Maximal quantum randomness in Bell tests

Chirag Dhara,[1,*] Giuseppe Prettico,[1] and Antonio Acín[1,2]
[1]*ICFO-Institut de Ciencies Fotoniques, Mediterranean Technology Park, 08860 Castelldefels (Barcelona), Spain*
[2]*ICREA-Institució Catalana de Recerca i Estudis Avançats, Lluis Companys 23, 08010 Barcelona, Spain*

(Received 12 November 2012; published 15 November 2013)

The nonlocal correlations exhibited when measuring entangled particles can be used to certify the presence of genuine randomness in Bell experiments. While nonlocality is necessary for randomness certification, it is unclear when and why nonlocality certifies maximal randomness. We provide a simple argument to certify the presence of maximal local and global randomness based on symmetries of a Bell inequality and the existence of a unique quantum probability distribution that maximally violates it. We prove the existence of N-party Bell tests attaining maximal global randomness by identifying those combinations of two-outcome measurements by each party providing N perfect random bits.

DOI: 10.1103/PhysRevA.88.052116 PACS number(s): 03.65.Ud, 03.65.Ta, 03.67.Mn, 05.40.−a

I. INTRODUCTION

Quantum theory radically departs from classical theory in many aspects. Quantum theory, for instance, predicts correlations among distant noncommunicating observers that cannot be reproduced classically. These correlations are termed nonlocal and violate those conditions known as Bell inequalities that, in contrast, are satisfied by classically correlated systems [1]. Quantum theory also incorporates a form of randomness in its framework that does not have a classical counterpart. There is no true randomness in Newtonian physics, as the complete knowledge of initial conditions along with interactions of a system allows one to predict its future dynamics deterministically. As is well known, however, predictions in quantum systems are necessarily probabilistic. Since the violation of Bell inequalities implies that quantum theory cannot be explained by local deterministic theories [2], the probabilistic nature must arise from intrinsic randomness. Hence, the violation of a Bell inequality certifies the existence of genuine randomness (for recent developments, see [3] and references therein).

Randomness constitutes a valuable information resource, with applications ranging from cryptographic protocols and gambling to numerical simulations of physical and biological systems. Recently, tools to quantify the presence of randomness in Bell tests have been presented in Ref. [4]. An important advantage of this approach is that it is derived in the device-independent scenario, where the system is characterized from an input-output perspective without regard for its internal working. Thus, although we now have tools to link quantum randomness and nonlocality, we are still far from understanding the exact relation between these two quantum properties. For instance, there exist probability distributions with maximal nonlocality but less than maximal randomness. Even more counterintuitively, distributions with arbitrarily small nonlocality can contain almost maximal randomness [5]. Along this direction, identifying those quantum setups, namely Bell tests, which offer the highest possible randomness would be a highly desirable result, both from a fundamental and practical point of view. This is the main goal of the present paper.

It is worth illustrating our motivations with an example. Consider the standard Clauser-Horne-Shimony-Holt (CHSH) inequality [6], $I_{\text{CHSH}} = \langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle$. At the point of maximal quantum violation, any measurement output by any of the parties provides a perfect random bit. That is, the corresponding probability distribution contains *locally* the maximum possible of one bit of randomness for every party and every measurement setting. However, there are strictly less than two random bits *globally*, as any pair of local measurements gives correlated results. Now, consider the following modification of the CHSH inequality, $I_\eta = \langle A_1 B_1\rangle + \langle A_1 B_2\rangle + \langle A_2 B_1\rangle - \langle A_2 B_2\rangle + \eta\langle A_1\rangle$. At the point of maximal quantum violation, only the measurement A_2 defines a perfect random bit [5]. Why this setting and not the others? Why all of them in the case of CHSH? More in general, when can we expect maximal local and global randomness in a Bell test?

Our main result consists of a simple argument that not only provides an answer to the preceding questions but also provides Bell tests certifying maximal global randomness in a robust manner. In fact, our argument allows one to identify measurements in a Bell test that provide maximally random outputs. We can state our recipe for randomness certification as a simple protocol. Given a Bell inequality, our argument (i) assumes that the quantum distribution attaining its maximal violation is unique and (ii) exploits the symmetries of the inequality by making transformations that leave the Bell inequality unchanged while permuting the outcomes of the measurements of interest. Uniqueness finally allows one to conclude that the permuted outcomes must be random. As seen, our argument crucially relies on the assumption that the quantum distribution attaining the maximal violation of the given Bell inequality is unique. We come back to this point later, but we just mention here that geometric arguments based on our understanding of the set of quantum correlations support the validity of this assumption in general.

II. DEFINITIONS

We start by explaining our notation and stating the basic definitions we use in the text.

[*] Now at the Max-Planck-Institute for Biogeochemistry, Hans-Knöll-Str. 10, 07745 Jena, Germany.

A. Bell tests

We denote by (N,M,d) a standard Bell experiment consisting of N separated and noncommunicating parties, where each of them can perform M local measurements of d outcomes. By repeating the experiment, it is possible to assign a probability distribution $P(a_1,\ldots,a_N|x_1,\ldots,x_N)$, where a_i is the outcome of a measurement x_i by party $1 \leqslant i \leqslant N$. We often consider cases with dichotomic measurements, i.e., $d=2$. In this case, we can use the following useful parametrization,

$$P(\mathbf{a}|\mathbf{x}) = \frac{1}{2^N}\left(1 + \sum_{i=1}^{N} a_i\langle A_i\rangle + \sum_{i<j} a_i a_j\langle A_i A_j\rangle + \sum_{i<j<k} a_i a_j a_k\langle A_i A_j A_k\rangle + \cdots + a_1 a_2 \ldots a_N\langle A_1 A_2 \ldots A_N\rangle\right). \tag{1}$$

Here, measurement outputs are labeled by ±1 and $\langle A_i \ldots A_j\rangle$ are the standard correlators $\langle A_i \ldots A_j\rangle = \Pr(A_i \ldots A_j = +1) - \Pr(A_i \ldots A_j = -1)$.

B. Randomness

We follow [4,5] and adopt an operational approach where randomness is related to the probability of correctly guessing the outcome of some joint measurement, $\mathbf{x} = (x_1,x_2,\ldots,x_N)$. We use the *guessing probability*, $P_G(P;\mathbf{x}) = \max_{\mathbf{a}} P(\mathbf{a}|\mathbf{x})$, where $\mathbf{a} = (a_1,a_2,\ldots,a_N)$. The proper measure of intrinsic randomness requires optimizing over all realizations of the observed correlations $G(P;\mathbf{x}) = \max \sum_i \lambda_i P_G(P_i;\mathbf{x})$, where the maximization is over all convex decompositions $P(\mathbf{a}|\mathbf{x}) = \sum_i \lambda_i P_i(\mathbf{a}|\mathbf{x})$. It is convenient to express the randomness in bits with the *min-entropy*, $H_\infty(P;\mathbf{x}) = -\log_2 G(P;\mathbf{x})$. Note that in a general (N,M,d) scenario there can be at most $\log_2 d$ bits of local and $N\log_2 d$ bits of global randomness at any given round of the experiment. For a given $\mathbf{x} = \mathbf{x_0}$, maximal randomness is obtained from a uniform distribution $P(\mathbf{a}|\mathbf{x_0}) = 1/d^N, \forall\, \mathbf{a}$. When $d=2$, this occurs if, and only if, all the correlators appearing in Eq. (1) are zero.

III. RANDOMNESS FROM BELL TESTS

While not detailed here, we implicitly work in the framework of randomness expansion introduced in Refs. [4,7], where randomness is certified from Bell violations. The measurements of the Bell test are chosen using an initial seed of random bits. These initial random bits are assumed to be uncorrelated to the particles measured in the Bell test (for a discussion on how the relaxation of this assumption may affect the randomness certified by the Bell violation, see [8] and references therein). However, as explained in [4], the bits used to choose the measurements can be highly biased so that (i) most of the time only a given combination of measurements is performed, yet (ii) the remaining measurements are performed a sufficient number of times to reliably estimate the Bell violation (we refer interested readers to [4] for details). This is why we say here that a Bell test generates maximal randomness whenever there exists a combination of measurements whose outputs are maximally random. As mentioned, the choice of measurements in the Bell test can be arbitrarily biased to these specific measurements without affecting their randomness.

Maximal randomness certification. We assume in what follows that the quantum distribution attaining the maximal quantum violation of the Bell inequality is unique (discussed later). Under this assumption, we show how symmetries in the Bell inequality under permutation of measurement results, possibly together with permutations of measurement settings, lead to maximal randomness. Our method, then, can be summarized as follows: *uniqueness plus symmetries implies maximal randomness.*

IV. METHODS AND RESULTS

In this section, we provide several examples of Bell inequalities and the corresponding certified randomness that demonstrate the applicability of our simple criterion.

A. Certifying maximal local randomness

First of all, it is worth re-examining the examples mentioned in the introduction. Consider again the CHSH inequality and denote by $\mathcal{P}^*$ the distribution attaining its maximal quantum violation, namely $I_{\text{CHSH}}(\mathcal{P}^*) = 2\sqrt{2}$. Note that in this case, this distribution is known to be unique [9]. The symmetry transformation $\mathcal{T}_s$: $a_{1,2} \mapsto -a_{1,2}$ and $b_{1,2} \mapsto -b_{1,2}$ flips the signs of all the one-body correlators, $\langle A_i\rangle$ and $\langle B_j\rangle$, while keeps unchanged all two-body correlators, $\langle A_i B_j\rangle$. Applying $\mathcal{T}_s$ to P^* we obtain the distribution $\mathcal{T}_s(\mathcal{P}^*) = \mathcal{P}^{**}$ with

$$\langle A_i\rangle^{**} = -\langle A_i\rangle^*, \quad \langle B_j\rangle^{**} = -\langle B_j\rangle^*, \tag{2}$$

and that also maximally violates CHSH. Because of the uniqueness of the distribution, $\mathcal{P}^* = \mathcal{P}^{**}$ and all one-body correlators (2) must be zero, which certifies one bit of *local* randomness (for both parties). Moving to I_η, the transformation $a_2 \mapsto -a_2$, $B_1 \leftrightarrow B_2$ flips the value of $\langle A_2\rangle$ without changing the value of I_η. Under the assumption of uniqueness, this proves that the setting A_2 is fully random. A little thought shows that it is impossible to construct similar transformations for the other local measurements. Our argument, then, easily reproduces the known results for these two inequalities.

As mentioned, our method applies to any Bell inequality with symmetries. The previous argument for the CHSH inequality can be easily generalized to all the chained inequalities of Refs. [10] and [11]. Under the assumption of uniqueness, these inequalities always certify one dit of local randomness. The chained Bell inequalities can be compactly represented as [11]:

$$C_d^M = \sum_{i=1}^{M}\langle[A_i - B_i]_d\rangle + \langle[B_i - A_{i+1}]_d\rangle \geqslant d-1, \tag{3}$$

where A_i, $B_j \in \{0,\ldots,d-1\}$ are measurement choices for Alice and Bob and $A_{M+1} = A_1 + 1$. The square brackets denote sum modulo d.

Let P attain the quantum maximum of C_d^M. The transformation $\mathcal{T}$: $a_i \mapsto a_i + 1$ and $b_i \mapsto b_i + 1$ for every i changes the value of the marginal distributions of Alice and Bob but leaves the terms in C_d^M unchanged. Applying $\mathcal{T}$ to P and assuming it to be unique, it follows that all local distributions of Alice and Bob must be uniform. In other words, the chained inequality certifies $\log_2 d$ bits of local randomness for every measurement by each party.

B. Certifying maximal global randomness

A natural open question is whether there exist Bell tests in the (N,M,d) scenario that allow certifying the maximal possible randomness, namely $N \log_2 d$ bits. Some progress on this question was obtained in Ref. [5], where it was shown how to get arbitrarily close to two random bits in the $(2,2,2)$ scenario. However the corresponding correlations are nonresistant to noise. Here, we show how our method can be easily applied to design Bell tests allowing *exact* maximal randomness certification in a *robust* manner.

We start with the bipartite case. Maximal global randomness is impossible in the CHSH case, as at the point of maximal violation all settings are correlated. Maximal global randomness, however, can be certified as soon as more measurements are included. For instance, consider adding a third measurement on Bob's side and the expression [12]

$$I_{\mathrm{CHSH}} + \langle A_1 B_3 \rangle. \tag{4}$$

Clearly, the maximal quantum violation is $2\sqrt{2}+1$ as the two terms above can be maximized independently (the classical value is 3). At the point of maximal violation, the two settings on Alice's side should maximize the CHSH violation and, thus, be orthogonal and act on a maximally entangled state. The third setting on Bob's should be parallel to A_1 and thus $\langle A_2 B_3 \rangle$ is equal to zero. This can alternatively be understood using our argument, as the transformation $\mathcal{T}$: $a_2 \mapsto -a_2$, $B_1 \leftrightarrow B_2$ leaves the inequality unchanged while flipping the sign of $\langle A_2 B_3 \rangle$. In this case, uniqueness is known to hold too and, therefore, $\langle A_2 B_3 \rangle = 0$. A similar argument holds for the term $\langle A_1 B_3 \rangle$ when the setting correlated with B_3 in inequality (4) is A_2 instead of A_1.

More in general, consider the chained inequalities for an odd number of two-outcome measurements. We move to the notation $a_i, b_j = \pm 1$ and reexpress (3) as follows:

$$C_2^M = \left| \sum_{i=1}^{M} \langle A_i B_i \rangle + \sum_{i=1}^{M-1} \langle A_{i+1} B_i \rangle - \langle A_1 B_M \rangle \right|, \tag{5}$$

where A_i, $B_j = \pm 1$. Let $M = 2k+1$. As above, we consider a transformation leaving C_2^M unchanged but under which $\langle A_1 B_{k+1} \rangle \mapsto -\langle A_1 B_{k+1} \rangle$. Such a transformation is: $\mathcal{T}$: $a_1 \mapsto -a_1$, $B_{1+i} \leftrightarrow B_{M-i}$, $A_{2+i} \leftrightarrow A_{M-i} \forall i \;\; 0 \leqslant i \leqslant k-1$. Assuming that the distribution maximally violating (5) is unique leads to $\langle A_1 B_{k+1} \rangle = 0$. The previous results show that $\langle A_1 \rangle = 0 = \langle B_{k+1} \rangle$. These together certify two bits of global randomness for (A_1, B_{k+1}). Similar arguments certify maximal randomness in all inputs of the form $(A_l, B_{k+l}) \;\forall\; 1 \leqslant l \leqslant M$. Analogous to the case for CHSH, maximal randomness cannot be certified for those measurement combinations appearing in the chained inequality, as they display nonzero correlations. The previous results rely on the assumption of uniqueness, which is unknown for the case of the chained inequality with $M > 2$. We then follow [4] and apply the techniques in Ref. [13] to get an upper bound on the randomness of (A_1, B_2) for the chained inequality with three measurement settings. The obtained results corroborate the presence of maximal global randomness, up to numerical accuracy.

The results for the chained inequality illustrate the power of our method. It is intuitive that for a measurement to be fully random, its correlator should not appear in the inequality, as at the point of maximal violation the correlators in the inequality are expected to have a nonzero value. For the chained inequality, $M^2 - 2M$ of all possible correlators do not appear in the inequality. Yet, the known realization of the maximal quantum violation, consisting of measurements equally distributed on an equator and acting on a two-qubit maximally entangled state, implies that only M of these $M^2 - 2M$ correlators can be zero, precisely those detected by our symmetry argument.

We now move to the multipartite case. More precisely, we consider the Mermin inequalities [14] and prove that they allow certifying up to N bits of global randomness for arbitrary odd N. Mermin inequalities of N parties are defined recursively as

$$M_N = \tfrac{1}{2} M_{N-1}(A_N + A'_N) + \tfrac{1}{2} M'_{N-1}(A_N - A'_N), \tag{6}$$

where M_2 is the CHSH inequality and M'_{N-1} is obtained from M_{N-1} by exchanging all A_j and A'_j.

Let M_N denote a Mermin inequality of $N = 2J+1$ sites. Party i, with $i = 1,\ldots,N$ has a choice between two dichotomic measurements, A_i and A'_i. It is easily checked that for odd N, M_N contains only full correlators with an odd number of primes. We show, using symmetry arguments, that at the point of maximal quantum violation every correlator $\langle A_i \ldots A_j \rangle$ (involving an arbitrary number of measurements) that does not appear in M_N is identically zero. This automatically implies that any combination of N settings not appearing in the inequality define N random bits.

To see this, first take a specific N-body correlator not appearing in M_N, $\langle X_1 X_2 \ldots X_N \rangle$, where $X_i = A_i$ or A'_i but such that the total number of primed A is an even number. Denote the outcome of X_i by x_i. Choose any of the parties, say the first one, and denote by $\mathrm{Corr}(X_1)$ the set of all correlators of arbitrary size containing X_1 plus possibly other settings X_i with $i > 1$. We would like to show that every element belonging to $\mathrm{Corr}(X_1)$ is equal to zero for the unique distribution maximally violating the inequality. Let us consider the transformation $\mathcal{S}_1 : \{x_1 \mapsto -x_1$, and x_j untouched $\forall j > 1\}$. This maps $\mathrm{Corr}(X_1) \mapsto -\mathrm{Corr}(X_1)$. The terms in M_N remains unchanged if we complement $\mathcal{S}_1$ with $\mathcal{S}'_1 : \{x'_j \mapsto -x'_j \forall j > 1\}$, where we use $(A'_i)' = A_i$. In fact, note that for the original even primed term we started with, $\mathcal{S}'_1 \circ \mathcal{S}_1 \langle X_1 X_2 \ldots X_N \rangle = -\langle X_1 X_2 \ldots X_N \rangle$. The Mermin inequality consists only of odd-parity full correlators. Any such term can be obtained from $\langle X_1 X_2 \ldots X_N \rangle$ by swapping inputs at an odd number of places. However, the transformation $\mathcal{S}'_1 \circ \mathcal{S}_1$ is such that at every site, either the outcome of A_i or A'_i flips sign but not both. Hence, $\mathcal{S}'_1 \circ \mathcal{S}_1$ applied on any correlator obtained by an odd number of local swaps on $\langle X_1 X_2 \ldots X_N \rangle$ gains an additional factor of -1 *for each swapped site* relative to $\mathcal{S}_1' \circ \mathcal{S}_1 \langle X_1 X_2 \ldots X_N \rangle$. Thus, M_N remains unchanged. It remains to study the effect of $\mathcal{S}_1'$ on $\mathrm{Corr}(X_1)$. Since $X_j' \notin \mathrm{Corr}(X_1)$, this set is unmodified under $\mathcal{S}_1'$, so $\mathcal{S}_1' \circ \mathcal{S}_1$ maps $\mathrm{Corr}(X_1) \mapsto -\mathrm{Corr}(X_1)$. We then conclude from uniqueness that all the correlators in $\mathrm{Corr}(X_1)$ must be zero. The same argument can be run for any party, and then for any full correlator with an even number of primes, proving the result.

It is worth mentioning that similar arguments when applied to the Mermin inequality for even N allow certifying $(N-1)$ bits of randomness.

C. The uniqueness assumption

Our method requires a unique quantum distribution attaining the maximal violation of a given Bell inequality. For some cases, such as Mermin $(N,2,2)$, uniqueness has been proven [15,16] analytically. For the chained inequality, we have numerical evidence using the techniques from [13] that the distribution saturating it is unique in the $(2,3,2)$ and $(2,4,2)$ cases. Beyond these specific cases, a proof of uniqueness valid for *any* Bell inequality is impossible, as it is known that uniqueness does not hold for all Bell inequalities. Lifted Bell inequalities [17] or Bell inequalities with no quantum violation [18] are examples of inequalities that have more than one quantum realization of the maximal violation.

Despite uniqueness not holding for all Bell inequalities, from a geometric point of view, it is natural to expect that the maximal violation of a *generic* Bell inequality is attained by a unique point. The set of quantum correlations defines a convex set in the space of probability distributions $P(a_1,\ldots,a_N|x_1,\ldots,x_N)$. A Bell inequality is a hyperplane in this space. The maximal quantum violation corresponds to the point in which the hyperplane, i.e., the Bell inequality, becomes tangent to the set of quantum correlations. Since the set is convex, this point is expected to be unique, in general.

The previous considerations, however, do not apply to our case, as our argument holds for inequalities that are not generic, but symmetric under permutation of some of the measurement results, possibly assisted by permutations of measurements. At this point, it is worth noting that in all the previous discussion we did not make use of any quantum property. In fact, the set of classical correlations is also convex and, thus, a generic hyperplane is expected to become tangent at a unique extremal point, see Fig. 1(a). However, randomness cannot be certified by classical correlations. The reason is that in the classical case, any symmetry under permutations of the results, necessary in our argument, can be immediately used to construct another extremal and deterministic point saturating the inequality as in Fig. 1(b). Therefore, in the classical case, uniqueness and the required symmetries are never satisfied simultaneously. In the quantum case, it is not expected that symmetries break the uniqueness of the maximal violation. The reason is that the set of quantum correlations is convex, but, contrary to the classical case, not a polytope; that is, it has an infinite number of extreme points, see Fig. 1(b).

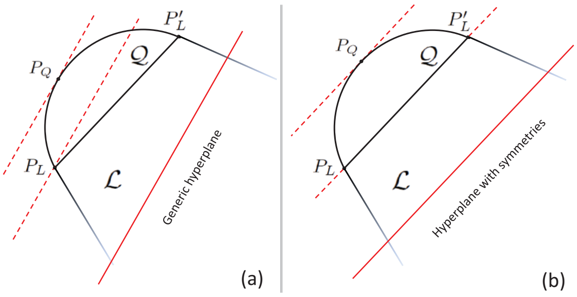

FIG. 1. (Color online) (a) A generic hyperplane generally does not have symmetries and has a unique maximum in both the sets of local and quantum correlations. (b) A hyperplane with symmetries (such as the one corresponding to the CHSH inequality) precludes uniqueness in the local set but still allows for a unique maximum in the quantum set. The reason is that the set of quantum correlations, contrary to its classical counterpart, has an infinite number of extreme points.

V. CONCLUSIONS

We have presented a very simple argument explaining why and when measurement outcomes in a Bell test are expected to provide maximally random bits. Our method does not constitute a formal proof of randomness as it requires an assumption of the uniqueness of the maximal quantum violation of a Bell inequality. We however provided a geometrical intuition of why this assumption may hold in most cases. We believe this intuition explains why our method works so well, as we are not aware of any Bell test leading to maximal randomness, local or global, that cannot be explained using our method. Our method is also useful to design good Bell tests for randomness generation, as the insight provided by it can later be confirmed in a rigorous way using the techniques from [4,13]. In fact, using our method, we easily demonstrated the existence of Bell tests allowing maximal global randomness certification.

ACKNOWLEDGMENTS

We acknowledge support from the ERC Starting Grant PERCENT, the EU Project QCS, and the Spanish MICIIN through an FPI grant and projects FIS2010-14830, Explora-Intrinqra and CHIST-ERA DIQIP.

[1] J. S. Bell, Physics **1**, 195 (1964).

[2] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[3] R. Gallego, L. Masanes, G. de la Torre, C. Dhara, L. Aolita, and A. Acín, Nat. Commun. **4**, 2654 (2013).

[4] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature **464**, 1021 (2010).

[5] A. Acín, S. Massar, and S. Pironio, Phys. Rev. Lett. **108**, 100402 (2012).

[6] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[7] R. Colbeck, Ph.D. thesis, University of Cambridge, 2007. Also available as arXiv:0911.3814.

[8] Dax Enshan Koh, Michael J. W. Hall, Setiawan, James E. Pope, Chiara Marletto, Alastair Kay, Valerio Scarani, and Artur Ekert, Phys. Rev. Lett. **109**, 160404 (2012).

[9] B. S. Tsirel'son, J. Sov. Math. **36**, 557 (1987).

[10] S. L. Braunstein and C. M. Caves, Ann. Phys. **202**, 22 (1990).

[11] J. Barrett, A. Kent, and S. Pironio, Phys. Rev. Lett. **97**, 170409 (2006).

[12] We thank V. Scarani for pointing out this inequality.

[13] M. Navascués, S. Pironio, and A. Acín, Phys. Rev. Lett. **98**, 010401 (2007); New J. Phys. **10**, 073013 (2008).

[14] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

[15] R. F. Werner and M. M. Wolf, Phys. Rev. A **64**, 032112 (2001).

[16] T. Franz, F. Furrer, and R. F. Werner, Phys. Rev. Lett. **106**, 250502 (2011).

[17] S. Pironio, J. Math. Phys. **46**, 062112 (2005).

[18] M. L. Almeida, J. D. Bancal, N. Brunner, A. Acin, N. Gisin, and S. Pironio, Phys. Rev. Lett. **104**, 230404 (2010); R. Augusiak, J. Stasińska, C. Hadley, J. K. Korbicz, M. Lewenstein, and A. Acín, *ibid.* **107**, 070401 (2011).