Time-energy trade-off in unambiguous-state-discrimination positive operator-valued measures

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Unambiguous-nonorthogonal-state discrimination has a fundamental importance in quantum information. Moreover, it can be used for entanglement distillation and secure communication. The discrimination is carried out by a positive operator-valued measure (POVM) generalized measurement, which is typically implemented by coupling the system to an ancilla. We find a trade-off between the needed energy resources and the evolution time needed to implement the POVM and express it in terms of an actionlike cost inequality. We find the realization that minimizes this actionlike cost and show that, in this case, the cost is determined by the maximal population transfer from the system to the ancilla. We demonstrate our findings in an example of a three-level system coupled to a laser.

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I. INTRODUCTION

The ability to discriminate and to separate nonorthogonal states can be used to construct simple secure quantum communication protocols [1,2] and entanglement distillation schemes [3,4]. Furthermore, investigation of this ability, commonly referred to as *unambiguous state discrimination* (USD), has led to a deeper understanding of what information can be extracted from a quantum system and at what probabilistic cost (Refs. [3,5–8], and references therein).

In an ideal USD, a system is prepared randomly in one of a set of prior known *nonorthogonal* states. The goal is to detect the state of the system with zero error probability. In this paper, we study the time-energy resources needed to implement a given USD task. Clearly, both energy and time are of prime importance when it comes to a realization in a specific physical system. For example, excessive external energy (e.g., laser light) eventually may lead to overheating, and/or recoil, whereas, long evolution times may expose the system to decoherence effects.

Nonstationary processes in quantum mechanics involve an intrinsic energy cost that is inversely proportional to the time duration of the process. The exact relation, however, depends on the details of the process. Anandan and Aharonov [9] established a relation between the energy variance of the Hamiltonian and the rotation time of a state in Hilbert space (e.g., spin-1/2 flips). A more general relation between the norm action of the Hamiltonian and the evolution operator of a process was derived by Lidar *et al.* [10]. A similar result holds for systems with absorption (loss of probability) [11]. In this paper, we derive the time-energy constraint for the fundamental quantum process of unambiguous-state discrimination.

Although USD of nonorthogonal states cannot be realized by standard (von Neumann) projective measurements (without dilating the Hilbert space), it can be implemented without errors by a generalized measurement known as the positive operator-valued measure (POVM) [5,12]. Unfortunately, POVM implementation of USD inherently involves some nonzero probability of obtaining an inconclusive result from which the input state cannot be inferred. The inconclusive result probability depends on the degree of nonorthogonality of the input states.

In this paper, we consider the unitary embedding POVM scheme [13–18] where the system subspace is coupled to ancillary levels by a unitary evolution and the state detection is carried out by standard von Neuman measurement in the system subspace only. Here, we quantify the minimal time-energy resources associated with the unitary evolution in this scheme. According to the Neumark dilation theorem [5], a POVM can also be implemented directly as a von Neumann measurement in a larger Hilbert space (without a premeasurement evolution). However, if we require that the conclusive measurement results appear exclusively in the original system subspace as in the unitary embedding scheme, a unitary evolution must be applied after the measurement. We show that the cost of the postmeasurement "information concentrating" unitary is exactly equal to the unitary evolution cost in the unitary embedding scheme described above. In the unitary embedding scheme, we find that the minimal time-energy cost is determined by the maximal population transfer from the system subspace to the ancilla subspace.

Note that previous studies about resources of unitary evolution, such as Ref. [10], cannot immediately be applied to the unitary embedding scheme studied here since the USD process provides only partial information on the unitary evolution operator. Finally, we comment that USD requires other resources besides energy. For example, the entanglement cost of a general rank-1 POVM embedding was studied in Ref. [19].

II. BACKGROUND AND PRELIMINARIES

A. USD POVM and lossy evolution

In a POVM measurement, each measurement result "i" is associated with a positive operator F_i. Given a density matrix ρ, the probability to get the result i is $p_i = \mathrm{tr}(\rho F_i)$. For a USD of N nonorthogonal states in a Hilbert space of dimension N, the $\{F_i\}_{i=1}^{N}$ rank-1 operators are constructed from the bi-orthogonal basis [20]. An additional operator (that is typically not rank 1) is defined as $F_{N+1} = I - \sum_{n=1}^{N} F_n$, and it describes the inconclusive result. Huttner *et al.* [13]

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first suggested and experimentally demonstrated that USD POVM can be implemented by a lossy evolution. Recently, the equivalence between USD POVM and lossy evolution was further studied [20]. An N-level state evolving under a lossy evolution operator K satisfies: $|\psi_{\text{final}}\rangle = K|\psi_{\text{initial}}\rangle$ where $K \in C^{N\times N}$ is not a unitary operator. Such an evolution does not conserve the angle between states. This fact can be exploited in order to transform the nonorthogonal states at the input to orthogonal states at the output. Once the states are orthogonal, they can be discriminated without errors using a regular projective measurement. The inherent losses in the system make the detection probability smaller than 1. This probability loss is mathematically equivalent to the inconclusive result in the POVM formalism. In fact, any USD POVM $\{F_i\}_{i=1}^{N+1}$ can be associated with a lossy evolution operator and vice versa [20]. Apart from the practical value of this equivalence for USD realization, the lossy evolution approach has theoretical merits as well. In particular, it was shown in Ref. [4] that the singular values of the lossy evolution operator capture the essence of USD and can be used to reveal interesting insights into multiple USDs.

B. Embedding of a lossy evolution

There are two different ways of implementing a lossy evolution as defined above. The first is to find a system described by some effective non-Hermitian Hamiltonian [21] that includes losses (e.g., optics with non-negligible absorption). The other way is to consider the evolution of a closed unitary system and to measure the evolution outcomes only in a subsystem. Assume that, initially, the total probability of finding a particle in a subsystem is unity. After a unitary evolution of the whole system is completed, the total probability in the subsystem is typically less than 1. From the point of view of the subsystem, the evolution is lossy. We refer to this second implementation as unitary embedding. The resources needed to implement the first scheme were studied in Ref. [11]. In this paper, we focus on the embedding scheme. Let K be a lossy evolution operator that implements some desired USD POVM, and let $|\psi\rangle \in \mathbb{C}^N$ be the input state. The embedding scheme we use is as follows:

$$U\begin{pmatrix}|\psi\rangle\\0\end{pmatrix} = \left(\begin{array}{c|c}K & B\\ \hline C & D\end{array}\right)\begin{pmatrix}|\psi\rangle\\0\end{pmatrix} = \begin{pmatrix}K|\psi\rangle\\C|\psi\rangle\end{pmatrix}\begin{matrix}\}\text{ System}\\\}\text{ Ancilla}\end{matrix}. \quad (1)$$

The conclusive discrimination results are contained in the first N levels so, in principle, the ancilla levels need not be measured for USD. The ancilla levels may be, for example, some additional atomic levels (see the example below and Ref. [22]) or auxiliary waveguides in optics. There are many degrees of freedom in choosing B, C, and D. The USD POVM at hand determines K up to a multiplication by a unitary matrix from the left. Given only K, our goal is to find the choice that minimizes the time-energy resources defined in the next section.

C. Resources and norm action

Let U be some unitary evolution operator generated by a Hamiltonian H so that $i\frac{d}{dt}U = HU$. The construction of H requires some physical resources, such as a magnetic field or coupling to laser radiation. To quantify the resources, the Hamiltonian must be mapped to a scalar. Here, we use unitarily invariant matrix norms for this purpose for the following reasons. First, it is natural to demand that a measure of resources will satisfy the defining properties of a norm [23]. Second, unitary invariance ensures that the resources are independent of the basis in which the Hamiltonian is expressed. Third, we will use an important result concerning unitarily invariant norms of the Hamiltonian [10]. Therefore, to quantify the time-energy cost, we will use the norm action that is defined as the time integral over the Hamiltonian norm. From the result of Ref. [10], it follows that the norm action satisfies

$$\int_0^T \|H(t)\|dt \geqslant \|\ln U(T)\|, \quad (2)$$

where, in the ln, the angles belong to the branch $(-\pi,\pi]$ and $\|\cdot\|$ may refer to any unitarily invariant matrix norm. When the Hamiltonian is time independent, the norm action integral on the left-hand side (LHS) of (2) reduces to time $\times$ energy, and the inequality becomes an equality. In this case, $U = e^{-iH_0T}$, where H_0 is the generating Hamiltonian and T is the duration of the evolution. At this point, it is still not clear what is the embedding U that yields a minimal norm action, but it is clear that this embedding must be generated by some time-independent Hamiltonian H_0. Actionlike quantities have been used before to analyze quantum evolution [24]. Unless stated otherwise, in this paper, we will use the spectral norm [23]. It has a clear physical interpretation [11,25], and it leads to compact and comprehensible results. The spectral norm of a matrix $A \in \mathbb{C}^{N\times N}$ is the largest singular value of A. The singular values s_i of A are as follows: $\{s_i\} = \sqrt{\text{eigenvalues }[A^\dagger A]}$, and therefore, $\|A\| = s_{\max} = \max(\sqrt{\text{eigenvalues }[A^\dagger A]})$.

The fact that lossy systems cannot amplify the amplitude of any state manifests itself in the condition $\|K\| \leqslant 1$ [11]. If $\|K\| = 1$, the system is called "marginally passive" [11]. Finally, when applied to *(Hermitian)* Hamiltonians, the *spectral norm is equal to the largest absolute-valued energy (eigenvalue) of the Hamiltonian*. This gives a clear time $\times$ energy interpretation to the norm action integral on the LHS of (2).

III. TIME-ENERGY COST OF UNITARY EMBEDDING

Our strategy of finding the USD-generating Hamiltonian with the minimal possible norm action is the following. We start by finding an explicit expression for the norm action associated with a specific embedding choice of a general USD task. Next, we show that this choice is the norm action minimizer among all possible unitary embeddings implementing the same USD task.

Until stated otherwise, we will assume that the number of ancilla levels and the number of system levels is equal. This is enough to allow the embedding of the most general lossy evolution operator. Using the polar decomposition of blocks K and D, any unitary U can be written as a product of two unitaries V and W, where V is a block-diagonal unitary and W is a positive diagonal block unitary,

$$U = VW, \quad (3)$$

$$V = \begin{pmatrix} u_s & 0 \\ 0 & u_a \end{pmatrix}, \quad W = \begin{pmatrix} \mathcal{K} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{4}$$

u_s and u_a are unitaries which operate on the system and ancilla space, respectively. $\mathcal{K}$ and $\mathcal{D}$ are positive matrices. For $\mathcal{K},\mathcal{D} > 0$, the unitarity constraints and the singular value decomposition of each block lead to the following general form:

$$W = \left(\begin{array}{c|c} u_{\mathcal{K}} \cos\,\Theta u_{\mathcal{K}}^{\dagger} & -i u_{\mathcal{K}} \sin\,\Theta u_{\mathcal{D}}^{\dagger} \\ \hline -i u_{\mathcal{D}} \sin\,\Theta u_{\mathcal{K}}^{\dagger} & u_{\mathcal{D}} \cos\,\Theta u_{\mathcal{D}}^{\dagger} \end{array} \right), \tag{5}$$

where Θ is a positive diagonal matrix satisfying

$$0 \leqslant \Theta_{ii} \leqslant \pi/2. \tag{6}$$

The $u_{\mathcal{K}}$ and $u_{\mathcal{D}}$ are unitaries whose column vectors are the orthogonal eigenstates of the positive $\mathcal{K}$'s and $\mathcal{D}$'s, respectively. Each of the blocks is now written in terms of its singular vectors, and therefore, the diagonal matrices $\cos\,\Theta$ and $\sin\,\Theta$ contain the singular values of blocks $\mathcal{K},\mathcal{D}$ and of blocks $\mathcal{B},\mathcal{C}$, respectively (or, alternatively, K, D and B, C). The time-independent Hamiltonian that generates W is given by

$$H_W = H_{\text{opt}} = \frac{1}{T} \left(\begin{array}{c|c} 0 & u_{\mathcal{K}} \Theta u_{D}^{\dagger} \\ \hline u_{\mathcal{D}} \Theta u_{K}^{\dagger} & 0 \end{array} \right). \tag{7}$$

We write H_{opt} since, later on, we show that H_W is the most efficient Hamiltonian that implements the desired USD characterized by K. A similar Hamiltonian has been used before in Ref. [18] for probabilistic evolution and for POVM embedding in Refs. [15,16]. Here, however, we focus on the resources of embedding. Furthermore, in our scheme, it is critical that $0 \leqslant \Theta_{ii} \leqslant \pi/2$ so that $\mathcal{K}$ and $\mathcal{D}$ are positive. As will be explained later, this is necessary for optimality. By inspecting $H^{\dagger}H$, it is easy to verify that

$$\|H_{\text{opt}}\|T = \max(\Theta_{ii}) = \arcsin(\|B\|). \tag{8}$$

Or, in terms of the smallest singular value of K (or $\mathcal{K}$), which is directly determined by the USD POVM,

$$\|H_{\text{opt}}\|T = \arcsin\left(\sqrt{1 - s_{\text{min}}^2}\right). \tag{9}$$

The quantity $1 - s_{\text{min}}^2$ is the maximal population transfer from the system to the ancilla.

A. W requires the lowest possible Hamiltonian resources

The goal of this section is to show that W requires the minimal norm action for the given USD task. Let us try to better understand the relation among the Hamiltonian H, U, and $\mathcal{K}$. An input state $|\psi_{\text{in}}\rangle$ is transformed by W according to (1). W rotates this vector in Hilbert space. From the overlap of the initial and final states, we can obtain the rotation angle in Hilbert space,

$$\cos\,\Omega = |\langle\psi_{\text{in}}|\psi_{\text{out}}\rangle| = |\langle\psi_{\text{in}}|\mathcal{K}|\psi_{\text{in}}\rangle|. \tag{10}$$

Since $\mathcal{K}$ is positive, the maximal angle is obtained for the singular vector $|\psi_{\text{min}}\rangle$ associated with the minimal singular vector. Using $|\psi_{\text{in}}\rangle = |\psi_{\text{min}}\rangle$ in (10), we get $\cos\,\Omega_{\max,\mathcal{K}} = s_{\text{min}}$, or

$$\Omega_{\max,\mathcal{K}} = \arcsin\sqrt{1 - s_{\text{min}}^2}, \tag{11}$$

which is exactly equal to (9). Furthermore, from the Hamiltonian variance [9,25], one can show that

$$\Omega \leqslant \int \left|\frac{d\Omega}{dt}\right| dt \leqslant \int \|H\| dt. \tag{12}$$

Applying this to $\Omega_{\max,\mathcal{K}}$, we get

$$\int \|H\| dt \geqslant \arcsin\sqrt{1 - s_{\text{min}}^2}. \tag{13}$$

However, for $\mathcal{K},\mathcal{D} > 0$ and the time-independent Hamiltonian, we have already shown that there is an equality (9). This provides a very intuitive picture of our claim. The needed resources in this embedding are determined by the state that experiences the largest population transfer to the ancilla.

To achieve the goal of the section, we will show that, when applying an extra block diagonal unitary V, $|\psi_{\text{min}}\rangle$ leads to a larger rotation in Hilbert space compared to the previous case (and, consequently, more norm action resources are needed). We repeat (10) but this time add a unitary u_s that operates on the system subspace and obtain

$$\cos\,\Omega_{\text{new}} = |\langle\psi_{\text{min}}|u_s\mathcal{K}|\psi_{\text{min}}\rangle| \leqslant |\langle\psi_{\text{min}}|u_s|\psi_{\text{min}}\rangle| s_{\text{min}}, \tag{14}$$

which is always smaller than s_{min} and, therefore, $\Omega_{\text{new}} > \Omega_{\max,\mathcal{K}}$. Using inequality (12), once again, we get $\int \|H_{\text{new}}\| dt \geqslant \Omega_{\text{new}} \geqslant \arcsin\sqrt{1 - s_{\text{min}}^2}$ or stated in a different way using (9),

$$\int \|H_{\text{new}}\| dt \geqslant H_{\text{opt}} T. \tag{15}$$

Repeating this for states that populate only the ancilla and using the fact that blocks $\mathcal{K}$ and $\mathcal{D}$ have the same s_{min}, we get that any block-diagonal rotation of the form V (3) only increases the Hamiltonian resources with respect to the H_{opt} that generates W. The same claim can be proved for the Hilbert-Schmidt norm ($\|A\|_{\text{HS}} = \sqrt{\sum_{ij} |A_{ij}|^2} = \sqrt{\text{tr}(A^{\dagger}A)} = \sqrt{\sum s_i^2}$). The optimal Hamiltonian is the same, but in expression (9) for the norm action, the right-hand side is replaced by $\sqrt{2\sum_{i=1}^{N} (\arcsin\sqrt{1 - s_i^2})^2}$.

B. The dimension of the ancilla

In principle, the ancilla dimension (the number of levels) N_a does not have to be equal to the dimension of the system N_s. Let us start with the $N_a = N_s$ case and see that N_a can be changed without any effect on the norm action as long as it still implements the same $\mathcal{K}$. The off-diagonal blocks of the Hamiltonian (7) have dimensions $N_a \times N_a$. However, if M singular values of K are equal to 1, the upper right block will contain M zero columns, and the bottom left block will contain M zero rows. Cropping out the zeros, rows, and columns, the new dimension of the reduced unitary is $[2N_s - M] \times [2N_s - M]$, which means that only $N_s - M$ ancilla levels are needed for the embedding. The converse of this claim is that, if $N_a = N_s - M$, then there are at least M singular values of K that are equal to 1. Note that, even if one singular value is equal to 1, the lossy evolution operator K must be marginally passive. An interesting case is $N_a = 1$ where only one singular value is *different* from 1. The

inconclusive result POVM operator, in this case, is a rank-1 matrix. Consequently, when an inconclusive result is obtained, the state of the system contains zero information on the input state (see Sec. III B of Ref. [4]). The $N_a > N_s$ case can be analyzed by replacing the states in $u_{\mathcal{D}}$ in (7) by orthogonal vectors of dimension $N_a > N_s$ (so that $u_{\mathcal{D}}$ has N_s row and N_a columns). This just adds zeros to the singular values of H_{opt}. Hence, extending the ancilla dimension in this way does not change the norm action with respect to the optimal $N_a = N_s$.

C. Relation to Neumark dilation

Using tensor product notation for the $N_a = N_s$ case, the embedding scheme can be written as

$$p_k = \text{tr}\{U(\rho_{\text{in}} \otimes \rho_{\uparrow})U^{\dagger}(\pi_k \otimes \rho_{\uparrow})\}, \tag{16}$$

$$\rho_{\uparrow} = |\uparrow\rangle\langle\uparrow| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{17}$$

where π_k's are von Neumann projection operators in the system subspace. Equation (16) can also be written as

$$p_k = \text{tr}\{(\rho_{\text{in}} \otimes \rho_{\uparrow})\Pi_k\}, \tag{18}$$

$$\Pi_k = U^{\dagger}(\pi_k \otimes \rho_{\uparrow})U, \tag{19}$$

where Π_k's are projection operators in the total system-ancilla space. These extended projectors constitute a realization of Neumark dilation. Despite this intimate relation between Neumark dilation and lossy evolution, they differ significantly when it comes to implementation and to the nature of the measurement needed to extract the conclusive information. In the embedding scheme, after the measurement, all the conclusive results are described by the *conclusive density matrix*,

$$\rho_c^{\text{Lossy}} = \sum_{k=1}^{N_s} p_k \pi_k \otimes \rho_{\uparrow}, \tag{20}$$

so only the system levels need to be measured. In contrast, in the Neumark scheme, the conclusive density matrix is

$$\rho_c^{\text{Neumark}} = \sum_{k=1}^{N_s} p_k \Pi_k, \tag{21}$$

and therefore, typically, $N_s + N_a$ levels must be measured. The fact that the conclusive results are contained in a smaller space is very useful since USD processes can be analyzed by studying the properties of $K_{N_s \times N_s}$ only [4].

In contrast to the minimal norm action found above (9), it appears that the Neumark scheme requires no norm action since a projective measurement Π_k is immediately carried out on the input states without any prior evolution. The reason for the resource's discordance stems from the fact that the information is encoded differently in the two schemes as explained above. In the lossy evolution, the conclusive results are contained in the N system levels, whereas, in the Neumark approach, the same information is contained in $N_s + N_a$ levels. To concentrate the conclusive Neumark detections to N levels (as in the lossy evolution scheme), another unitary must be applied to the system *after the measurement has been completed*. That is, we want to transform Π_k in the Neumark conclusive density matrix to $\pi_k \otimes \rho_{\uparrow}$. This transformation is exactly U. Hence, the minimal cost of concentrating the conclusive Neumark information to N levels is exactly equal to the minimal cost of the unitary embedding scheme (9).

IV. AN EXAMPLE—USD IN AN ATOMIC SYSTEM COUPLED TO A LASER

In this section, we wish to explicitly show a USD realization where a laser field is used to select the normalized nonorthogonal input states $|\alpha_{\pm}\rangle \in \mathbb{C}^2$ that the system can discriminate. Our goal is to relate the laser and system parameters to the given input vectors $|\alpha_{\pm}\rangle$ and to evaluate the needed resources in terms of physical quantities (e.g., the laser power). Consider a three-level atomic system in a time-dependent external electric field $\varepsilon(t)$ (a laser). The first and second levels are dipole coupled to the third level but not coupled to each other. The Hamiltonian is as follows:

$$H_0 = \begin{pmatrix} E_1 & 0 & d_1\varepsilon(t) \\ 0 & E_2 & d_2\varepsilon(t) \\ d_1^*\varepsilon(t) & d_2^*\varepsilon(t) & E_3 \end{pmatrix}, \tag{22}$$

where the d_i are the dipole coupling coefficients. Setting the time-dependent electric field to be $\varepsilon(t) = a_1 \cos[(E_3 - E_1)t + \varphi_1] + a_2 \cos[(E_3 - E_2)t + \varphi_2]$ and applying the rotating-wave approximation (RWA), we get

$$H_{\text{RWA}} = \begin{pmatrix} 0 & 0 & A_1 \\ 0 & 0 & A_2 \\ A_1^* & A_2^* & 0 \end{pmatrix}, \tag{23}$$

where $A_i = \frac{d_i}{2} a_i e^{-i\varphi_i}$. H_{RWA} has the form of H_{opt} (7), so the final result will be expressed in terms of the equality (9) rather than (13). Levels one and two will constitute the system, whereas, the third level will be used as an ancilla. Note that the rotated state is related to the actual state via $|\psi\rangle = \exp[-i\,\text{diag}\{E_1, E_2, E_3\}]|\psi_{\text{RWA}}\rangle$. However, this rotation has a block-diagonal structure with respect to the system and the ancilla, and therefore, this transformation will not effect the orthogonality of the final states in the system subspace. Notice that the given input states $|\alpha_{\pm}\rangle$ do not populate the ancilla level.

The relation between the singular values of K and the two-state USD was studied in Ref. [4]. The singular values and the angle between $|\alpha_{\pm}\rangle$ must satisfy $\tan\frac{\phi}{2} = \frac{s_{\min}}{s_{\max}}$ where $\cos\phi = |\langle\alpha_-|\alpha_+\rangle|$. Since $s_{\max} = 1$ in this problem, we use the result of Ref. [4] and get $s_{\min} = \sqrt{\frac{1-|\langle\alpha_+|\alpha_-\rangle|}{1+|\langle\alpha_+|\alpha_-\rangle|}}$. The weighted laser amplitudes A_1, A_2 are given by the first and second components of the vector $\frac{|\alpha_+\rangle+|\alpha_-\rangle}{2s_{\min}}$ in the standard basis. After calculating the spectral norm of H_{RWA}, we use (9) and get

$$T\sqrt{|A_1|^2 + |A_2|^2} = \arcsin\sqrt{\frac{2|\langle\alpha_+|\alpha_-\rangle|}{1 + |\langle\alpha_+|\alpha_-\rangle|}}, \tag{24}$$

where $|A_1|^2 + |A_2|^2$ is the optical power weighted by the dipole coefficients (in larger systems, the Hilbert-Schmidt norm should be used to keep the optical power interpretation of the norm). This trade-off relation between time and effective optical power ($|A_1|^2 + |A_2|^2$) demonstrates the main point of this article: The time × energy cost of implementing the discrimination grows when the overlap of the input states is larger.

Although more complicated coupling can be studied along the same lines, here we choose the simplest couplings that provide general two-state discrimination. Furthermore, this choice requires the minimal possible resources needed for two-state discrimination in the unitary embedding scheme.

V. CONCLUSION

In this article, we have shown that the unitary embedding of a USD POVM has an intrinsic time × energy cost which depends on the degree of nonorthogonality of the input states. We have found that the lowest possible embedding cost is obtained when the diagonal blocks of the unitary are positive. Physically, this optimal cost is determined by the maximal population transfer from the system to the ancilla. The optimal time-energy cost depends only on the singular values associated with the desired USD and not on the size of the ancilla. As shown in the example studied above, this cost has a clear physical significance, and we expect the results obtained here to become useful in any specific USD realization.

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