Diffraction in time for tunneling invisibility in quantum systems

Sergio Cordero[1] and Gastón García-Calderón[2,*]
[1]*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, México 04510, Distrito Federal, Mexico*
[2]*Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal 20-364, México 01000, Distrito Federal, Mexico*
(Received 3 June 2013; published 18 November 2013)

We show that tunneling invisibility in quantum systems holds also in the transient regime by demonstrating that it is indistinguishable from the phenomenon of diffraction in time of a free evolving particle.

DOI: 10.1103/PhysRevA.88.052118 PACS number(s): 03.65.Ca, 03.65.Xp, 03.65.Db

I. INTRODUCTION

In a recent work [1], it is shown that in a time-independent description, an appropriate choice of the potential parameters in one-dimensional quantum systems may allow unity transmission for tunneling at all incident energies except for controllable, exceedingly small incident energies. See related work, concerning non-Hermitian tight-binding lattices, in Ref. [2]. The corresponding transmission amplitude and phase are indistinguishable from that of a free particle for energies along the tunneling region. The analysis holds for coherent (elastic) processes, i.e., no inelastic or dissipative processes, and is robust against a small variation of the potential parameters and the mass of the incident particle. We refer to this situation as *tunneling invisibility*.

The above situation may be contrasted with the so-called *transparent* systems [3], which exhibit also unity transmission at all energies, including the threshold energy, but that in general suffer a shift in the value of the transmission phase. A well-known example is the Pöschl-Teller potential [4]. Transparent potentials have escaped experimental verification, to the best of our knowledge, and are mainly of interest in mathematical-oriented studies. A possible reason is that transparency in these potentials is tightly bound to the functional dependence of the potential. One finds also in the literature a class of systems which exhibit unity transmission through a resonant tunneling process. This occurs at specific energies, the so-called resonance energies, and is a process that involves also a phase shift with respect to the free case. Resonant tunneling systems have been studied both theoretically and experimentally [5]. It is precisely the occurrence of a phase shift that is the relevant feature that distinguishes resonant tunneling from invisible systems.

It is worth mentioning that here invisibility refers to a different process from studies that involve the design of a cloak surrounding a system that then becomes invisible to light within a range of frequencies [6,7]. These approaches are based on ideas from transformation optics that refer only to a time-independent description.

As shown in Ref. [1], resonant tunneling and tunneling invisibility systems are closely related to each other. One may go from one to the other by modifying appropriately the potential parameters. This, of course, follows because the behavior of the transmission amplitude with energy depends on the potential parameters. The characterization of invisible systems given in Ref. [1] has been obtained by studying analytically the transmission amplitude in terms of its complex poles. Although this corresponds to monochromatic waves, numerical calculations for tunneling of Gaussian wave packets corroborated tunneling invisibility in these systems. However, such time-dependent calculations were considered at very long distances (compared with the length of the system) and times, and hence leave unanswered the question of whether tunneling invisibility remains away from these asymptotic conditions, as in a nonstationary regime. This has motivated us to investigate the transient regime in tunneling invisible systems.

It is known that quantum transients refer to the time evolution of matter waves before they reach a stationary or steady regime. They may occur as a result of a sudden change in the initial conditions of a system described by the time-dependent Schrödinger equation and are usually amenable to exact analytical treatments. They were first discussed by Moshinsky [8], who considered the transient behavior that follows after the sudden opening of a quantum shutter that initially keeps an incident beam of particles from evolving freely through space. He determined that the transient probability density had a close mathematical resemblance to the intensity of light in the Fresnel diffraction by a sharp edge and, for that reason, he named this transient phenomenon diffraction in time. The essential feature of the diffraction-in-time phenomenon consists of spatial and temporal oscillations of Schrödinger matter waves released in one or several pulses from a preparation region in which the wave was originally confined [8,9]. The experimental verification of diffraction in time has stimulated a great deal of work, both theoretical and experimental, on this subject [10,11].

In this work, we address the issue of the transient behavior of tunneling invisible systems in an analytical fashion. The aim is to find out whether tunneling invisibility remains in such a nonstationary regime. We find that this is the case by showing that tunneling invisibility is indistinguishable from the corresponding transient regime of a free evolving particle, the so-called *diffraction-in-time* phenomenon.

II. TRANSIENT SOLUTION ALONG THE TRANSMITTED REGION OF A POTENTIAL

The quantum shutter setup for tunneling through a potential of finite range corresponds to a situation where, upon the instantaneous release of the shutter, the initial state impinges on the potential. The dynamics that follows involves reflected,

*gaston@fisica.unam.mx

tunneling, and transmitted transient solutions. Most early works concentrated on analyzing the tunneling and transmitted solutions [12–15], but recently, however, the reflected solution has also been discussed, leading to a full description of transients in one dimension [16]. All of these works exploit the analytical properties of the outgoing Green's function to the problem which possesses an infinite number of complex poles $\{\kappa_n\}$ distributed in a well-known manner on the complex k plane [17] and their corresponding residues $\{r_n\}$. The above works consider also a variety of initial states.

The transient solution to the time-dependent Schrödinger equation for a finite range potential of arbitrary shape $V(x)$, extending from $x = 0$ to $x = L$ with the same initial condition at $t = 0$ as considered by Moshinsky [8],

$$\Psi(x,0) = \begin{cases} e^{ik_0x}, & x < 0 \\ 0, & x > 0, \end{cases} \tag{1}$$

reads along the transmitted region of the potential $x > L$ as [12]

$$\Psi_T(x,t) = \mathbf{t}(k_0)M\left(y_{k_0}\right) - \sum_{n=-\infty}^{\infty} \mathbf{t}_n(k_0)M\left(y_{\kappa_n}\right), \tag{2}$$

where $\mathbf{t}(k_0)$ corresponds to the transmission amplitude, $\mathbf{t}_n(k_0)$ is given by

$$\mathbf{t}_n(k_0) = \frac{r_n}{k_0 - \kappa_n}, \tag{3}$$

and the M functions refer to the Moshinsky function,

$$M(y_q) \equiv \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx - i\hbar k^2 t/2m}}{k - q}\, dk = \frac{1}{2} e^{(imx^2/2\hbar t)} e^{y_q^2} \operatorname{erfc}(y_q), \tag{4}$$

with argument y_q given by

$$y_q = e^{-i\pi/4}\left(\frac{m}{2\hbar t}\right)^{1/2}\left[x - \frac{\hbar q}{m}t\right], \tag{5}$$

where q stands for k_0 or κ_n.

It is convenient to mention that the transmission amplitude may be expanded in several forms in terms of its poles [18]. In particular, it may be expanded as

$$\mathbf{t}(k_0) = k_0 \sum_{n=-\infty}^{\infty} \frac{r_n}{\kappa_n(k_0 - \kappa_n)}. \tag{6}$$

The absence of a potential implies that there are no poles and that the transmission amplitude $\mathbf{t}(k_0) = 1$. Hence, the solution given by Eq. (2) reduces to the free case discussed by Moshinsky [8], namely,

$$\Psi_F(x,t) = M\left(y_{k_0}\right). \tag{7}$$

Substitution of (7) into (2) shows that the transmitted solution is proportional to the free solution minus a term that depends on the resonance poles of the system,

$$\Psi_T(x,t) = \mathbf{t}(k_0)\Psi_F(x,t) - \sum_{n=-\infty}^{\infty} \mathbf{t}_n(k_0)M\left(y_{\kappa_n}\right). \tag{8}$$

It is interesting to see that both resonant tunneling and tunneling invisibility are dominated by a single pole in Eq. (8).

For the case of a sharp isolated tunneling resonance of the system, we denote the corresponding complex pole by $\kappa_r = \alpha_r - i\beta_r$. At resonance, $k_0 = \alpha_r$ and hence $|\mathbf{t}(\alpha_r)| = 1$; then one may write $\mathbf{t}(\alpha_r) = \exp[i\theta(\alpha_r)]$. As a consequence, Eq. (8) may be written as

$$\Psi_T(x,t) = e^{i\theta(\alpha_r)}\Psi_F(x,t) - \mathbf{t}_r(\alpha_r)M\left(y_{\kappa_r}\right) - \sum_{n\neq r}^{\infty} \mathbf{t}_n(\alpha_r)M\left(y_{\kappa_n}\right), \tag{9}$$

where $\theta(\alpha_r)$ represents the phase shift of the transmission amplitude. It is well known that at fixed distance $x = x_0$ and long times, the Moshinsky functions corresponding to complex poles seated on the fourth quadrant of the k plane decay exponentially with time except at very short and extremely long times compared with the lifetime of the system [10,12]. We recall that the decay widths are defined as $\Gamma_n = (\hbar^2/2m)4\alpha_n\beta_n$ and the corresponding lifetimes by $\tau_n = \hbar/\Gamma_n$. The lifetime τ of the system is that referring to the smallest width. Since the decay widths increase with n away from the resonance at $n = r$, one sees that the contribution of the sum on the right-hand side of Eq. (9) becomes much smaller than the resonant contribution at $n = r$, and hence one may write Eq. (9) as

$$\Psi_T(x,t) \approx e^{i\theta(\alpha_r)}\Psi_F(x,t) - \mathbf{t}_r(\alpha_r)M\left(y_{\kappa_r}\right), \tag{10}$$

which implies that the transmitted time-dependent solution becomes proportional to the free solution. It turns out, however, that $|\Psi_T(x_0,t)|^2$ is delayed with respect to the free evolving solution $|\Psi_F(x_0,t)|^2$ [12]. As mentioned before, this arises from interference between the two terms in Eq. (10) [19,20]. It is important to stress that the overall phase of $\Psi_T(x,t)$ in Eq. (9) differs from that of $\Psi_F(x,t)$, and hence an interference experiment could distinguish between the free and transmitted solutions in spite of the fact that both exhibit unity transmission.

For the case of tunneling invisibility, it has been shown recently [1] that this phenomenon arises provided all the complex poles are seated far away from the real k axis except an antibound or a bound pole seated very close to the threshold energy. In fact, any of these poles is responsible for unity transmission of tunneling at all energies except those very close to the threshold energy, and, also, for the vanishing of the corresponding transmitted phase. Denoting these poles by

$$\kappa_q = i\gamma_q, \tag{11}$$

where $q = a$ and $q = b$ stand, respectively, for antibound ($\gamma_a < 0$) and bound ($\gamma_b > 0$) poles, it turns out that for tunneling invisibility, the transmission amplitude may be written as [1]

$$\mathbf{t}(k_0) \approx \frac{1}{(1 - \kappa_q/k_0)} \approx 1, \tag{12}$$

with $k_0 \gg |\kappa_q|$ which implies that the phase θ is

$$\theta(\kappa_q) \approx \frac{\gamma_q}{k} \approx 0. \tag{13}$$

The behavior of $\mathbf{t}(k_0)$ given by Eq. (12) follows, in view of (6), because $r_q \approx \kappa_q$. Surprisingly, this implies, however, that $\mathbf{t}_q(k_0)$, defined by (3), behaves as

$$\mathbf{t}_q(k_0) \approx \frac{\kappa_q}{(k_0 - \kappa_q)} \approx 0, \qquad (14)$$

since $|\kappa_q| \ll 1$ is a negligible quantity. Hence, using Eqs. (13) and (14), one may write Eq. (8) as

$$\Psi_T(x,t) \approx \Psi_F(x,t) - \sum_{n \neq q}^{\infty} \mathbf{t}_n(k_0) M\left(y_{\kappa_n}\right). \qquad (15)$$

Now, from the properties of the Moshinsky functions [10,12] for complex poles, $n \neq q$, for fixed $x = x_0$ and long times, i.e., $t \gg \tau$, as discussed previously for resonant tunneling, one sees that the last term in (15) is also negligible. The difference here with the resonant tunneling case is that there are no poles close to the real k axis, since all the decay widths are very large [1]. It follows then that Eq. (15) becomes

$$\Psi_T(x,t) \approx \Psi_F(x,t), \qquad (16)$$

which, for a given $x = x_0$, holds provided the time t is larger than the lifetime of the system.

In summary, the above discussion for resonant tunneling and tunneling invisibility systems indicates that for unity transmission in the transient regime, resonant tunneling systems exhibit a phase dependence with time that allows one to distinguish them, whereas tunneling invisibility systems remain undetected except at energies very close to the threshold and very short times.

For the sake of completeness of the discussion, we discuss briefly here the limit as the time t goes to infinity for the free and transmitted solutions. The free transient solution given by (7) tends to the stationary solution [8],

$$\Psi_F(x,t) \to e^{ik_0x} e^{-iE_0t/\hbar}. \qquad (17)$$

As discussed above, the resonance terms of the transmitted solution tend to a vanishing value at long times and therefore the transmitted solution (8) behaves as [12]

$$\begin{aligned}\Psi_T(x,t) &\to \mathbf{t}(k_0) e^{ik_0x} e^{-iE_0t/\hbar}\\ &= |\mathbf{t}(k_0)| e^{i\theta(k_0)} e^{ik_0x} e^{-iE_0t/\hbar}, \qquad (18)\end{aligned}$$

where $E_0 = (\hbar^2/2m)k_0^2$. Recalling that $\kappa_q = i\gamma_q$, we define $E_q = (\hbar^2/2m)\gamma_q^2$, and hence using (12), one may write the transmission coefficient for tunneling invisible systems as

$$T(E_0) = |\mathbf{t}(E_0)|^2 \approx \frac{1}{1 + E_q/E_0}. \qquad (19)$$

III. ANALYSIS OF THE INTERFERENCE OF THE FREE AND TRANSMITTED SOLUTIONS

As discussed by Moshinsky [8,10], a typical plot of the behavior of the density profile $|\Psi_F(x_0,t)|^2$ for the free case at a fixed distance $x = x_0$ from the shutter as a function of time is shown in Fig. 1. From the point of view of classical mechanics, one detects the particle until the time of flight $t_0 = x_0/v_0$, where $v_0 = \hbar k_0/m$ is the velocity of the particle. The diffraction in time pattern, however, grows from the start in a monotonic way up to a time slightly larger than the time of flight t_0. As time increases further, the density profile exhibits a damped oscillation around unity and tends to this value as $t \to \infty$.

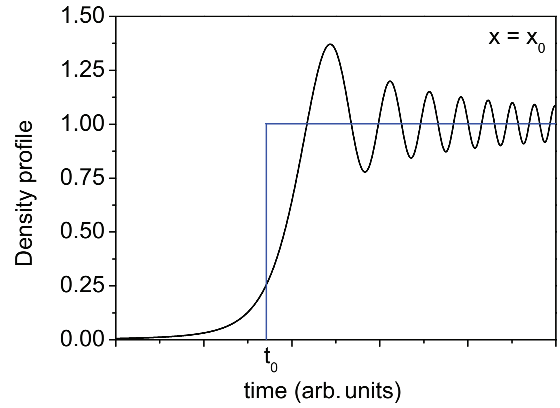

FIG. 1. (Color online) Unnormalized probability density as a function of time at a given distance $x = x_0$, showing a typical diffraction in time pattern (solid line) and its corresponding classical analog (dotted line). See text.

In general, in the presence of a potential, a diffraction pattern similar to that shown in Fig. 1 forms at long distances and times. This occurs due to the vanishing of the resonance pole contribution. However, it may be shown that the interference between the freelike term and the resonance pole contribution in (8) yields a wave front that exhibits a delay or a time advance with respect to the free evolving case [12,19,20]. In the case of systems that exhibit unity transmission, i.e., $|\mathbf{t}(k_0)|^2 = 1$, the above occurs at a resonance energy of the system, as in double-barrier resonant tunneling systems [12,21], but also it may occur due to strong interference between two resonance levels, as in triple-barrier resonant tunneling systems, that results in a plateau of energies that yield unity transmission [22]. As pointed out before, in the case of an invisible system, one expects, in addition to unity transmission at all tunneling energies except those very close to the threshold energy, a zero phase shift as time evolves.

For systems that show unity transmission, i.e., $|\mathbf{t}(k_0)|^2 = 1$, an interference experiment could distinguish if there is a phase shift due to the presence of the potential. From a theoretical point of view, we find it convenient to consider the quantity

$$\rho_{TF}(x,t) = \tfrac{1}{4}|\Psi_T(x,t) + \Psi_F(x,t)|^2, \qquad (20)$$

where F and T stand, respectively, for the free and transmitted time-dependent solutions given, respectively, by Eqs. (7) and (8).

On the other hand, for a free evolving system, $\Psi_T(x,t) = \Psi_F(x,t)$ and hence Eq. (20) clearly becomes

$$\rho_{FF}(x,t) = |\Psi_F(x,t)|^2. \qquad (21)$$

In the case of a nonzero phase shift, as in resonant tunneling systems, one expects that at a given distance $x = x_0$, at resonance energy as a function of time,

$$\rho_{TF}(x_0,t) \neq \rho_{FF}(x_0,t), \qquad (22)$$

whereas for invisible systems, where the phase shift essentially vanishes at all energies except too close to threshold, one should obtain

$$\rho_{TF}(x_0,t) = \rho_{FF}(x_0,t). \tag{23}$$

It is worth noticing, using Eqs. (17) and (18), that at asymptotically long times, which correspond to the stationary regime,

$$\rho_{TF}(x,t) \to \cos^2\left[\frac{\theta(k_0)}{2}\right], \tag{24}$$

which is an expression that due to the interference contribution, exhibits explicitly the dependence on the phase shift $\theta(k_0)$. On the other hand, in the same limit, it follows immediately from Eq. (17) that $\rho_{FF}(x,t) \to 1$. It follows therefore that at asymptotically long times, a calculation of Eqs. (20) and (21) allows one to distinguish, for systems with unity transmission, whether or not they are invisible.

IV. EXAMPLES

In order to illustrate the distinction between resonant tunneling and invisible systems discussed in the previous section, we discuss here, without loss of generality, two examples: a double-barrier (DB) resonant tunneling system and a barrier-well-barrier (BWB) invisible system. In order to keep ourselves on physical grounds, we refer to systems characterized by typical parameters of semiconductor heterostructures [5]. In both cases, we consider an effective electronic mass $m = 0.067\,m_e$, with m_e the electron mass. The parameters of the DB resonant tunneling system are barrier heights $V_0 = 0.23$ eV, barrier widths $b = 50.0$ Å, and well width $w = 50.0$ Å. These parameters correspond to the system used in Ref. [12] to discuss transient effects in resonant tunneling. There, one learns that at asymptotically long distances and times, the transient transmitted probability density, at resonance energy, suffers a delay time with respect to the free evolving case. The parameters of the BWB invisible system are barrier heights $V_0 = 0.12$ eV, barrier widths $b = 4.0$ Å, well depth $U_0 = -0.12$ eV, and well width $w = 8.0$ Å.

Figure 2(a) shows the transmission coefficients, calculated using the transfer matrix method, for the DB resonant tunneling (full line) and BWB invisible (dotted line) systems as a function of energy in units of the corresponding barrier heights V_0. In both systems, the energy threshold is at $E = 0$. Notice that the DB system possesses a sharp isolated resonance below the barrier heights and some resonant structures above the barrier heights. On the other hand, as discussed in detail in Ref. [1], the BWB system exhibits almost unity transmission at all energies except very close to energy threshold, where the coefficient rises sharply to reach a value very close to unity.

Figure 2(b) displays the distribution of the complex poles κ_n for the DB resonant tunneling system (dots) and the BWB invisible system (stars) on the $\beta = kL$ plane. Here k stands for the wave number and L is the length of the corresponding system. Notice that in the former case, $L = 150$ Å, and for the latter one, $L = 16$ Å. Using the pole expansion given by Eq. (6), one may evaluate the transmission coefficient [18] which provides a relationship between the complex pole distributions and the resonance spectra of the systems. In particular, for the DB resonant tunneling system, the first two poles are close to the real β axis and one sees that they correspond to the two resonance lines displayed in Fig. 2(a), whereas at higher energies, the poles seat away from the real β axis and hence the corresponding transmission coefficient exhibits an overlapping resonance structure that oscillates close to unity. On the other hand, for the BWB invisible system, all of the poles, except a bound pole seated very close to the energy threshold at $\beta_b \equiv \gamma_b L = i\,0.00314$, are located very far away from the real β axis. In this case, the transmission coefficient is governed by the bound pole, whereas the contribution of the distant complex poles is negligible. In fact, using the corresponding value of $E_b = (\hbar^2/2m)\gamma_b^2 = 2.19 \times 10^{-6}$ eV in Eq. (19) shows that indeed the rising of the transmission coefficient from a vanishing value up to unity occurs within a very short energy interval, as illustrated in Fig. 2(a). For example, using the value of E_b given above into (19), writing there $E_b/E_0 = (E_b/V_0)/(E_0/V_0)$, one sees that $E_b/V_0 = 1.82 \times 10^{-5}$ and hence already a small value of E_0/V_0 as $E_0/V_0 = 0.01$ yields a value of $T(E_0) = 0.998$ very close to unity.

Let us now consider the comparison of the transient expressions given by Eqs. (20) and (21) for both the DB resonant tunneling system and the BWB invisible system with the initial state given by Eq. (1).

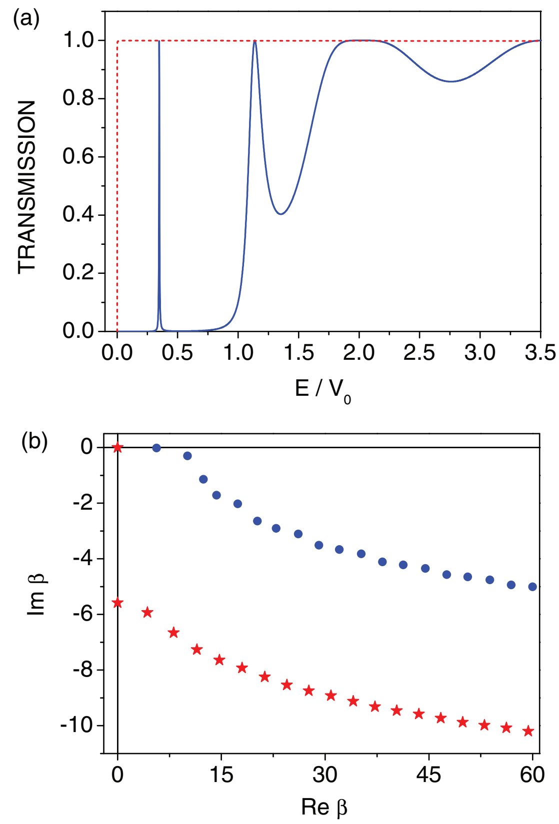

FIG. 2. (Color online) (a) The transmission coefficient as a function of the energy in units of the corresponding potential height V_0 for the resonant tunneling system (full line) and the invisible system (dashed line). (b) The distribution of the complex poles for the previous systems: resonant tunneling system (dots) and the invisible system (stars). The star seated very close to the threshold stands for a bound pole. See text.

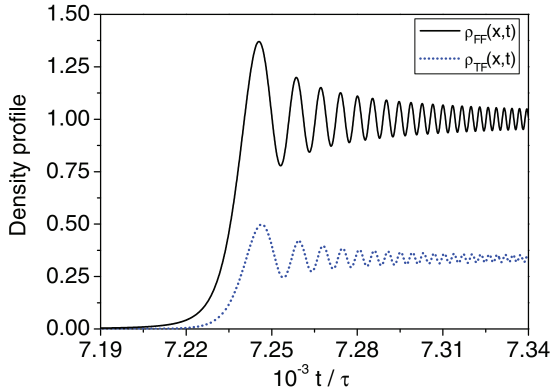

FIG. 3. (Color online) Comparison of $\rho_{FF}(x,t)$ (full line) and $\rho_{TF}(x,t)$ (dotted line) for the resonant tunneling system that shows a nonzero phase shift at all times, which shows that $\rho_{FF}(x_0,t) \neq \rho_{TF}(x_0,t)$. See text.

Figure 3 exhibits a comparison of $\rho_{FF}(x_0,t)$ (full line) and $\rho_{TF}(x_0,t)$ (dotted line) as a function of time in units of the lifetime τ of the DB resonant tunneling system at the fixed distance $x_0/L = 2 \times 10^5$ for an incident energy $E_0/V_0 = 0.35$ that corresponds to the first resonance of the DB system, which yields unity transmission as shown by Fig. 2(a). Notice that the calculation is made at long distances and times, so that both the free evolving and DB density profiles exhibit a well-defined transient pattern. Clearly, $\rho_{FF}(x_0,t) \neq \rho_{TF}(x_0,t)$, which shows that there is a nonzero phase shift between the free and transmitted solutions.

Figure 4(a) exhibits a similar comparison as in the previous case of $\rho_{FF}(x_0,t)$ (full line) and $\rho_{TF}(x_0,t)$ (dotted line) for the BWB invisible system. The calculation is made at the same position as in the previous case, so that $x_0/L = 1.875 \times 10^6$ for an incident energy $E_0/V_0 = 0.5$. One sees that both quantities are indistinguishable, namely, $\rho_{FF}(x_0,t) = \rho_{TF}(x_0,t)$. In fact, an analogous situation occurs also at any distance $x_0/L \geqslant 1$. This is exemplified by Fig. 4(b) where $x_0/L = 1$ for the same incident energy as in the previous example. Again one sees that $\rho_{FF}(x_0,t) = \rho_{TF}(x_0,t)$. Notice that the width of the main diffraction peak is of the order of millions of a lifetime in Fig. 4(a) and thousands of a lifetime in Fig. 4(b). The insets to both figures exhibit, respectively, a region where the transient term on the right-hand side of (8) yields a contribution that allows one to distinguish the free and transmitted solutions. Notice, however, that the corresponding density profiles have an extremely small value and that as time evolves become completely indistinguishable from the free evolving cases. Notice also that the above occurs very far away from the corresponding density profile wave fronts.

The above two situations corroborate indeed that the invisibility nature of the BWB system remains in the transient regime, and, hence, for invisible systems, one cannot distinguish between the free and transmitted time-dependent solutions via an interference experiment.

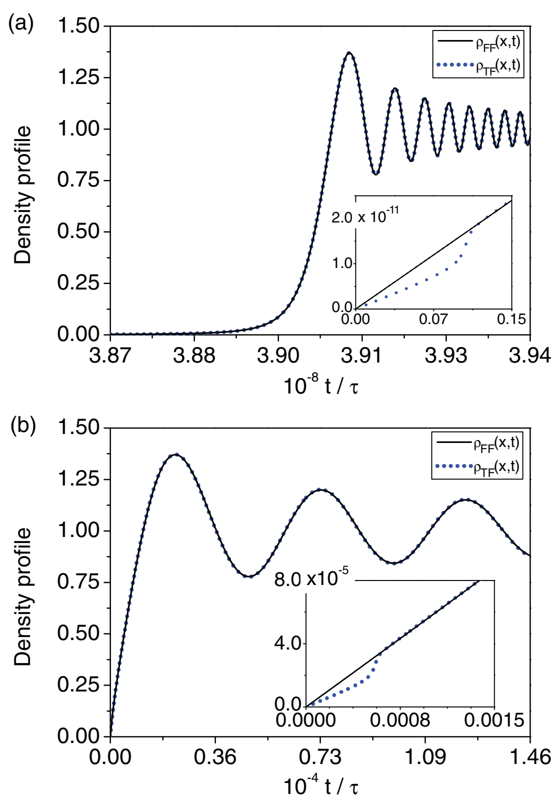

FIG. 4. (Color online) Comparison of $\rho_{FF}(x,t)$ (full line) and $\rho_{TF}(x,t)$ (dotted line) for a barrier-well-barrier invisible system exhibiting a zero phase shift at all times (a) at $x_0/L = 1.875 \times 10^6$ and (b) $x_0/L = 1$, which shows in both cases that $\rho_{FF}(x_0,t) = \rho_{TF}(x_0,t)$. The insets to both figures exhibit, respectively, the corresponding behaviors for short times. See text.

V. CONCLUSIONS

In this work, we have considered the quantum shutter setup to show that the time evolving wave function for tunneling invisibility exhibits a diffraction-in-time phenomenon that is indistinguishable from that for the free evolving case [Eq. (16)]. We find of interest that the tunneling-invisibility phenomenon remains in time domain. In order to emphasize the differences between tunneling invisibility (unity transmission and vanishing transmission phase) and transparent systems (unity transmission and nonvanishing transmission phase), we have also analyzed the time-dependent solution for resonant tunneling systems at resonance energy [Eq. (10)]. The difference between these systems is exhibited in a numerical example by calculating Eqs. (20) and (21), as displayed in Figs. 3 and 4. It is worth mentioning that the results of our analysis hold also for distinct initial states. Tunneling invisibility refers to an almost unexplored realm of quantum mechanics. The properties of these systems, their possible construction by suitable engineering in the growing field of artificial quantum structures, as well as the analogies with optic systems of a negative-refractive index that exhibit zero phase delay [23], in addition to its possible applications, might deserve further study.

ACKNOWLEDGMENTS

S.C. acknowledges the financial support of a fellowship from CONACyT-México under Project No. 101541 and G.G-C. acknowledges the partial financial support of DGAPA-UNAM, México under Grant No. IN103612.

[1] S. Cordero and G. García-Calderón, Phys. Rev. A **79**, 052103 (2009).
[2] S. Longhi, Phys. Rev. A **82**, 032111 (2010).
[3] G. A. Kerimov and A. Ventura, J. Math. Phys. **47**, 082108 (2006).
[4] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977), para. 25.
[5] D. K. Ferry and S. M. Goodnick, *Transport in Nanostructures* (Cambridge University Press, Cambridge, UK, 1997).
[6] J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
[7] U. Leonhardt, Science **312**, 1777 (2006).
[8] M. Moshinsky, Phys. Rev. **88**, 625 (1952).
[9] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, UK, 1995).
[10] A. del Campo, G. García-Calderón, and J. Muga, Phys. Rep. **476**, 1 (2009).
[11] A. D. Cronin, J. Schmiedmayer, and D. E. Pritchard, Rev. Mod. Phys. **81**, 1051 (2009).
[12] G. García-Calderón and A. Rubio, Phys. Rev. A **55**, 3361 (1997).
[13] G. García-Calderón and J. Villavicencio, Phys. Rev. A **64**, 012107 (2001).
[14] R. Romo, Phys. Rev. B **66**, 245311 (2002).
[15] N. Yamada, G. García-Calderón, and J. Villavicencio, Phys. Rev. A **72**, 012106 (2005).
[16] S. Cordero and G. García-Calderón, J. Phys. A **44**, 305302 (2011).
[17] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Dover, New York, 2002), Chap. 12.
[18] S. Cordero and G. García-Calderón, J. Phys. A: Math. Theor. **43**, 415303 (2010).
[19] A. Hernández and G. García-Calderón, Phys. Rev. A **68**, 014104 (2003).
[20] G. García-Calderón and A. Hernández-Maldonado, Phys. Rev. A **86**, 062118 (2012).
[21] B. Ricco and M. Y. Azbel, Phys. Rev. B **29**, 1970 (1984).
[22] R. Romo and G. García-Calderón, Phys. Rev. B **49**, 14016 (1994).
[23] S. Kocaman, M. S. Aras, P. Hsieh, J. F. McMillan, C. G. Biris, N. C. Panoiu, M. B. Yu, D. L. Kwong, A. Stein, and C. W. Wong, Nat. Photon. **5**, 499 (2011).