Criterion for remote clock synchronization with Heisenberg-scaling accuracy

Yong-Liang Zhang,[1] Yu-Ran Zhang,[2] Liang-Zhu Mu,[1,*] and Heng Fan[2,†]
[1]*School of Physics, Peking University, Beijing 100871, China*
[2]*Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China*
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We propose a quantum method to judge whether two spatially separated clocks have been synchronized within a specific accuracy σ. If the measurement result of the experiment is clearly a nonzero value, the time difference between two clocks is smaller than σ; otherwise, the difference is beyond σ. Upon sharing the $2N$-qubit bipartite maximally entangled state in this scheme, the accuracy of judgment can be enhanced to $\sigma \sim \pi/[\omega(N+1)]$. This criterion is consistent with Heisenberg scaling, which can be considered as the beating standard quantum limit; moreover, the unbiased estimation condition is not necessary. In particular, we demonstrate that this scheme is still feasible when suffering the loss of qubits.

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I. INTRODUCTION

Clock synchronization with high precision is at the heart of many modern technologies and researches, such as global positioning systems (GPSs), long baseline interferometry, synchronous data transfer, laser interferometer gravitational wave observation (LIGO), tests of theory of general relativity, and distributed computation. There are two standard methods for synchronizing two spatially separated clocks in the frame of the special theory of relativity. One is based on Einstein's synchronization procedure, which uses an operational line-of-sight exchange of light pulses between two spatially separated clocks [1]. The other method is based on the internal time evolution of quantum systems, such as Eddington's infinitesimally slow clock transport [2].

The quantum clock synchronization method based on the strength of sharing prior entanglement has also been proposed [3] and has been generalized to several multiparty clock synchronization protocols [4–6]. Independent of the participants' knowledge of their relative locations or of the properties of the intervening media, these procedures utilize the instantaneity of a wave function collapsing after the measurement performed on the shared entangled states. These protocols are tantamount to the Eddington protocol as the process of distributing entanglement is adiabatic. An experiment focusing on the quantum clock synchronization implementation has also been reported [7]. In addition, the progressive techniques of multiphoton entanglement [8,9] are basic and promising for the realization of quantum clock synchronization.

Applying the technique of quantum entanglement-enhanced parameter estimation in the quantum clock synchronization is a direct and natural idea to enhance the accuracy of synchronization. It has been shown that standard quantum limit $1/\sqrt{N}$, where N is the number of particles used in the measurement, can be overcome using coherent light with squeezed vacuum [10]. In the study of quantum metrology, quantum Fisher information theory and quantum Cramér-Rao bound based on the statistical distance of states have been proposed and developed in [11–16]. The NOON state has been demonstrated to be able to achieve a phase sensitivity that saturates the Heisenberg limit $1/N$ [17]. Some related strategies have been proposed to attain high precision in the quantum metrology framework [18–20], and many experiments have also been performed on this topic [21–26]. A high-efficiency quantum ticking qubits handshake protocol is presented that allows two remote clocks to be synchronized independent of the message transport time [27]; a similar protocol has been proposed to overcome the standard quantum limit [28].

In this article we relate the quantum clock synchronization protocol to the problem of estimating an unknown parameter. We investigate the performance of the bipartite maximally entangled spin-zero singlet in the scheme of two-clock synchronization and offer a standard to judge whether two spatially separated clocks have been synchronized to a specific accuracy. Our scheme may offer the following advantages: this criterion is practical and consistent with Heisenberg scaling accuracy ($\sim 1/N$), and this scheme does not rely on the unbiased estimation condition, which is a fundamental hypothesis in quantum Fisher information theory. We also demonstrate that this protocol remains workable against loss of qubits, which is in sharp contrast to other known protocols. Additionally, our scheme can avoid the 2π periodical problem in estimating an unknown phase parameter.

This paper is organized as follows. In Sec. II we present a general framework of Jozsa's quantum clock synchronization scheme. In Sec. III we discuss two synchronization schemes with Bell state and GHZ state and estimate their accuracy. The GHZ state is the superposition of all spin up and all spin down which involves at least three qubits. In Sec. IV we investigate the scheme by using bipartite maximally entangled spin-zero singlet in both ideal case and qubits loss case. Finally, a conclusion is given in Sec. V.

II. GENERAL FRAMEWORK OF QUANTUM CLOCK SYNCHRONIZATION

In this section we present a review of Jozsa's quantum clock synchronization scheme [3] and discuss the sensitivity of measurements via the Fisher information and Cramér-Rao inequality.

*muliangzhu@pku.edu.cn
†hfan@iphy.ac.cn

Suppose two spatially separated parties, Alice and Bob, resting on the same reference frame, both possess high-precision clocks, such as Cs atomic clocks, running at precisely the same rate. These clocks do not agree on a common time at the same readout, for example, 12 o'clock. The difference of time t_D between Alice and Bob's clocks can be expressed as

$$t_D = t_B - t_A|_{\text{Alice and Bob have the same readouts}}.$$

In this scheme, to eliminate the relative phase that may emerge during the transport of qubits to the spatially separated locations, the entangled states, with a normalized form $|\psi\rangle = \sum_{i,j} c_{ij}|i_A\rangle|j_B\rangle$, should be distributed to Alice and Bob adiabatically. $|i_A\rangle$ and $|j_B\rangle$ are the orthonormal basis of measurements, which satisfy the completeness $\sum_{i,j}|i_A\rangle\langle i_A|\otimes|j_B\rangle\langle j_B| = \mathbb{I}$, where $\mathbb{I}$ denotes the identity. Suppose that the unitary time evolution is $U_{AB}(t) = e^{-i\hat{H}_A t/\hbar}\otimes e^{-i\hat{H}_B t/\hbar}$ and the entangled states Alice and Bob select should be merely changed with an overall unobservable phase under this unitary evolution. After the entanglement distribution, Alice and Bob perform measurements on all of their qubits "simultaneously" when their clocks point to the same readout. Suppose Alice performs the measurement before Bob ($t_D > 0$) and obtains a result $|i_A\rangle$ with probability $P(i_A) = \sum_j |c_{ij}|^2$; then, the collapsed state evolves as

$$e^{-\mathrm{i}\hat{H}_B t_D/\hbar}\sum_j c_{ij}|j_B\rangle = \sum_{k,j} c_{ij}U_{kj}|k_B\rangle, \tag{1}$$

where $U_{kj} = \langle k_B|e^{-i\hat{H}_B t_D/\hbar}|j_B\rangle$. Then, Bob will do the measurement on his qubits and obtain a result $|k_B\rangle$ with probability $P(k_B|i_A) = |\sum_j c_{ij}U_{kj}|^2/P(i_A)$. Thus, Fisher information $\mathcal{F}_{t_D} = \sum_\xi P(\xi|t_D)[\partial_{t_D}\ln P(\xi|t_D)]^2$ and Cramér-Rao bound [11–16] $\delta t_D \geqslant 1/(\nu\mathcal{F}_{t_D})^{1/2}$, with ξ the readout of the measurement and ν the number of repetitions of experiment, can be utilized in this clock synchronization situation when the estimation is asymptotically unbiased.

Via comparing the ratio of observed measurement outcomes with the probability distribution that is determined by the parameter t_D, two issues may prevent one from estimating t_D with a high precision. First, the number of experimental trials is finite; thus, the ratio of measured outcomes may deviate from the distribution. The Cramér-Rao bound can be reached only when the number ν is sufficiently large and the estimation is unbiased. Second, a one-to-one mapping $P(\xi|t_D)\leftrightarrow t_D$ between the probability distributions and parameter is essential.

III. QUANTUM CLOCK SYNCHRONIZATION WITH BELL STATE AND GHZ STATE

In this article we assume that Alice and Bob perform measurements $\hat{X} = |\widetilde{0}\rangle\langle\widetilde{0}| - |\widetilde{1}\rangle\langle\widetilde{1}|$ on all of their own qubits simultaneously when their own clock points to a specific value, where $|\widetilde{0}\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|\widetilde{1}\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$, and $|0\rangle,|1\rangle$ are the orthogonal eigenstates of each qubit. Each qubit has the identical Hamiltonian $\hat{H}_A = \hat{H}_B = \hat{H}$ satisfying $\hat{H}|0\rangle = E_0|0\rangle$, $\hat{H}|1\rangle = E_1|1\rangle$, and $\omega = (E_1 - E_0)/\hbar > 0$. One constructive approach to implement these ticking qubits in an experiment is to place some spin-1/2 particles into the magnetic fields with the same field strength.

We suppose that Alice and Bob share $N\nu$ pairs of entangled qubits with form $|\Psi^{(-)}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, which is invariant under the unitary evolution $e^{-i\hat{H}_A t/\hbar}\otimes e^{-i\hat{H}_B t/\hbar}$, as discussed in Ref. [3]. Alice and Bob perform measurements expressed as an operator $\hat{M}_2 = \hat{X}(t_A)\otimes\hat{X}(t_B)$ in the Heisenberg picture, which can also be described as a set of positive operator valued measurements (POVMs) with element $\hat{E}(\xi_2) = |\widetilde{x}\rangle\langle\widetilde{x}|\otimes|\widetilde{y}\rangle\langle\widetilde{y}|$ performed on the state $\hat{\rho}(t_D)$. Here $\xi_2 = (x,y)$ is the readout for measuring $|\widetilde{x}\rangle_A|\widetilde{y}\rangle_B$, $x,y = 0,1$ and $\hat{\rho}(t_D)$ is the density matrix of the pure state $e^{-i\hat{H}_A t/\hbar}\otimes e^{-i\hat{H}_B(t+t_D)/\hbar}|\Psi^{(-)}\rangle$. Then, the probability distribution of each measurement result is

$$\begin{aligned}P(\xi_2|t_D) &= \mathrm{Tr}[\hat{E}(\xi_2)\hat{\rho}(t_D)]\\ &= \frac{1}{2}\left(\delta_{x,(y+1)\mathrm{mod}2}\cos^2\frac{\beta}{2} + \delta_{x,y}\sin^2\frac{\beta}{2}\right),\end{aligned} \tag{2}$$

where $\beta = \omega|t_D|$, and $\delta_{x,y}$ is Kronecker's delta. The Fisher information is calculated as $\mathcal{F}_{t_D} = \omega^2$.

In addition, the average of the measurement operator for the state $|\Psi^{(-)}\rangle$ can be calculated as $\langle\hat{M}_2\rangle = \sum_{\xi_2} g_2(\xi_2)P(\xi_2|t_D) = -\cos\beta$, where $g_2(\xi_2) = (-1)^{x+y}$. If $|t_D| < \pi/\omega$ holds, we can obtain the difference of time $|t_D^{\mathrm{est}}|$ from the observed expectation value $\overline{M_2} = \frac{1}{N\nu}\sum_{i=1}^{N\nu} g_2(\xi_2^i)$ after local measurements and classical communication, where ξ_2^i denotes the result of the measurement on the ith pair of qubit. The sign of t_D^{est} can be determined by the outcomes of Alice's and Bob's measurements because the one who first performed $N\nu$ measurements would obtain dual results with probabilities $P(|\widetilde{0}\rangle) = P(|\widetilde{1}\rangle) = 1/2$. Furthermore, the uncertainty of the estimation of t_D could reach the Cramér-Rao bound $\delta t_D^{\mathrm{est}} = 1/(\sqrt{N\nu\mathcal{F}_{t_D}}) = 1/(\omega\sqrt{N\nu})$ which is the standard quantum limit. Therefore, in this scheme, Alice and Bob can synchronize their clocks with accuracy $1/(\omega\sqrt{N\nu})$.

Some studies have also focused on using multiphoton entanglement strategies and employing NOON states (or GHZ states) to enhance the precision of parameter estimation [17–20]. Although GHZ states are not energy eigenstates and should not be used in quantum clock synchronization [6], N pairs of qubits can be arranged as a GHZ-type state that indeed is an energy eigenstate:

$$|\Psi_{2N}\rangle = \left(|0\rangle_A^{\otimes N}|1\rangle_B^{\otimes N} + |1\rangle_A^{\otimes N}|0\rangle_B^{\otimes N}\right)/\sqrt{2}, \tag{3}$$

where Alice and Bob each own N qubits. It is easy to verify that this type of state is merely changed with an overall phase under the unitary evolution $(e^{-i\hat{H}t/\hbar})_A^{\otimes N}\otimes(e^{-i\hat{H}t/\hbar})_B^{\otimes N}$ and is suitable for quantum clock synchronization. The probability distribution in this protocol takes the form $P(\xi_{2N}|t_D) = [1 + g_{2N}(\xi_{2N})\cos(N\beta)]/2^{2N}$, where $g_{2N}(\xi_{2N}) = (-1)^{\sum_k(x_k+y_k)}$ and the readout $\xi_{2N} = (x_1,\ldots,x_N,y_1,\ldots,y_N)$, $x_i,y_j = 0,1$; and $\hat{M}_{2N} = \hat{X}(t_A)^{\otimes N}\otimes\hat{X}(t_B)^{\otimes N}$ is the measurement operator in the Heisenberg picture. The average of the operator is calculated as $\cos(N\beta)$, and the Fisher information $\mathcal{F}_{t_D}$ is $N^2\omega^2$. Considering that the probability distributions and expectation value are all functions with periodicity $2\pi/N$, one can unambiguously obtain $|t_D^{\mathrm{est}}|$ from the observed expectation value after local measurements and classical communication only when the condition $|t_D| < \pi/(\omega N)$ is satisfied. The sign of t_D^{est} can also be determined from the outcomes of Alice's and Bob's measurements because the one who first performed $N\nu$ times measurement would obtain the probability $P(|\widetilde{x}_1 \cdots \widetilde{x}_N\rangle) = 1/2^N$ for all his/her qubits and $P(|\widetilde{x}_i\rangle) = 1/2$ for an arbitrary one. When the number of trials ν is sufficiently large, the uncertainty can reach the Cramér-Rao bound $\delta t_D^{\text{est}} = 1/\sqrt{\nu \mathcal{F}_{t_D}} = 1/(\omega N \sqrt{\nu})$, which is a Heisenberg scaling accuracy.

Despite this optimal local distinguishability in Hilbert space, this GHZ-type state is inappropriate to obtain more advantageous information from any values of the parameter t_D in this single procedure because the condition $|t_D| < \pi/(\omega N)$ is required [29].

IV. QUANTUM CLOCK SYNCHRONIZATION WITH BIPARTITE MAXIMALLY ENTANGLED STATES

We next consider a scheme that exploits different entanglement resources, which is the core content of our article. The bipartite maximally entangled spin-zero singlet has been proposed as a resource for quantum-enhanced metrology [30], with the following form:

$$|\chi\rangle = \frac{1}{\sqrt{2J+1}} \sum_{M=-J}^{J} (-1)^{J-M} |J,M\rangle_{z,A} |J,-M\rangle_{z,B}, \tag{4}$$

where $J = N/2$, and $|J,M\rangle_z$ is a completely symmetric normalized state (Dicke state) with $(J-M)$ qubits being $|0\rangle$ and $(J+M)$ qubits being $|1\rangle$. There is an explicit mapping between the two-symmetric entangled state and a direct product of N maximally entangled states, which is presented in [31]; then, one obtains

$$\begin{aligned}|\chi\rangle &= \frac{2^{N/2}}{\sqrt{N+1}} \mathbb{I}^{\otimes N} \otimes \hat{S} |\Psi^{(-)}\rangle^{\otimes N} \\ &= \frac{2^{N/2}}{N!\sqrt{N+1}} \sum_{\vec{\sigma}} |\Psi^{(-)}\rangle_{A_1 B_{\sigma_1}} \cdots |\Psi^{(-)}\rangle_{A_N B_{\sigma_N}},\end{aligned} \tag{5}$$

where $\mathbb{I}$ is the identity operator on Hilbert space $\mathcal{H} = \{|0\rangle,|1\rangle\}$, $\hat{S} = \sum_{M=-J}^{J} |J,M\rangle_z\langle J,M|$ is the symmetric projector that maps states in $\mathcal{H}^{\otimes N}$ onto its symmetric subspace $\mathcal{H}_+^{\otimes N}$, and $\sum_{\vec{\sigma}}$ denotes the summation of all permutations of integers 1 to N.

After the adiabatic distribution of $N\nu$ entanglement pairs $|\Psi^{(-)}\rangle$, Alice or Bob can perform the symmetric projector $\hat{S}$ to obtain ν pairs $|\chi\rangle$. Because $|\Psi^{(-)}\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$ changes only with an overall unobservable phase under any unitary evolution of form $e^{-i\hat{H}_A t/\hbar} \otimes e^{-i\hat{H}_B t/\hbar}$ in two-qubit space, this singlet exhibits the rotational invariance property under unitary evolution $(e^{-i\hat{H}t/\hbar})_A^{\otimes N} \otimes (e^{-i\hat{H}t/\hbar})_B^{\otimes N}$ and has identical expression in any spin basis, e.g., $z \mapsto x \mapsto y$, when ignoring the overall phase. Further evidence has been presented [32], and this invariance property has been tested experimentally [33].

These bipartite maximally entangled states play an important role in the quantum information distribution and concentration [34–36]. Recently, these states used in our scheme have been generated experimentally using stimulated parametric down-conversion and have been used in the 1 to 3 + 2 information distribution [33,37,38]. Additional experiments also produced similar entanglement and realized the quantum information distribution [39–41].

A. Ideal situation

Next, we will demonstrate that these technologies can also be utilized to implement our scheme for quantum clock synchronization.

The pure state evolved as

$$\begin{aligned}&(e^{-i\hat{H}_A t_A/\hbar})^{\otimes N} \otimes (e^{-i\hat{H}_B t_B/\hbar})^{\otimes N} |\chi\rangle \\ &\quad= \big(\mathbb{H}_A^{\otimes N} \otimes \mathbb{H}_B^{\otimes N}\big)\big[\mathbb{I}_A^{\otimes N} \otimes (e^{-i t_D \mathbb{H}_B \hat{H}_B \mathbb{H}_B/\hbar})^{\otimes N}\big]|\chi\rangle \\ &\quad= \big(\mathbb{H}_A^{\otimes N} \otimes \mathbb{H}_B^{\otimes N}\big)\big[\mathbb{I}_A^{\otimes N} \otimes U_B(\pi/2,\beta,-\pi/2)^{\otimes N}\big]|\chi\rangle,\end{aligned} \tag{6}$$

where the overall phase is ignored; $\mathbb{H}_{A,B} = \left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)/\sqrt{2}$ is the Hadmard matrix; and the unitary operator $U(\alpha,\beta,\gamma)$ is expressed by three Euler angles in the basis $\{|0\rangle,|1\rangle\}$ as follows:

$$U(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\frac{\beta}{2} e^{i(\alpha+\gamma)/2} & \sin\frac{\beta}{2} e^{-i(\alpha-\gamma)/2} \\ -\sin\frac{\beta}{2} e^{i(\alpha-\gamma)/2} & \cos\frac{\beta}{2} e^{-i(\alpha+\gamma)/2} \end{pmatrix}. \tag{7}$$

Moreover, according to group theory of the irreducible representation, we obtain an analytical expression of the unitary operator in N-qubit space [42]

$$U(\alpha,\beta,\gamma)^{\otimes N}|JM\rangle = \sum_{M'} e^{-i(M\alpha+M'\gamma)} d^J_{M',M}(\beta)|JM'\rangle, \tag{8}$$

where $d^J_{M',M}(\beta) = \langle JM'|\exp(-i\beta J_y)|JM\rangle$. Thus we can obtain the probability distribution of measurement outcomes $\xi_{2N} = (x_1,\ldots,x_N,y_1,\ldots,y_N)$, with $x_i,y_j = 0,1$:

$$P(\xi_{2N}|t_D) = \frac{\big[d^J_{M',-M}(\beta)\big]^2}{(2J+1)C_{2J}^{J-M}C_{2J}^{J-M'}}, \tag{9}$$

where $(J-M)$ is the number of 0s in $\{x_1,\ldots,x_N\}$ and $(J-M')$ is the number of 0s in $\{y_1,\ldots,y_N\}$. As before, the expectation value of measurement operator $\hat{M}_{2N} = \hat{X}(t_A)^{\otimes N} \otimes \hat{X}(t_B)^{\otimes N}$ is calculated as

$$\begin{aligned}M(\beta) := \langle \hat{M}_{2N}\rangle &= \sum_{M,M'=-J}^{J} (-1)^{N+M+M'} \frac{\big[d^J_{M',-M}(\beta)\big]^2}{2J+1} \\ &= \frac{(-1)^N}{N+1}\frac{\sin(N+1)\beta}{\sin\beta}.\end{aligned} \tag{10}$$

The function $M(\beta)$ against its argument $\beta = \omega|t_D|$ is presented in Fig. 1 for various numbers of qubits.

Furthermore, Fisher information reads [30]

$$\mathcal{F}_{t_D} = \sum_{M,M'=-J}^{J} \frac{4\big[\partial_{t_D} d^J_{M',-M}(\beta)\big]^2}{2J+1} = \frac{4J(J+1)\omega^2}{3}, \tag{11}$$

with which it is straightforward to determine that the lower bound $\delta t_D^{\text{est}} = \sqrt{3}/[\omega\sqrt{N(N+2)\nu}]$ clearly breaks the quantum standard limit and attains Heisenberg scaling accuracy for large N.

Nevertheless, this scheme has some special properties that differ from the previous schemes. We can observe that $M(\beta)$ clearly "peaks" around $\beta = 0$ with width $\sim\pi/(N+1)$ (see Fig. 1). This property can avoid the periodical phase 2π

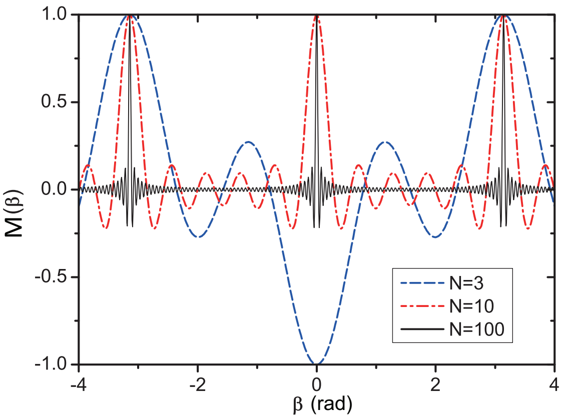

FIG. 1. (Color online) The functional relationship between the expectation value $M(\beta)$ and parameter β. The dashed blue line, dash-dotted red line, and solid black line denote the cases $N=3$, $N=10$, and $N=100$, respectively. When N is an odd number, $M(\beta)$ is negative with β near 0. The peak narrows as N grows.

problem in identifying an unknown phase parameter. We then can confirm that the uncertainty of t_D^{est} could reach the Cramér-Rao bound $1/(\omega\sqrt{\nu\mathcal{F}_{t_D}})$ for a large number ν when the condition $|t_D| < \pi/[(N+1)\omega]$ is satisfied. Although the bipartite maximally entangled spin-zero singlet fails to gain more advantageous information about the parameter t_D from the observed expectation value against the GHZ-type state (3). More importantly, we will acquire a quantum enhanced criterion to judge whether two remote clocks have been synchronized with the accuracy $\pi/[\omega(N+1)]$ even when the number ν is not sufficiently large, i.e., the unbiased estimation hypothesis is not fulfilled. When the expectation value $\overline{M_{2N}} = \frac{1}{\nu}\sum_{i=1}^{\nu} g(\xi_{2N}^i)$ obtained from the outcomes of experiments after classical communication satisfies $|\overline{M_{2N}} - (-1)^N| \leqslant 1 - 1/\sqrt{2}$, one obtains

$$\Pr\left(|t_D| \leqslant \frac{\pi}{(N+1)\omega}\right) \gtrsim \Pr\left(\left|\overline{M_{2N}} - \langle \hat{M}_{2N}\rangle_{t_D}\right| \leqslant \frac{1}{\sqrt{2}}\right) \geqslant 1 - 2e^{-\frac{\nu}{4}}, \tag{12}$$

which is the Hoeffding's inequality [43,44]. For example, suppose that $|\overline{M_{2N}} - (-1)^N| \leqslant 1 - 1/\sqrt{2}$ and $\nu = 10$, we can infer that the inequality $|t_D| \leqslant \pi/[(N+1)\omega]$ holds with fiducial probability larger than 84%.

This strategy can therefore yield a determinate criterion on remote clock synchronization within a Heisenberg scaling accuracy if the measurement results $\overline{M_{2N}}$ are clearly nonzero. Furthermore, as only one high peak exists with β near zero and $M(\beta)$ has a large periodicity 2π (π for N being an even number), this criterion does yield a certain judgment, unlike the strategy using GHZ-type states, which is only valid under the prior condition $|t_D| < \pi/(\omega N)$.

B. Qubits loss

In practical analysis we must consider relevant decoherence in particular, for example, loss qubits. Here we consider the loss of qubits during the clock synchronization procedure.

After the adiabatical distribution of entangled state $|\chi\rangle$, Alice and Bob obtain $N_1 = 2j_1$ and $N_2 = 2j_2$ ticking qubits, respectively; here N_1 and N_2 are not necessarily the same because of different loss numbers. The model of qubits loss we demonstrated is different from the photon loss using two-mode entangled number states of light in the interferometer, which is modeled by placing a beam splitter with partial transmission $\eta < 1$ [30,45]. Then, Alice and Bob perform measurements on all of their qubits simultaneously when their clocks point to the same readout. This measurement can be expressed as an operator $\hat{M}^{\text{loss}} = \hat{X}(t_A)^{\otimes N_1} \otimes \hat{X}(t_B)^{\otimes N_2}$ performed on the reduced density matrix $\hat{\rho}_{N_1,N_2}$ in the Heisenberg picture; then, one obtains the expectation

$$\begin{aligned} M^{\text{loss}}(\beta) &:= \text{Tr}\left[\hat{M}^{\text{loss}}\hat{\rho}_{N_1,N_2}\right] \\ &= \langle\chi|\hat{X}(t_A)^{\otimes N_1} \otimes \mathbb{I}_A^{N-N_1} \otimes \hat{X}(t_B)^{\otimes N_2} \otimes \mathbb{I}_B^{N-N_2}|\chi\rangle. \end{aligned} \tag{13}$$

The symmetric state $|J,M\rangle$ of $N=2J$ qubits can be divided into two parts with $N_1 = 2j_1$ qubits and $2(J-j_1)$ qubits as follows [42]:

$$|J,M\rangle = \sum_{m_1}\sqrt{\frac{C_{J+M}^{j_1+m_1}C_{J-M}^{j_1-m_1}}{C_{2J}^{2j_1}}}|j_1,m_1\rangle|J-j_1,M-m_1\rangle, \tag{14}$$

where the summation is assumed to run over all possible values with the constraint that all of the numbers appearing inside the expression as $(\cdots)!$ are nonnegative integers.

After some calculations we obtain

$$\begin{aligned} M^{\text{loss}}(\beta) = &\frac{(-1)^{2j_1}+(-1)^{2j_2}}{2(2J+1)C_{2J}^{2j_1}C_{2J}^{2j_2}}\sum_{m_1=-a}^{a} e^{i2m_1\beta}\sum_{M=-J+b+m_1}^{J-b+m_1} \\ &\times C_{J+M}^{j_1+m_1}C_{J-M}^{j_1-m_1}C_{J+M}^{j_2+m_1}C_{J-M}^{j_2-m_1}, \end{aligned} \tag{15}$$

where $a = \min\{j_1, j_2\}$ and $b = \max\{j_1, j_2\}$. It is apparent from the above discussion that $M^{\text{loss}}(\beta)$ is an even function for parameter β.

Considering the particular circumstances, it is convenient to verify that

$$M^{\text{loss}}(\beta) = \frac{(-1)^{2J}}{2J+1}\sum_{m_1=-J}^{J} e^{i2m_1\beta} = M(\beta) \tag{16}$$

when $j_1 = j_2 = J$. Furthermore, if $j_1 = 0$, then m_1 can only be 0 in the summation, and we have

$$\begin{aligned} M^{\text{loss}}(\beta) &= \frac{1+(-1)^{2j_2}}{2(2J+1)C_{2J}^{2j_2}}\sum_{M=-J+j_2}^{J-j_2} C_{J+M}^{j_2}C_{J-M}^{j_2} \\ &= \frac{1+(-1)^{N_2}}{2(N_2+1)}, \end{aligned} \tag{17}$$

where we have used the identity on combinatorics [42]. Thus, it is consistent with the assumption that $M^{\text{loss}}(\beta)$ is independent of the difference of time t_D when $j_1 = 0$ or $j_2 = 0$.

Then we demonstrate that our strategy is also workable when considering that some qubits are lost during the distribution process. The function $M^{\text{loss}}(\beta)$ is plotted against the parameter β for different numbers $N = 3$, 10, and 100 in Fig. 2. We study two types of loss cases: equally bilateral loss and unilateral loss. It is demonstrated that for both cases,

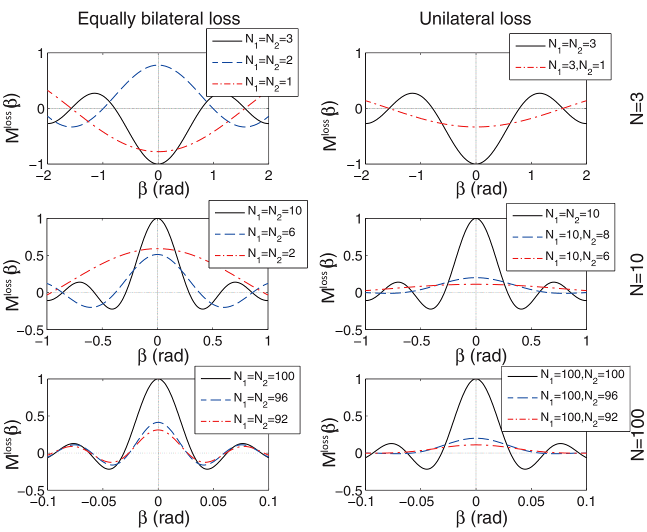

FIG. 2. (Color online) Function $M^{\text{loss}}(\beta)$ against the parameter β. The first column presents the case of equally bilateral loss: $N_1 = N_2$; and the second column presents the case of unilateral loss: $N_1 = N$ and $N_2 \leqslant N_1$. The first, second, and third line illustrate cases $N = 3$, $N = 10$, and $N = 100$, respectively.

the criterion for the synchronization of two remote clocks is still available by adjusting the confidence interval of the measurement results. As the loss is enhanced, the peak widens, decreasing the precision of the estimation. Moreover, even if twice as many qubits are missing in the equally bilateral loss case, the height of the peak is higher than that in the unilateral loss case because it maintains higher symmetry. We then conclude that to a large extent the clock synchronization criterion presented in this article is still feasible when suffering the loss of qubits although it may not always maintain the Heisenberg scaling accuracy.

V. CONCLUSION

In many cases of quantum metrology, the quantum Cramér-Rao bound can merely be achieved asymptotically, i.e., it holds for an unbiased estimation with an infinite number ν and zero error $\delta t_D^{\text{est}} \rightarrow 0$ [46,47]; unfortunately, this problem also exists in certain quantum clock synchronization strategies. However, by applying the bipartite maximally entangled spin-zero singlet, one can obtain a standard to judge whether two spatially separated clocks have been synchronized within a specific uncertainty even when the number ν is not sufficiently large. Thus we can step further to obtain the difference between two clocks with a Heisenberg scaling accuracy in accordance with the expectation of the measurement results by increasing the number ν.

In conclusion, we propose a quantum scheme for remote clock synchronization within a specific accuracy. This bound of accuracy scales with the Heisenberg limit, which is the ultimate limit of precision measurements under all conditions. Although the accuracy may be worse, this scheme is still workable against qubit loss, which is one of the main obstacles for long-distance state transferring. With developments in experimentally creating entanglement resources, this quantum scheme of remote clocks synchronization may be implemented and may possess unprecedented precision.

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