf{K})=C_2K_y$. Let us first consider mode $n=1$. To first order in $q$, Eq. (86) becomes

$$\begin{aligned}&\omega_1^{(1)}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}C_1K_x\\&=\mathcal{B}_KK_z\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}C_1K_x+\frac{1}{\left(\omega_1^{(1)}\right)^2-\mathcal{A}}\frac{\hbar}{m}\Gamma_K^{\rm eq}\mathcal{J}\frac{1}{V}\\&\quad\times\sum_{\mathbf{K}'}{}^{'}K_z'\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}C_1K'_x-K_z\Theta_K^{\rm eq}\frac{1}{V}\sum_{\mathbf{K}'}\left(u_{K'}^2+v_{K'}^2\right)\\&\quad\times\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}C_1K'_x-\frac{i}{\hbar}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\hat{G}_{II}\psi_1^{(1)}(\mathbf{K}).\end{aligned}\tag{87}$$

The second and third terms on the right are 0 because of the summation involving $K'_x$. Therefore, Eq. (87) reduces to

$$\omega_1^{(1)}C_1K_x=\mathcal{B}_KC_1K_zK_x-\frac{i}{\hbar}\hat{G}_{II}\psi_1^{(1)}(\mathbf{K}).\tag{88}$$

Now multiply on the left by $C_1K_x\mathcal{N}_{\mathbf{K}}^{\rm eq}\mathcal{F}_{\mathbf{K}}^{\rm eq}$ and sum over $\mathbf{K}$. Since $C_1^2\langle K_x^2\rangle=1$, we find

$$\omega_1^{(1)}=\sum_K\mathcal{B}_KK_z\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}C_1^2K_x^2=0\tag{89}$$

and

$$\psi_1^{(1)}(\mathbf{K})=-\frac{C_1\hbar}{i\hat{G}_{II}}\mathcal{B}_KK_zK_x.\tag{90}$$

Let us now consider terms that are second order in $q$. The second and third terms on the right in Eq. (86) again do not contribute due to the summation $\sum_{K'}$. Therefore, Eq. (86) reduces to

$$\omega_1^{(2)}C_1K_x=\mathcal{B}_KK_z\psi_1^{(1)}(\mathbf{K})-\frac{i}{\hbar}\hat{G}_{II}\psi_1^{(1)}(\mathbf{K}).\tag{91}$$

Now multiply on the left by $C_1K_x\mathcal{N}_{\mathbf{K}}^{\rm eq}\mathcal{F}_{\mathbf{K}}^{\rm eq}$ and sum over $\mathbf{K}$ to get

$$\omega_1^{(2)}=-\sum_{\mathbf{K}}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\mathcal{B}_KK_zC_1K_x\frac{\hbar}{i\hat{G}_{II}}\mathcal{B}_KK_zC_1K_x.\tag{92}$$

It is straightforward to show that $\omega_1^{(2)}=\omega_2^{(2)}$. Thus, the transverse modes are decoupled from each other and the frequencies associated with the microscopic transverse modes are of the form $\omega=\omega_n^{(2)}q^2$ for $n=1,2$. Because $\omega_1^{(1)}=\omega_2^{(1)}=0$, the transverse modes cannot propagate. Transverse fluctuations in the gas will relax at a rate determined by $\omega_1^{(2)}$ and $\omega_2^{(2)}$.

### B. Longitudinal modes

The longitudinal modes are coupled to each other and are degenerate for $q=0$ so we need to use degenerate perturbation theory to determine the correct combination of modes $\psi_3^{(0)}(\mathbf{K})=C_3K_z$ and $\psi_4^{(0)}(\mathbf{K})=C_4E_K$ to use in the perturbation expansion. Also, because the equations are not symmetric, the left and right zeroth order (in $q$) states will be different. Let us write

$$\begin{aligned}\Psi_L^{(0)}(\mathbf{K})&=\Gamma_{3,L}\psi_3^{(0)}(\mathbf{K})+\Gamma_{4,L}\psi_4^{(0)}(\mathbf{K})\\\text{and}\quad\Psi_R^{(0)}(\mathbf{K})&=\Gamma_{3,R}\psi_3^{(0)}(\mathbf{K})+\Gamma_{4,R}\psi_4^{(0)}(\mathbf{K})\end{aligned}\tag{93}$$

for the left and right eigenvectors, respectively. Keeping terms first order in $q$, Eq. (86) becomes

$$\begin{aligned}&\omega^{(1)}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\Psi_R^{(0)}(\mathbf{K})\\&=\mathcal{B}_KK_z\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\Psi_R^{(0)}(\mathbf{K})+\frac{1}{((\omega^{(1)})^2-\mathcal{A})}\frac{\hbar}{m}\Gamma_K^{\rm eq}\mathcal{J}\frac{1}{V}\\&\quad\times\sum_{\mathbf{K}'}{}^{'}K_z'\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}\Psi_R^{(0)}(\mathbf{K}')-K_z\Theta_K^{\rm eq}\frac{1}{V}\sum_{\mathbf{K}'}\left(u_{K'}^2+v_{K'}^2\right)\\&\quad\times\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq}\Psi_R^{(0)}(\mathbf{K}')-\frac{i}{\hbar}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\hat{G}_{II}\psi_R^{(1)}(\mathbf{K}).\end{aligned}\tag{94}$$

Although the collision operator only admits two longitudinal modes with zero eigenvalue, Eq. (94), because of its nonlinear dependence on the frequency $\omega^{(1)}$ has four solutions. These four solutions correspond to the four longitudinal hydrodynamic modes.

Let us multiply Eq. (94) on the left by $\Psi_L^{(0)}(\mathbf{K})$ and sum over $\mathbf{K}$. The term involving the collision operator gives 0 when $\Psi_L^{(0)}(\mathbf{K})$ acts on it from the left. We then obtain

$$\begin{pmatrix}\Gamma_{1,L} & \Gamma_{2,L}\end{pmatrix}\begin{pmatrix}-\omega^{(1)} & \alpha\\ \beta(\omega^{(1)}) & -\omega^{(1)}\end{pmatrix}\begin{pmatrix}\Gamma_{1,R}\\ \Gamma_{2,R}\end{pmatrix}=0,\tag{95}$$

where

$$\beta(\omega^{(1)})=\alpha+\frac{\gamma}{(\omega^{(1)})^2-\mathcal{A}},\tag{96}$$

$$\begin{aligned}\alpha&=\sum_{\mathbf{K}}\mathcal{B}_KK_z^2C_3C_4E_K\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\\&\quad-\sum_{\mathbf{K}}C_3K_z^2\Theta_K^{\rm eq}\frac{1}{V}\sum_{\mathbf{K}'}\left(u_{K'}^2+v_{K'}^2\right)C_4E_{K'}\mathcal{N}_{K'}^{\rm eq}\mathcal{F}_{K'}^{\rm eq},\end{aligned}\tag{97}$$

and

$$\gamma=\mathcal{J}\sum_{\mathbf{K}}\frac{\hbar}{m}C_4E_K\Gamma_K^{\rm eq}\frac{1}{V}\sum_{\mathbf{K}'}{}^{'}(K_z')^2C_3\mathcal{N}_{\mathbf{K}'}^{\rm eq}\mathcal{F}_{\mathbf{K}'}^{\rm eq}.\tag{98}$$

Equation (95) has solutions if

$$\det\begin{pmatrix} -\omega^{(1)} & \alpha \\ \beta(\omega^{(1)}) & -\omega^{(1)} \end{pmatrix} = \frac{(\omega^{(1)})^4 - (\alpha^2 + \mathcal{A})(\omega^{(1)})^2 + \alpha^2\mathcal{A} - \alpha\gamma}{(\omega^{(1)})^2 - \mathcal{A}} = 0. \quad (99)$$

This equation has two pairs of solutions for $\omega^{(1)}$. The solutions undergo an avoided crossing at low temperatures. We call them $\omega^{(1)}_{\mathrm{I},\pm}$ and $\omega^{(1)}_{\mathrm{II},\pm}$ so that

$$\omega^{(1)}_{\mathrm{I},\pm} = \pm\sqrt{\frac{1}{2}[\alpha^2 + \mathcal{A} + \sqrt{(\mathcal{A} - \alpha^2)^2 + 4\alpha\gamma}]} \quad (100)$$

and

$$\omega^{(1)}_{\mathrm{II},\pm} = \pm\sqrt{\frac{1}{2}[\alpha^2 + \mathcal{A} - \sqrt{(\mathcal{A} - \alpha^2)^2 + 4\alpha\gamma}]}. \quad (101)$$

Note that $\gamma \to 0$ as $T \to 0$ K so that at very low temperatures the two sound modes decouple. Then we obtain $\lim_{T\to 0}\omega^{(1)}_{\mathrm{I},\pm} = \sqrt{\mathcal{A}} = \sqrt{\frac{gN_0^{\mathrm{eq}}}{m}}$ (which is the Bogoliubov speed of sound at $T = 0$) K and $\lim_{T\to 0}\omega^{(1)}_{\mathrm{II},\pm} = \alpha = \sqrt{\frac{gN_0^{\mathrm{eq}}}{3m}}$ at $T = 0$ K. These results agree with those obtained by Lee and Yang [25] using a very different method. In Fig. 1, we plot the speeds of these two types of sound as a function of $T/T_c$.

We can construct orthonormal pairs of left and right eigenvectors for the first sound and second sound modes. Let us first consider the first sound waves. The left and right eigenvectors, $\bar{\Psi}^T_{\mathrm{I}\pm}$ and $\bar{\Psi}_{\mathrm{I}\pm}$, respectively, take the form

$$\bar{\Psi}_{\mathrm{I}\pm} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm\sqrt{\alpha/\beta(\omega_{\mathrm{I},\pm})} \\ 1 \end{pmatrix}$$

$$\text{and} \quad \bar{\Psi}^T_{\mathrm{I}\pm} = \frac{1}{\sqrt{2}}(\pm\sqrt{\beta(\omega_{\mathrm{I},\pm})/\alpha}, 1). \quad (102)$$

Therefore, the right and left eigenvectors can be written

$$\Psi^R_{\mathrm{I},\pm}(\mathbf{K}) = \pm\frac{\sqrt{\alpha}}{\sqrt{2\beta(\omega_{\mathrm{I},\pm})}}\psi_3^{(0)}(\mathbf{K}) + \frac{1}{\sqrt{2}}\psi_4^{(0)}(\mathbf{K}) \quad (103)$$

and

$$\Psi^L_{\mathrm{I},\pm}(\mathbf{K}) = \pm\frac{\sqrt{\beta(\omega_{\mathrm{I},\pm})}}{\sqrt{2\alpha}}\psi_3^{(0)}(\mathbf{K}) + \frac{1}{\sqrt{2}}\psi_4^{(0)}(\mathbf{K}), \quad (104)$$

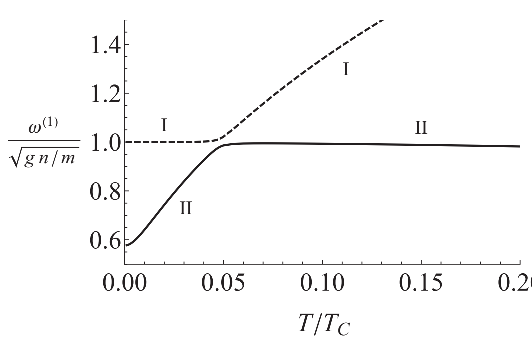


FIG. 1. Plot of the speed of the first sound (dashed line) and second sound (solid line) versus temperature at a particle number density $n \equiv \mathrm{N}^{\mathrm{eq}}$ such that $na^3 = 1.0 \times 10^{-5}$. Speeds are plotted in units of $\sqrt{gn/m}$, which is the Bogoliubov speed of sound at zero temperature.

respectively. Analogous expressions can be written for the second sound modes. Namely,

$$\Psi^R_{\mathrm{II},\pm}(\mathbf{K}) = \pm\frac{\sqrt{\alpha}}{\sqrt{2\beta(\omega_{\mathrm{II},\pm})}}\psi_3^{(0)}(\mathbf{K}) + \frac{1}{\sqrt{2}}\psi_4^{(0)}(\mathbf{K}) \quad (105)$$

and

$$\Psi^L_{\mathrm{II},\pm}(\mathbf{K}) = \pm\frac{\sqrt{\beta(\omega_{\mathrm{II},\pm})}}{\sqrt{2\alpha}}\psi_3^{(0)}(\mathbf{K}) + \frac{1}{\sqrt{2}}\psi_4^{(0)}(\mathbf{K}). \quad (106)$$

Let us now return to the perturbation expansion for the four longitudinal microscopic modes. We can solve Eq. (94) for the first-order (in $q$) correction to the right eigenvector. We obtain

$$\begin{aligned} &\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}) \\ &= -\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\frac{\hbar}{i\hat{G}_{II}}[(\omega^{(1)} - \mathcal{B}_K K_z)\Psi_R^{(0)}(\mathbf{K})] \\ &\quad - \frac{\hbar}{i\hat{G}_{II}}\Bigg[K_z\Theta_K^{\mathrm{eq}}\frac{1}{V}\sum_{\mathbf{K}'}(u_{K'}^2 + v_{K'}^2)\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}\Psi_R^{(0)}(\mathbf{K}') \\ &\quad - \frac{1}{((\omega^{(1)})^2 - \mathcal{A})}\frac{\hbar}{m}\Gamma_K^{\mathrm{eq}}\mathcal{J}\frac{1}{V}\sum_{\mathbf{K}'}K_z'\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}\Psi_R^{(0)}(\mathbf{K}')\Bigg]. \end{aligned} \quad (107)$$

We next find the correction to $\omega$ of order $q^2$. From Eq. (86) we can write

$$\begin{aligned} &\omega^{(2)}\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\Psi_R^{(0)}(\mathbf{K}) \\ &= \mathcal{B}_K K_z\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}) - K_z\Theta_K^{\mathrm{eq}}\frac{1}{V}\sum_{\mathbf{K}'}(u_{K'}^2 + v_{K'}^2) \\ &\quad \times \mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}') + \frac{1}{(\omega^{(1)})^2 - \mathcal{A}}\frac{\hbar}{m}\Gamma_K^{\mathrm{eq}}\mathcal{J}\frac{1}{V}\sum_{\mathbf{K}'} \\ &\quad - K_z'\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}') - \frac{i}{\hbar}\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\hat{G}_{II}\psi_R^{(2)}(\mathbf{K}). \end{aligned} \quad (108)$$

If we multiply on the left by $\Psi_L^{(0)}(\mathbf{K})$ and integrate over $\mathbf{K}$, we obtain

$$\begin{aligned} \omega^{(2)} &= \sum_{\mathbf{K}}\Psi_L^{(0)}(\mathbf{K})\mathcal{B}_K K_z\mathcal{N}_K^{\mathrm{eq}}\mathcal{F}_K^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}) \\ &\quad - \sum_{\mathbf{K}}\Psi_L^{(0)}(\mathbf{K})K_z\Theta_K^{\mathrm{eq}}\frac{1}{V}\sum_{\mathbf{K}'}(u_{K'}^2 + v_{K'}^2)\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}} \\ &\quad \times \psi_R^{(1)}(\mathbf{K}') + \sum_{\mathbf{K}}\Psi_L^{(0)}(\mathbf{K})\frac{1}{(\omega^{(1)})^2 - \mathcal{A}}\frac{\hbar}{m}\Gamma_K^{\mathrm{eq}}\mathcal{J}\frac{1}{V} \\ &\quad \times \sum_{\mathbf{K}'}K_z'\mathcal{N}_{K'}^{\mathrm{eq}}\mathcal{F}_{K'}^{\mathrm{eq}}\psi_R^{(1)}(\mathbf{K}'). \end{aligned} \quad (109)$$

This expression applies for all four longitudinal modes. We simply insert the appropriate values of $\omega$ and $\psi_R^{(1)}(\mathbf{K})$ into Eq. (109). The detailed behavior of the decay rates given by Eq. (109) is rather involved and will be considered elsewhere.

## VIII. VISCOSITY

We can obtain a relation between the viscosity and the microscopic hydrodynamic modes from the bogolon kinetic equation. The bogolon kinetic equation can be written

$$\frac{\partial \delta \mathcal{N}(\mathbf{K},\mathbf{R},t)}{\partial t} = -\nabla_{\mathbf{R}} \cdot \frac{(\epsilon_K + \Delta)}{E_K}\frac{\hbar \mathbf{K}}{m}\delta \mathcal{N}(\mathbf{K},\mathbf{R},t) + \nabla_{\mathbf{R}} \cdot \Theta_K \mathbf{K} \delta \mathrm{N}(\mathbf{R},t) - \nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R},t)\Gamma_K^{\mathrm{eq}} + \frac{i}{\hbar}\delta \mathcal{G}_{11}(\mathbf{K},\mathbf{R},t). \tag{110}$$

The balance equation for the momentum density is obtained by multiplying the bogolon kinetic equation by the momentum $\hbar\mathbf{K}$ and summing over all $\mathbf{K}$. As shown earlier, the bogolon current is equal to the particle current $[\mathcal{J}(\mathbf{R},t) = \sum_{\mathbf{K}} \hbar \mathbf{K} \delta \mathcal{N}(\mathbf{K},\mathbf{R},t) = \sum_{\mathbf{K}} \hbar \mathbf{K} \delta F_{11}(\mathbf{K},\mathbf{R},t)]$ and we obtain

$$\frac{\partial \delta \mathcal{J}(\mathbf{R},t)}{\partial t} = -\nabla_{\mathbf{R}} \cdot \sum_{\mathbf{K}} \frac{(\epsilon_K + \Delta)}{E_K}\frac{\hbar^2}{m}\mathbf{K}\mathbf{K}\delta \mathcal{N}(\mathbf{K},\mathbf{R},t) + \hbar \nabla_{\mathbf{R}} \cdot \sum_{\mathbf{K}} \Theta_K \mathbf{K}\mathbf{K}\delta \mathrm{N}(\mathbf{R},t). \tag{111}$$

The two-fluid hydrodynamic expression for the total particle current in the laboratory frame is $\mathbf{J} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n$, where $\rho_n(\rho_s)$ is the hydrodynamic expression for the normal fluid (superfluid) density, and $\mathbf{v}_n$ is the normal fluid velocity [26,27]. The time derivative of the total particle current then can be written (to first order in deviations from equilibrium) as

$$\frac{\partial \delta \mathbf{J}}{\partial t} = \rho_s \frac{\partial \mathbf{v}_s}{\partial t} + \rho_n \frac{\partial \mathbf{v}_n}{\partial t} = m N^{\mathrm{eq}} \frac{\partial \mathbf{v}_s}{\partial t} + \frac{\partial \delta \mathcal{J}(\mathbf{R},t)}{\partial t} \equiv -\nabla_{\mathbf{R}} \cdot \delta P(\mathbf{R},t) - \nabla_{\mathbf{R}} \cdot \bar{\Pi}_D(\mathbf{R},t), \tag{112}$$

where $\delta P(\mathbf{R},t)$ is the pressure and $\bar{\Pi}_D(\mathbf{R},t)$ is the dissipative momentum current (stress tensor).

We now introduce the coefficient of shear viscosity $\eta$. The dissipative part of the stress tensor can be written [22] $\bar{\Pi}^D = \bar{\Pi}^{(s)} + \Pi \bar{\mathbf{U}}$, where $\bar{\Pi}^{(s)}$ is a symmetric tensor, $\Pi$ is a scalar, and $\bar{\mathbf{U}}$ is the unit tensor. The shear viscosity is the generalized conductivity, which relates gradients in the transverse particle current, $\mathbf{J}_\perp = \rho_n \mathbf{v}_{n,\perp}$ to the symmetric stress tensor so that

$$\rho_n \frac{\partial \mathbf{v}_{n,\perp}}{\partial t} = \frac{\partial \mathcal{J}_\perp}{\partial t} = \left(\nabla_{\mathbf{R}} \cdot \bar{\Pi}_D^{(s)}\right)_\perp \equiv -2\eta \nabla_{\mathbf{R}}^2 \mathbf{v}_{n,\perp}. \tag{113}$$

It is now useful to transform from variables $(\mathbf{R},t)$ to the Fourier components of the (linearized) hydrodynamic equation with wave vector $\mathbf{q}$ and frequency $\omega$ via a transformation of the form

$$\delta \mathcal{J}(\mathbf{R},t) \sim e^{+i\omega t} e^{-i\mathbf{q}\cdot\mathbf{R}} \delta \mathcal{J}(\mathbf{q},\omega). \tag{114}$$

In order to pull out the transverse parts of Eqs. (111) and (113), we assume that $\mathbf{q} = q\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is a unit vector in the $z$ direction. Eq. (111), then separates into two independent equations for $\mathcal{J}_x$ and $\mathcal{J}_y$ (the transverse modes) and an equation for the longitudinal part of the flow $\mathcal{J}_z$. The two transverse hydrodynamic modes each satisfy the equation

$$\omega \delta \mathcal{J}_\perp(\mathbf{q},\omega) = i\frac{\eta q^2}{\rho_n}\mathcal{J}_\perp(\mathbf{q},\omega), \tag{115}$$

where $\delta \mathcal{J}_\perp(\mathbf{q},\omega)$ is a component of the momentum current perpendicular to the $z$ direction.

If we now equate the microscopic expression [Eq. (92)] and the macroscopic expression [Eq. (115)] for the transverse hydrodynamic modes, we obtain the following microscopic expression for the shear viscosity:

$$\eta = -\rho_n \sum_{\mathbf{K}} \mathcal{N}_{\mathbf{K}}^{\mathrm{eq}} \mathcal{F}_{\mathbf{K}}^{\mathrm{eq}} \mathcal{B}_K K_z C_1 K_x \frac{\hbar}{\hat{G}_{11}} \mathcal{B}_K K_z C_1 K_x. \tag{116}$$

Numerical values of the shear viscosity for a dilute gas can be explicitly calculated using this expression. To carry out this calculation, we use a spectral decomposition of the linearized collision operator [10,18]. The viscosity depends on the particle properties and the equilibrium density and temperature of the system. The particle density is parameterized by $a^3 N^{\mathrm{eq}}$, where $N^{\mathrm{eq}}$ is the particle number density, $a = \frac{mg}{4\pi\hbar^2}$ is the $s$-wave scattering length, and the temperature is parameterized by $T/T_C$, where $T_C$ is the BEC transition temperature of an ideal gas. In this parametrization scheme, the viscosity can be written as a pure function of $a^3 N^{\mathrm{eq}}$ and $T/T_C$ times the quantity

$$\xi = \frac{1}{8\pi a^2}\sqrt{\frac{\pi m k_B T}{2}}. \tag{117}$$

The results of the viscosity calculation are shown in Fig. 2. We have plotted the viscosity as a function of $T/T_C$ for two densities, $a^3 N^{\mathrm{eq}} = 2.5 \times 10^{-6}$ and $a^3 N^{\mathrm{eq}} = 1.0 \times 10^{-5}$. Compared to the noncondensed case, where $\eta/\xi \sim 0.5$, the viscosity of the BEC is much lower. This is a result of the higher relaxation rates as discussed in Ref. [10].

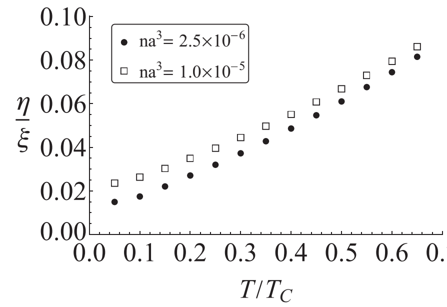


FIG. 2. Variation of the shear viscosity with temperature for a dilute BEC. Circles represent the density $na^3 = 2.5 \times 10^{-6}$ and squares represent the density $na^3 = 1.0 \times 10^{-5}$ ($n$ is the particle number density).

## IX. CONCLUSIONS

We have derived the kinetic equation for a dilute BEC using Bogoliubov mean-field theory and we have retained terms to second order in perturbations $\hat{H}_1$ to the mean-field Hamiltonian. The superfluid velocity is given by the gradient of the macroscopic phase of the condensate wave function. In all our calculations, the equilibrium state of the gas is evaluated in the "Popov" approximation $\Delta = g N_0^{\mathrm{eq}}$ and are expected to give good results for the transverse modes and the viscosity in the temperature range $0 \leqslant T \leqslant 0.6 T_c$. To obtain the longitudinal modes, we use a closure relation that has the form of a nonequilibrium Hugenholtz-Pines relation, and we evaluate it using the Bogoliubov approximation $\delta\tilde{\Delta}(\mathbf{R},T) = g\delta N(\mathbf{R},T)$. We expect our expressions for the longitudinal modes to give good results for temperatures $0 \leqslant T \leqslant 0.3 T_c$.

We have used the bogolon kinetic equation to derive microscopic expressions for the six hydrodynamic modes of a dilute

BEC gas: two transverse modes and four longitudinal modes. Using our microscopic expressions for the four longitudinal modes, we have computed the speeds of the first and second sound. We obtain sound speeds that are in agreement with those obtained by Lee and Yang [25] for $0 \leqslant T \leqslant 0.3T_c$.

We have computed the shear viscosity for a dilute BEC of neutral atoms that interact via a contact potential, using the full information contained in the three collision operators (see [10] and Appendix B) that contribute to the relaxation of the gas. For the temperature region we consider, we find that the shear viscosity of a Bose-Einstein condensed gas is about 10% of its value just above the critical temperature.

## ACKNOWLEDGMENT

The authors thank the Robert A. Welch Foundation (Grant No. F-1051) for support of this work.

## APPENDIX A: WIGNER FUNCTIONS

Wigner functions are distribution functions in phase space for quantum systems [28]. They are particularly useful in dealing with transport processes because in the classical limit they reduce to classical probability distributions in phase space. Let us introduce center-of-mass and relative coordinates $\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, respectively. Then we write the one-body density matrix in the form

$$\begin{pmatrix} F_{11}(\mathbf{r},\mathbf{R}) & F_{12}(\mathbf{r},\mathbf{R}) \\ F_{21}(\mathbf{r},\mathbf{R}) & F_{22}(\mathbf{r},\mathbf{R}) \end{pmatrix} \equiv \begin{pmatrix} \langle \hat{\psi}_1^\dagger \hat{\psi}_2 \rangle & \langle \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \rangle \\ \langle \hat{\psi}_1 \hat{\psi}_2 \rangle & \langle \hat{\psi}_1 \hat{\psi}_2^\dagger \rangle \end{pmatrix}. \quad \text{(A1)}$$

If we next introduce center-of-mass and relative wave vectors $\mathbf{K} = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$ and $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$, respectively, the Wigner functions for the BEC are defined

$$\begin{pmatrix} F_{11}(\mathbf{K},\mathbf{R}) & F_{12}(\mathbf{K},\mathbf{R}) \\ F_{21}(\mathbf{K},\mathbf{R}) & F_{22}(\mathbf{K},\mathbf{R}) \end{pmatrix} = \frac{1}{V}\int d\mathbf{r} e^{+i\mathbf{K}\cdot\mathbf{r}} \begin{pmatrix} \langle \hat{\psi}_1^\dagger \hat{\psi}_2 \rangle & \langle \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \rangle \\ \langle \hat{\psi}_1 \hat{\psi}_2 \rangle & \langle \hat{\psi}_1 \hat{\psi}_2^\dagger \rangle \end{pmatrix}, \quad \text{(A2)}$$

where $\hbar\mathbf{K}$ is the momentum of particles in the gas and $\mathbf{R}$ is their position. In the classical limit, $F_{11}(\mathbf{K},\mathbf{R})$ is the particle number density in the interval $\mathbf{K} \to \mathbf{K} + d\mathbf{K}$ and $\mathbf{R} \to \mathbf{R} + d\mathbf{R}$.

The field operators $\hat{\psi}_1^\dagger$ and $\hat{\psi}_1$ are related to the operators $\hat{a}_{\mathbf{k}_1}^\dagger$ and $\hat{a}_{\mathbf{k}_1}$, which create and annihilate, respectively, a particle with momentum $\hbar\mathbf{k}_1$, via the Fourier transforms

$$\hat{\psi}_1^\dagger = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}_1} e^{-i\mathbf{k}_1\cdot\mathbf{r}_1}\hat{a}_{\mathbf{k}_1}^\dagger \quad \text{and} \quad \hat{\psi}_1 = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}_1} e^{+i\mathbf{k}_1\cdot\mathbf{r}_1}\hat{a}_{\mathbf{k}_1}. \quad \text{(A3)}$$

We can relate the configuration space distributions to momentum space distributions via the Fourier transformation

$$\begin{pmatrix} \langle \hat{\psi}_1^\dagger \hat{\psi}_2 \rangle & \langle \hat{\psi}_1^\dagger \hat{\psi}_2^\dagger \rangle \\ \langle \hat{\psi}_1 \hat{\psi}_2 \rangle & \langle \hat{\psi}_1 \hat{\psi}_2^\dagger \rangle \end{pmatrix} = \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2} e^{-i\mathbf{k}_1\cdot\mathbf{r}_1} e^{+i\mathbf{k}_2\cdot\mathbf{r}_2} \begin{pmatrix} \langle \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \rangle & \langle \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{-\mathbf{k}_2}^\dagger \rangle \\ \langle \hat{a}_{-\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \rangle & \langle \hat{a}_{-\mathbf{k}_1} \hat{a}_{-\mathbf{k}_2}^\dagger \rangle \end{pmatrix}. \quad \text{(A4)}$$

This, in turn, allows us to write

$$\begin{pmatrix} F_{11}(\mathbf{r},\mathbf{R}) & F_{12}(\mathbf{r},\mathbf{R}) \\ F_{21}(\mathbf{r},\mathbf{R}) & F_{22}(\mathbf{r},\mathbf{R}) \end{pmatrix} = \frac{1}{V}\sum_{\mathbf{K},\mathbf{q}} e^{-i\mathbf{K}\cdot\mathbf{r}} e^{-i\mathbf{q}\cdot\mathbf{R}} \begin{pmatrix} F_{11}(\mathbf{K},\mathbf{q}) & F_{12}(\mathbf{K},\mathbf{q}) \\ F_{21}(\mathbf{K},\mathbf{q}) & F_{22}(\mathbf{K},\mathbf{q}) \end{pmatrix} \quad \text{(A5)}$$

and

$$\begin{pmatrix} F_{11}(\mathbf{K},\mathbf{R}) & F_{12}(\mathbf{K},\mathbf{R}) \\ F_{21}(\mathbf{K},\mathbf{R}) & F_{22}(\mathbf{K},\mathbf{R}) \end{pmatrix} = \frac{1}{V}\sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}} \begin{pmatrix} F_{11}(\mathbf{K},\mathbf{q}) & F_{12}(\mathbf{K},\mathbf{q}) \\ F_{21}(\mathbf{K},\mathbf{q}) & F_{22}(\mathbf{K},\mathbf{q}) \end{pmatrix}, \quad \text{(A6)}$$

where

$$\begin{pmatrix} F_{11}(\mathbf{K},\mathbf{q}) & F_{12}(\mathbf{K},\mathbf{q}) \\ F_{21}(\mathbf{K},\mathbf{q}) & F_{22}(\mathbf{K},\mathbf{q}) \end{pmatrix} = \begin{pmatrix} \langle \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{\mathbf{k}_2} \rangle & \langle \hat{a}_{\mathbf{k}_1}^\dagger \hat{a}_{-\mathbf{k}_2}^\dagger \rangle \\ \langle \hat{a}_{-\mathbf{k}_1} \hat{a}_{\mathbf{k}_2} \rangle & \langle \hat{a}_{-\mathbf{k}_1} \hat{a}_{-\mathbf{k}_2}^\dagger \rangle \end{pmatrix}. \quad \text{(A7)}$$

In terms of Wigner functions, the particle number density $N(\mathbf{R})$ at point $\mathbf{R}$ is

$$N(\mathbf{R}) = \sum_{\mathbf{K}} F_{11}(\mathbf{K},\mathbf{R}) = \langle \hat{\psi}_{\mathbf{R}}^\dagger \hat{\psi}_{\mathbf{R}} \rangle = \sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{R}} N(\mathbf{q}), \quad \text{(A8)}$$

where $N(\mathbf{q}) = \sum_{\mathbf{K}} F_{11}(\mathbf{K},\mathbf{q})$ is the component of the particle number density whose spatial variation has wave vector $\mathbf{q}$. The number of particles $N(\mathbf{K})$ with momentum $\hbar\mathbf{K}$ is

$$N(\mathbf{K}) = \int d\mathbf{R} F_{11}(\mathbf{K},\mathbf{R}) = \langle \hat{a}_{\mathbf{K}}^\dagger \hat{a}_{\mathbf{K}} \rangle. \quad \text{(A9)}$$

The component of the order parameters whose spatial variation has wave vector $\mathbf{q}$ is given by

$$\Delta^\dagger(\mathbf{q}) = g\sum_{\mathbf{K}} F_{12}(\mathbf{K},\mathbf{q}) \quad \text{and} \quad \Delta(\mathbf{q}) = g\sum_{\mathbf{K}} F_{21}(\mathbf{K},\mathbf{q}). \quad \text{(A10)}$$

## APPENDIX B: LINEARIZED COLLISION INTEGRALS

In this Appendix we give the linearized collision integrals that appear in the particle kinetic equations and the bogolon kinetic equations. We first give the linearized particle collision integrals.

### 1. Linearized particle collision integrals

The linearized particle collision integral $\delta\mathcal{I}_{11}(\mathbf{q},\mathbf{K})$ is the sum of three contributions:

$$\delta\mathcal{I}_{11}(\mathbf{q},\mathbf{K}) = \delta\mathcal{C}_{\mathbf{K}_1,\mathbf{q}}^{12} + \delta\mathcal{C}_{\mathbf{K}_1,\mathbf{q}}^{22} + \delta\mathcal{C}_{\mathbf{K}_1,\mathbf{q}}^{31}. \quad \text{(B1)}$$

They are obtained by linearizing the bogolon collision integrals given in [10] and [21]. All of the particle collision integrals depend on the bogolon distribution functions. To linearize the collision integrals, we let $\mathcal{F}_j = \mathcal{F}_j^{\text{eq}} + \mathcal{N}_j^{\text{eq}}\mathcal{F}_j^{\text{eq}}\Phi_j$ and $\mathcal{N}_j = \mathcal{N}_j^{\text{eq}} + \mathcal{N}_j^{\text{eq}}\mathcal{F}_j^{\text{eq}}\Phi_j$, where $\Phi_j \equiv \Phi_{\mathbf{K}_j,\mathbf{q}}$, and keep only those terms of first order in the small parameter $\Phi_j$. The energy conservation $\delta$ function ensures that the zeroth-order terms cancel. The linearized particle collision integrals are

$$\delta\mathcal{C}^{12}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi N_0 g^2}{\hbar V^2}\sum_{2,3}{}' \delta_{1,2+3}\delta(\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3)W^{12}_{3,2,1}\big[\Upsilon^A_{1,2,3}\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\big)(\Phi_2+\Phi_3-\Phi_1)$$
$$+\tilde{\Upsilon}^A_{1,2,3}\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\big)(\Phi_{-2}+\Phi_{-3}-\Phi_{-1})\big] + \frac{8\pi N_0 g^2}{\hbar V^2}\sum_{2,3}{}'\delta_{1+2,3}\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3)W^{12}_{1,2,3}$$
$$\times\big[\Upsilon^B_{1,2,3}\big(\mathcal{F}^{\rm eq}_3\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_1\big)(\Phi_3-\Phi_1-\Phi_2)+\tilde{\Upsilon}^B_{1,2,3}\big(\mathcal{F}^{\rm eq}_3\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_1\big)(\Phi_{-3}-\Phi_{-1}-\Phi_{-2})\big], \tag{B2}$$

$$\delta\mathcal{C}^{22}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi g^2}{\hbar V^2}\sum_{2,3,4}{}'\delta_{1+2,3+4}\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)W^{22}_{1,2,3,4}\big[\Upsilon^C_{1,2,3,4}\big(\mathcal{F}^{\rm eq}_1\mathcal{F}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(\Phi_3+\Phi_4-\Phi_1-\Phi_2)$$
$$-\tilde{\Upsilon}^C_{1,2,3,4}\big(\mathcal{F}^{\rm eq}_1\mathcal{F}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(\Phi_{-3}+\Phi_{-4}-\Phi_{-1}-\Phi_{-2})\big], \tag{B3}$$

and

$$\delta\mathcal{C}^{31}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi g^2}{3\hbar V^2}\sum_{2,3,4}{}'\delta_{1,2+3+4}\delta(\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)W^{31}_{1,2,3,4}\big[\Upsilon^D_{1,2,3,4}\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(\Phi_2+\Phi_3+\Phi_4-\Phi_1)$$
$$-\tilde{\Upsilon}^D_{1,2,3,4}\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(\Phi_{-2}+\Phi_{-3}+\Phi_{-4}-\Phi_{-1})\big]+\frac{4\pi g^2}{\hbar V^2}\sum_{2,3,4}{}'\delta_{1+2+3,4}\delta(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3-\mathcal{E}_4)W^{31}_{4,3,2,1}$$
$$\times\big[\Upsilon^E_{1,2,3,4}\big(\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{F}^{\rm eq}_4\big)(\Phi_4-\Phi_1-\Phi_2-\Phi_3)-\tilde{\Upsilon}^E_{1,2,3,4}\big(\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{F}^{\rm eq}_4\big)(\Phi_{-4}-\Phi_{-1}-\Phi_{-2}-\Phi_{-3})\big], \tag{B4}$$

where the weighting functions are given in terms of $u_i$ and $v_i$ by

$$W^{12}_{1,2,3} = u_1u_2u_3 - u_1v_2u_3 - v_1u_2u_3 + u_1v_2v_3 + v_1u_2v_3 - v_1v_2v_3, \tag{B5}$$

$$W^{22}_{1,2,3,4} = u_1u_2u_3u_4 + u_1v_2u_3v_4 + u_1v_2v_3u_4 + v_1u_2u_3v_4 + v_1u_2v_3u_4 + v_1v_2v_3v_4, \tag{B6}$$

and

$$W^{31}_{1,2,3,4} = u_1u_2u_3v_4 + u_1u_2v_3u_4 + u_1v_2u_3u_4 + v_1v_2v_3u_4 + v_1v_2u_3v_4 + v_1u_2v_3v_4. \tag{B7}$$

Each of these weighting functions has specific symmetry with respect to interchanges of its indices that is shared by its collision operator. Also,

$$\Upsilon^A_{1,2,3} = u_1u_2u_3 - u_1v_2u_3 - u_1u_2v_3, \tag{B8}$$

$$\Upsilon^B_{1,2,3} = u_1u_2u_3 + u_1v_2v_3 - u_1v_2u_3, \tag{B9}$$

$$\Upsilon^C_{1,2,3,4} = u_1u_2u_3u_4 + u_1v_2v_3u_4 + u_1v_2u_3v_4, \tag{B10}$$

$$\Upsilon^D_{1,2,3,4} = u_1v_2u_3u_4 + u_1u_2v_3u_4 + u_1u_2u_3v_4, \tag{B11}$$

$$\Upsilon^E_{1,2,3,4} = u_1u_2v_3u_4 + u_1v_2u_3u_4 + u_1v_2v_3v_4, \tag{B12}$$

and $\tilde{\Upsilon}$ is $\Upsilon$ with each $u$ and $v$ interchanged.

### 2. Linearized bogolon collision integrals

The linearized bogolon collision integral $\delta\mathcal{J}_{II}(\mathbf{q},\mathbf{K})$ is the sum of three contributions,

$$\delta\mathcal{J}_{II}(\mathbf{q},\mathbf{K}) = \delta\mathcal{G}^{12}_{\mathbf{K}_1,\mathbf{q}} + \delta\mathcal{G}^{22}_{\mathbf{K}_1,\mathbf{q}} + \delta\mathcal{G}^{31}_{\mathbf{K}_1,\mathbf{q}}, \tag{B13}$$

defined below. They are obtained by linearizing the bogolon collision integrals given in [10] and [21]. To linearize the bogolon collision integrals, we have let $\mathcal{F}_j = \mathcal{F}^{\rm eq}_j + \mathcal{N}^{\rm eq}_j\mathcal{F}^{\rm eq}_j h_j$ and $\mathcal{N}_j = \mathcal{N}^{\rm eq}_j + \mathcal{N}^{\rm eq}_j\mathcal{F}^{\rm eq}_j h_j$, where $h_j \equiv h_{\mathbf{K}_j,\mathbf{q}}$, and keep only those terms of first order in the small parameter $h_j$. The energy conservation $\delta$ function ensures that the zeroth-order terms cancel. The linearized particle-bogolon collision integrals are

$$\delta\mathcal{G}^{12}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi N_0 g^2}{\hbar V^2}\sum_{2,3}{}'\delta_{1,2+3}\delta(\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3)\big(W^{12}_{3,2,1}\big)^2\big[\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\big)(h_2+h_3-h_1)\big] + \frac{8\pi N_0 g^2}{\hbar V^2}\sum_{2,3}{}'\delta_{1+2,3}\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3)\big(W^{12}_{1,2,3}\big)^2\big[\big(\mathcal{F}^{\rm eq}_3\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_1\big)(h_3-h_1-h_2)\big], \tag{B14}$$

$$\delta\mathcal{G}^{22}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi g^2}{\hbar V^2}\sum_{2,3,4}{}'\delta_{1+2,3+4}\delta(\mathcal{E}_1+\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)\big(W^{22}_{1,2,3,4}\big)^2\big[\big(\mathcal{F}^{\rm eq}_1\mathcal{F}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(h_3+h_4-h_1-h_2)\big], \tag{B15}$$

and

$$\delta\mathcal{G}^{31}_{\mathbf{K}_1,\mathbf{q}} = \frac{4\pi g^2}{3\hbar V^2}\sum_{2,3,4}{}'\delta_{1,2+3+4}\delta(\mathcal{E}_1-\mathcal{E}_2-\mathcal{E}_3-\mathcal{E}_4)\big(W^{31}_{1,2,3,4}\big)^2\big[\big(\mathcal{F}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{N}^{\rm eq}_4\big)(h_2+h_3+h_4-h_1)\big] + \frac{4\pi g^2}{\hbar V^2}\sum_{2,3,4}{}'\delta_{1+2+3,4}\delta(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3-\mathcal{E}_4)\big(W^{31}_{4,3,2,1}\big)^2\big[\big(\mathcal{N}^{\rm eq}_1\mathcal{N}^{\rm eq}_2\mathcal{N}^{\rm eq}_3\mathcal{F}^{\rm eq}_4\big)(h_4-h_1-h_2-h_3)\big]. \tag{B16}$$

## APPENDIX C: GRADIENT TERMS IN THE KINETIC EQUATION

In this Appendix we examine terms involving $\nabla_{\mathbf{K}}\bar{F}$ that appear in Eq. (40). We can evaluate them if we express them in terms of bogolon distributions. Note that

$$\nabla_{\mathbf{K}}\begin{pmatrix}F^{\rm eq}_{11}(\mathbf{K}) & F^{\rm eq}_{12}(\mathbf{K})\\ F^{\rm eq}_{21}(\mathbf{K}) & F^{\rm eq}_{22}(\mathbf{K})\end{pmatrix} = \nabla_{\mathbf{K}}\left[\begin{pmatrix}u_K & -v_K\\ -v_K & u_K\end{pmatrix}\cdot\begin{pmatrix}\mathcal{N}^{\rm eq}_K & 0\\ 0 & \mathcal{F}^{\rm eq}_K\end{pmatrix}\cdot\begin{pmatrix}u_K & -v_K\\ -v_K & u_K\end{pmatrix}\right] = -\frac{\hbar^2}{2m}\mathbf{K}\frac{2u_Kv_K}{E_k}\big(\mathcal{N}^{\rm eq}_K+\mathcal{F}^{\rm eq}_K\big)\begin{pmatrix}2u_Kv_K & -(u_K^2+v_K^2)\\ -(u_K^2+v_K^2) & 2u_Kv_K\end{pmatrix} - \frac{\hbar^2}{2mk_BT}\mathbf{K}\frac{2\epsilon_K}{E_k}\mathcal{N}^{\rm eq}_K\mathcal{F}^{\rm eq}_K\begin{pmatrix}u_K^2+v_K^2 & -2u_Kv_K\\ -2u_Kv_K & u_K^2+v_K^2\end{pmatrix}, \tag{C1}$$

where $\epsilon_K = \frac{\hbar^2K^2}{2m}+\Delta$.

It is useful to change to dimensionless variables and write $K_j = \frac{\sqrt{2mk_BT}}{\hbar}c_j$ for $(j=x,y,z)$, $b=\frac{\Delta}{k_BT}$, $\epsilon_c = c^2+b$, and $E_c=\sqrt{\epsilon_c^2-b^2}$, where $c_j$ is the $j$th component of the dimensionless wave vector. If we then use the results of Eq. (C1), we can write

$$[\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q})]\mathbf{q}\cdot\nabla_{\mathbf{K}}F^{\rm eq}_{11}(\mathbf{K}) = -(\mathbf{c}\cdot\mathbf{v}_s(\mathbf{q}))(\mathbf{c}\cdot\mathbf{q})\left[\frac{2u_c^2v_c^2}{E_c}\big(\mathcal{N}^{\rm eq}_c+\mathcal{F}^{\rm eq}_c\big)+\frac{2\epsilon_c}{E_c}\big(u_c^2+v_c^2\big)\mathcal{N}^{\rm eq}_c\mathcal{F}^{\rm eq}_c\right]. \tag{C2}$$

In order to get an estimate of the size of this term, let us consider, as an example, a condensate made of $^{23}$Na atoms at temperature $T=10^{-7}$ K, with $b=1$ and number density $\frac{N}{V}=10^{20}$ m$^{-3}$. Then $\big(\frac{2mk_BT}{h^2}\big)^{3/2}=1.2\times10^{17}$. Without loss of generality, let us assume that $\mathbf{q}$ lies along the $z$ axis so that $\mathbf{q}=q\hat{e}_z$, where $\hat{e}_z$ is a unit vector. Then $\mathbf{v}_s(\mathbf{q})=v_s(\mathbf{q})\hat{e}_z$. If we integrate over momentum we obtain

$$\int d\mathbf{K}\big[\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q})\big]\mathbf{q}\cdot\nabla_{\mathbf{K}}F^{\rm eq}_{11}(\mathbf{K}) = -qv_s(\mathbf{q})\left(\frac{2mk_BT}{h^2}\right)^{3/2}2\pi\int_0^\pi d\theta\sin(\theta)\cos^2(\theta)\int_0^\infty dcc^4\left[\frac{2u_c^2v_c^2}{E_c}(\mathcal{N}^{\rm eq}(\mathbf{c})+\mathcal{F}^{\rm eq}(\mathbf{c})) + \frac{\epsilon_c}{E_c}u_cv_c\big(u_c^2+v_c^2\big)\mathcal{N}^{\rm eq}(\mathbf{c})\mathcal{F}^{\rm eq}(\mathbf{c})\right] \approx -7.2qv_s(\mathbf{q})\left(\frac{2mk_BT}{h^2}\right)^{3/2}. \tag{C3}$$

We compare that to

$$\int d\mathbf{K}\big[\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q})\big]F^{\rm eq}_{11}(\mathbf{K}) = qv_s\frac{N}{V}\approx qv_s\frac{N_0}{V}+3.1qv_s(\mathbf{q})\left(\frac{2mk_BT}{h^2}\right)^{3/2}, \tag{C4}$$

where $N_0$ is the number of particles in the condensate and $N$ is the total particle number. Thus, Eq. (C4) is three orders of magnitude greater than Eq. (C3), so the contribution from Eq. (C3) can be neglected in Eq. (47).

We next consider the effect of the Hugenholtz-Pines relation on Eq. (63). For the case $\Phi(\mathbf{q}) = \delta\tilde{\Delta}(\mathbf{q})$, the bogolon kinetic equation matrix takes the form

$$-i\hbar\frac{\partial\delta\mathcal{N}(\mathbf{K},\mathbf{q})}{\partial t} = (E_1 - E_2)\delta\mathcal{N}(\mathbf{K},\mathbf{q}) + \hbar\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q})\mathcal{N}_K^{\rm eq} - \frac{\hbar^2}{2m}\mathbf{q}\cdot\mathbf{K}\frac{2u_K v_K}{E_K}(u_K - v_K)^2\left(\mathcal{N}_K^{\rm eq} + \mathcal{F}_K^{\rm eq}\right)\delta\tilde{\Delta}(\mathbf{q})$$
$$+ \frac{\hbar^2}{2mk_BT}\mathbf{K}\cdot\mathbf{q}\frac{2\epsilon_K}{E_k}(u_K - v_K)^2\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq}\delta\tilde{\Delta}(\mathbf{q}) + \hbar(\mathbf{K}\cdot\delta\mathbf{v}_s(\mathbf{q}))(\mathbf{q}\cdot\mathbf{K})\frac{\hbar^2}{2mk_BT}\frac{2\epsilon_K}{E_K}\mathcal{N}_K^{\rm eq}\mathcal{F}_K^{\rm eq} + \delta\mathcal{G}_{11}(1,2) \quad \text{(C5)}$$

and now includes the Hugenholtz-Pines equation.


[1] M. H. Anderson, J. R. Ensher, M. R. Mathews, C. E. Weiman, and E. A. Cornell, Science **269**, 198 (1995).
[2] K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
[3] N. N. Bogoliubov, J. Phys. USSR **11**, 23 (1947).
[4] D. A. W. Hutchinson, E. Zaremba, and A. Griffin, Phys. Rev. Lett. **78**, 1842 (1997).
[5] R. J. Dodd, M. Edwards, C. W. Clark, and K. Burnett, Phys. Rev. A **57**, R32 (1998).
[6] S. A. Morgan, J. Phys. B **33**, 3847 (2000).
[7] A. Griffin, T. Nikuni, and E. Zaremba, *Bose-Condensed Gases at Finite Temperatures* (Cambridge University Press, Cambridge, 2009).
[8] P. C. Hohenberg and P. C. Martin, Ann. Phys. (NY) **34**, 291 (1965).
[9] T. R. Kirkpatrick and J. R. Dorfman, J. Low Temp. Phys. **58**, 301 (1985); **58**, 399 (1985); **59**, 1 (1985).
[10] E. D. Gust and L. E. Reichl, J. Low Temp. Phys. **170**, 43 (2013).
[11] S. Peletminskii and A. Yatsenko, Sov. Phys. JETP **26**, 773 (1968).
[12] A. I. Akhiezer and S. V. Peletminskii, *Methods of Statistical Physics* (Pergamon, Oxford, UK, 1981).
[13] V. P. Galaiko, Sov. Phys. JETP **34**, 203 (1972).
[14] V. S. Shumeiko, Sov. Phys. JETP **36**, 330 (1973).
[15] L. E. Reichl, J. Stat. Phys. **23**, 83 (1980); **23**, 111 (1980).
[16] R. Walser, J. Williams, J. Cooper, and M. Holland, Phys. Rev. A **59**, 3878 (1999).
[17] E. D. Gust and L. E. Reichl, Phys. Scripta T **151**, 014057 (2012).
[18] E. D. Gust and L. E. Reichl, Phys. Rev. E **87**, 042109 (2013).
[19] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, 1970).
[20] P. Resibois, J. Stat. Phys. **2**, 21 (1970).
[21] E. D. Gust and L. E. Reichl, arXiv:1202.3418.
[22] L. E. Reichl, *A Modern Course in Statistical Physics*, 3rd ed. (Wiley-VCH, Weinheim, 2009).
[23] J. W. Kane and L. P. Kadanoff, J. Math. Phys. **6**, 1902 (1965).
[24] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).
[25] T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1406 (1959).
[26] S. J. Putterman, *Superfluid Hydrodynamics* (North-Holland, Amsterdam, 1974).
[27] L. Pitaevski and S. Stringari, *Bose-Einstein Condensation* (Clarendon Press, Oxford, 2003).
[28] E. Wigner, Phys. Rev. **40**, 749 (1932).