Time-domain resonances and the ultimate fate of a decaying quantum state

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We derive an analytical expression for the propagation of a quantum particle at asymptotic long distances and times from the potential where the particle was initially confined. We obtain that the particle is described by an evolving resonance in time domain that possesses a peaked shape characterized by the resonance parameters of the dominant resonance coefficient of the decaying wave function. The above situation corresponds to an unexplored postexponential regime that represents the ultimate fate of a decaying particle.

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I. INTRODUCTION

The subject of quantum decay is as old as quantum mechanics. Around two years after Schrödinger published his papers on wave mechanics, Gamow [1,2] considered the problem of α decay in atomic nuclei and obtained an analytical expression that describes the disintegration rate of α particles with time, i.e., the *exponential decay law*. The above paradigmatic case exemplifies a distinguished class of decay problems that model the full Hamiltonian of the system. This is usually the case when the decay originates from tunneling through a classically forbidden region as in the decay of a particle out of a potential having a barrier and refers to the type of quantum decay problems that we shall consider here. The above approach differs from approximate treatments where the Hamiltonian is separated into a part with stationary states and a part responsible for the decay, usually treated to some order of perturbation, an approach initiated by Weisskopf and Wigner [3,4], that has set another paradigm for treating decay problems in which an initial discrete (bound) state is coupled to a perturbing continuum of states to form a resonance that thereafter decays exponentially.

Subsequent theoretical work pointed out the approximate validity of the exponential decay law. Khalfin demonstrated that if the energy spectra E of the system is bounded from below, i.e., $E \in (0,\infty)$, the exponential decay law cannot hold at long times [5]. At short times there is also a departure from the exponential behavior that, however, has a dependence on the initial state of the problem [6–8]. The experimental verification of the above departures from the exponential decay law remained elusive for decades but has been finally corroborated by experiment [9,10]. In particular, the Zeno effect [11], which is a relevant consequence of the short-time deviation from the exponential decay law, has been also experimentally verified [12,13]. The above results contradict theoretical claims made in the last century that asserted that the interaction with the environment would force the validity of exponential decay at all times [14].

The analytical expression for the exponential decay derived by Gamow requires us to impose outgoing boundary conditions to the solutions to the Schrödinger equation of the problem. As a consequence the energy eigenvalues are complex and the corresponding decaying wave functions grow exponentially with distance [1,2,15]. This implies that the so-called resonant states cannot be normalized in the usual sense and also that the usual rules concerning orthogonality and completeness do not apply. The above old developments, however, have evolved over the years into a consistent theoretical framework which is particularly useful for the description of time-dependent problems [16,17]. It may be proved that resonant states may be obtained also from the residues at the complex poles of the outgoing Green's function to the problem which also provide the normalization rule for these states. It is worth emphasizing that in this approach one only needs to know the resonant states along the internal region and at the boundary value of the interaction potential. The time dependence of the decaying solution is given as a pole expansion [16,17] in which each term is given by the product of an expansion coefficient, a resonant state, and a Moshinsky function [18]. Each expansion coefficient represents the overlap between the initial state and the corresponding resonant state. It might be of interest to point out that the Moshinsky functions are characteristic of the description of many transient phenomena [19].

In the present paper we analyze the time evolution of the decaying wave function along the external region of an interaction of finite range using the formalism of resonant states. The exact analytical decaying solution is used to demonstrate that at asymptotically long distances and times, the wave function becomes an evolving structure that has a peaked shape in time domain that depends only on the resonance parameters of the largest coefficient in the resonance expansion of the decaying wave function. This solution corresponds to the ultimate fate of the transient decaying quantum state. The above result is exemplified for the exact solvable problem of the δ-shell potential, and is corroborated by solving numerically the time-dependent Schrödinger equation around the above structure.

To the best of our knowledge no previous work on decay has addressed the investigation on the asymptotic behavior of the decaying wave function mentioned above. Most treatments on decay refer to the nondecay or survival probability, which

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yields the probability that at time t the decaying particle remains in the initial state, as discussed in the 1978 review paper by Fonda *et al*. [14]. Another related quantity that has been considered in decay problems, is the nonescape probability, which provides the probability that at time t the decaying particle remains within the interaction region. Since the initial state is confined within the internal region of the potential, both the survival and the nonescape probabilities involve, respectively, integration along the internal region and hence are functions that depend only on time. One should also mention the monograph by Goldberger and Watson [20], that uses the resolvent method to discuss the long-time nonexponential behavior of the decay law of a sharp isolated resonance, and the large body of studies concerned with the decay of quantum systems involving different numbers of channels. These studies, which are based on the statistical assumptions of random matrix theory and are of relevance in studies of quantum chaotic systems [21,22], lie beyond the scope of the present work.

Following the work by García-Calderón on resonant states and the decay process [23] and with Rubio on transient phenomena in resonant tunneling [24], both involving linear combinations of resonant states and Moshinsky functions, there appeared in the literature an approach by van Dijk and Nogami where the decaying wave function is also written as a linear combination of Moshinsky functions but without using resonant states [25,26]. The authors of Refs. [25,26] argue that the exact decaying wave function always contains incoming waves, a situation that is prevented when using outgoing boundary conditions, and affirm, consequently, that the resonant-state approach is not exact. However, these authors miss the point that using the outgoing boundary condition with the poles that seat on the third quadrant of the k plane yields in fact, by time reversal considerations, incoming wave contributions, a fact that invalidates their argument. Further analysis has demonstrated that the resonant-state formalism provides an exact description of decay [27]. One should also comment that the controversial issue regarding the long-time behavior of the nonescape probability mentioned in Ref. [26] has been settled down as discussed in Ref. [28] and will not be repeated here. At long times the nonescape probability goes as t^{-3} and not as t^{-1} as was claimed in Ref. [29].

Although the resonant-state approach is non-Hermitian, it yields exactly the same results as a Hermitian description based on the continuum wave functions to the problem [27,30]. The essential difference, and also the main advantage of the resonant-state approach, is that the non-Hermitian formulation provides explicit analytical expressions for both exponential and nonexponential contributions to decay, whereas the Hermitian description corresponds to a black-box type of calculation that requires numerical integration over the momenta at every instant of time and hence it is difficult to foresee its behavior as a function of time. The notion of resonant state has been particularly useful in understanding the physics of decay in multibarrier quantum systems [31,32].

It is worth mentioning a recent work that explores the exponential to postexponential transition of quantum decay at long distances which is based on a *source model* [33]. This model consists of a simple exponentially decaying source (with complex frequency) that mimics the behavior of the decay of a single resonance and allows to derive a time scale that characterizes the transition from the exponential to the postexponential regime. An interesting contribution of that work is that the observability of the above transition increases with the position where the detector is placed up to a critical distance beyond which exponential decay is no longer observed. We confirm the above results with our multiresonance formalism. However, we have gone beyond, at asymptotically long times and distances, to obtain the ultimate fate of the decaying particle.

The paper is organized as follows: in Sec. II we recount the main equations for the description of the decay of a quantum state using the resonant-state formalism and we consider the steepest descent method to derive an exact analytical formula that provides the main contribution to the decaying state at very long distances and times. In Sec. III we recall the formulation of the description of a decaying particle using the continuum wave functions. In Sec. IV we analyze the dynamics of the decaying probability density along the external region using the δ-shell potential. Finally, in Sec. V we present the conclusions.

II. DYNAMICS OF THE DECAYING WAVE FUNCTION USING RESONANT STATES

The theoretical treatments of quantum decay refer to the time evolution, $|\psi_t\rangle = \exp(-iHt/\hbar)|\psi_0\rangle$, of an initial state $|\psi_0\rangle$ in a system characterized by a Hamiltonian H. We shall refer to a description of the decay process that involves real potentials of arbitrary shape that vanish beyond a distance. This is well justified since most effective potentials in physics are of finite range, and refer to decaying states which initially are confined within the interaction region. The above conditions are commonly found in quantum systems designed artificially [34,35]. A relevant feature of these systems is that at very low temperatures the decay process is essentially coherent (elastic). We assume that such is the case here. One may then exploit the analytical properties of the outgoing Green's function to the problem on the whole complex k plane, where it possesses an infinite number of poles. These poles are in general simple and are distributed in a well-known fashion [36]. The above considerations have led to a formulation of the time evolution of decay in terms of a purely discrete expansion involving the residues at the poles of the outgoing Green's function to the problem. These residues are proportional to the resonant states of the system that satisfy the Schrödinger equation with outgoing boundary conditions [16,17,29].

Let us consider the time evolution of decay of an initial wave function $\Psi(r,0)$ confined initially, at $t = 0$, along the internal region of a spherically symmetric potential of finite range, i.e., $V(r) = 0$ for $r > a$, where, for the sake of simplicity, we restrict the discussion to s waves. The units employed here are $\hbar = 2m = 1$, with m being the mass of the decaying particle. Hence the energy of the particle reads $E = k^2$, with k the corresponding wave number.

The time-dependent wave function $\Psi(r,t)$ may be written in terms of the retarded Green's function of the problem $g(r,r';t)$ and the initial state $\Psi(r,0)$ as

$$\Psi(r,t) = \int_0^a g(r,r';t)\Psi(r',0)\,dr', \quad t > 0. \tag{1}$$

The retarded Green's function $g(r,r';t)$ may be expressed, using Laplace transform techniques, in terms of the outgoing Green's function $G^+(r,r';k)$ of the problem, namely,

$$g(r,r';t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} G^+(r,r';k)e^{-ik^2t}\,2k\,dk. \tag{2}$$

Using the analytical properties of $G^+(r,r';k)$ one may then calculate the retarded Green's function $g(r,r';t)$ either as an expansion in terms of resonant states or in terms of the continuum wave functions to the problem.

For propagation along the external region of the potential it is convenient to write the outgoing Green's function as [16,17]

$$G^+(r,r';k) = G^+(a,r';k)e^{ik(r-a)}, \quad r' < a, \quad r \geqslant a. \tag{3}$$

Substitution of (3) into (34), and the resulting expression into (1), allows us then to express the time-dependent solution as

$$\Psi(r,t) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\Phi(k)e^{ik(r-a)}\,e^{-ik^2t}\,dk, \tag{4}$$

where we have defined

$$\Phi(k) = 2k\int_0^a \Psi(r',0)G^+(a,r';k)\,dr'. \tag{5}$$

The above expression is very convenient because then we may expand $G^+(a,r';k)$ in terms of the resonant states $\{u_n(r)\}$ and the corresponding complex poles $\{\kappa_n = \alpha_n - i\beta_n\}$ of the problem. The expansion reads [17]

$$G^+(a,r';k) = \frac{1}{2k}\sum_{n=-\infty}^{\infty}\frac{u_n(r')u_n(a)}{k-\kappa_n}, \quad r' < a. \tag{6}$$

It must be recalled that Eq. (6) holds provided the resonant states are normalized according to the condition [17,37]

$$\int_0^a u_n^2(r)dr + i\frac{u_n^2(a)}{2\kappa_n} = 1. \tag{7}$$

These states fulfill the closure relationship [16,17]

$$\frac{1}{2}\sum_{n=-\infty}^{\infty} u_n(r)u_n(r') = \delta(r-r'), \tag{8}$$

and the sum rules

$$\sum_{n=-\infty}^{\infty} u_n(r)u_n(r')\kappa_n = 0; \qquad \sum_{n=-\infty}^{\infty}\frac{u_n(r)u_n(r')}{\kappa_n} = 0. \tag{9}$$

Equations (8) and (9) hold provided $(r,r')^\dagger \leqslant a$, a notation that implies that the expansions do not hold for $r = r' = a$.

Substitution of (6) into (5) and the resulting expression into (4) allows us to express the time-dependent decaying solution as the resonant expansion [16,17]

$$\Psi(r,t) = \sum_{n=-\infty}^{\infty} C_n\,u_n(a)\,M(y_n), \quad r \geqslant a, \tag{10}$$

where the Moshinsky functions $M(y_n)$ are defined as

$$M(y_n) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik(r-a)}e^{-ik^2t}}{k-\kappa_n}\,dk = \frac{1}{2}\,e^{i(r-a)^2/4t}\,w(iy_n), \tag{11}$$

and the functions $w(iy_n) = \exp(y_n^2)\mathrm{erfc}(y_n)$ stand for the complex error function [38], the argument y_n is given by

$$y_n = (4it)^{-1/2}\,[(r-a) - 2\kappa_n t], \tag{12}$$

and the coefficients C_n by the expressions

$$C_n = \int_0^a \Psi(r',0)\,u_n(r')\,dr'. \tag{13}$$

It is of interest to see, in view of (8), that the coefficients C_n fulfill the relation [17]

$$\mathrm{Re}\left\{\sum_{n=1}^{\infty} C_n\bar{C}_n\right\} = 1, \tag{14}$$

where $\bar{C}_n$ follows by replacing in Eq. (13) $\Psi(r',0)$ by its conjugate.

Equation (10) tell us that the evolving decaying wave function carries out information on the resonant structure of the system. Notice that each resonant level of the system contributes with a weight or strength C_n and, in view of (14), that implies that some resonant terms might be more relevant than others for the behavior of the time evolving solution.

A. Behavior of the decaying state at a fixed distance and long times

It might be of interest to recall the expression for the wave function at a fixed value of the distance $r = r_0$ and very long times. This follows by considering the analytical properties of the complex error function and yields the expression [17],

$$\Psi(r_0,t) = \sum_{n=1}^{\infty} C_n u_n(a)e^{i\kappa_n(r_0-a)}e^{-i\mathcal{E}_n t}e^{-\Gamma_n t/2} - \frac{i}{2(\pi i)^{1/2}}\mathrm{Im}\left\{\sum_{n=1}^{\infty}\frac{C_n u_n(a)}{\kappa_n^3}\right\}\frac{1}{t^{3/2}}, \quad r \geqslant a, \tag{15}$$

where we have used that the complex energy $\kappa_n^2 = \mathcal{E}_n - i\Gamma_n/2$. Notice that since $\kappa_n = \alpha_n - i\beta_n$, one may write the resonance energy $\mathcal{E}_n$ and the decay width Γ_n, respectively, as

$$\mathcal{E}_n = \alpha_n^2 - \beta_n^2, \quad \Gamma_n = 4\alpha_n\beta_n. \tag{16}$$

B. Behavior of the decaying state at very long distances and times

One may ask what happens to the wave function as both the distance r and the time t increase to very large values. In such a case it is convenient to make use of the *phase stationary phase* method [39]. This involves applying the condition on the argument of the exponential in Eq. (4), $d[k(r-a) - k^2t]/dk = 0$, which determines the value,

$$k_s \equiv \frac{r-a}{2t}. \tag{17}$$

Then, performing a first-order Taylor expansion around $k = k_s$ on the function $\Phi(k)$ and integrating the resulting expression allows us to write Eq. (4) as

$$\Psi(r,t) \approx \sqrt{\frac{i}{4\pi t}}\,e^{ik_s^2t}\,\Phi(k_s). \tag{18}$$

Notice that substitution of Eq. (6) into Eq. (5) allows to write, in view of (13), a resonant expansion for $\Phi(k_s)$, that is

$$\Phi(k_s) = \sum_{n=-\infty}^{\infty} \frac{C_n u_n(a)}{k_s - \kappa_n}. \tag{19}$$

We make the common assumption that the initial state $\Psi(r,0)$ overlaps strongly with a resonant state $u_\ell(r)$ of the system, so that, in view of Eq. (14), $\mathrm{Re}\{C_\ell^2\} \approx 1$. This allows us to write (19) as

$$\Phi(k_s) \approx \frac{C_\ell\, u_\ell(a)}{k_s - \kappa_\ell}. \tag{20}$$

Hence, we may write Eq. (18) as

$$\Psi(r,t) \approx \sqrt{\frac{i}{4\pi t}}\, e^{ik_s^2 t}\, \frac{C_\ell\, u_\ell(a)}{k_s - \kappa_\ell}. \tag{21}$$

The above requires us to choose adequately, in view of (17), the values of r and t. The probability density $|\Psi(x,t)|^2$ calculated from Eq. (21) yields the expression

$$|\Psi(r,t)|^2 \approx \frac{1}{2\pi t}\, |C_\ell|^2 I_\ell \frac{\beta_\ell}{(k_s - \alpha_\ell)^2 + \beta_\ell^2}, \tag{22}$$

where we have used that $\kappa_\ell = \alpha_\ell - i\beta_\ell$, and the expressions [17]

$$\beta_\ell = \frac{|u_\ell(a)|^2}{2I_\ell}, \quad I_\ell = \int_0^a |u_\ell(r)|^2\, dr. \tag{23}$$

Notice, recalling Eq. (17), that for a fixed time $t = t_0$, the above expression exhibits a Lorentzian shape as a function of κ_s. On the other hand, for a fixed distance $r = r_0$, a simple manipulation allows us to write Eq. (22) as a function of time t that consists of a Lorentzian formula multiplied by t, namely,

$$|\Psi(r_0,t)|^2 \approx |C_\ell|^2 \frac{I_\ell}{(r_0 - a)}\, \frac{1}{\pi}\left[\frac{(\gamma_\ell/2)}{(t - t_\ell)^2 + (\gamma_\ell/2)^2}\right] t, \tag{24}$$

where $\gamma_\ell = \beta_\ell(r_0 - a)/(\alpha_\ell^2 + \beta_\ell^2)$, and $t_\ell = \alpha_\ell \gamma_\ell/(2\beta_\ell)$.

One sees therefore that the resonant pole corresponding to the largest expansion coefficient in Eq. (10) is responsible for the peaked behavior exhibited by Eq. (24) at very long times and distances from the decaying system.

III. DECAYING WAVE FUNCTION USING THE STATES OF THE CONTINUUM

We find it of interest to confront the above findings with the expression for the time evolution of the decaying wave function using the continuum wave solutions $\psi^+(k,r)$ to the problem, which reads [17]

$$\Psi(r,t) = \int_0^\infty C(k)\psi^+(k,r)e^{-ik^2 t}\, dk, \tag{25}$$

where the coefficient $C(k)$ is given by

$$C(k) = \int_0^a \psi^{+*}(k,r')\Psi(r',0)\, dr'. \tag{26}$$

Outside the range of the potential the continuum wave solutions may be written as

$$\psi^+(k,r) = \sqrt{\frac{2}{\pi}}\, \frac{i}{2}[e^{-ikr} - \mathbf{S}(k)e^{ikr}], \quad r > a, \tag{27}$$

where $\mathbf{S}(k)$ stands for the $\mathbf{S}$ matrix to the problem.

IV. EXAMPLES

Here we consider as an example of the dynamics of the decaying wave function [Eq. (10)] along the external region, the s-wave δ-shell potential,

$$V(r) = \lambda\delta(r - a), \tag{28}$$

with λ the intensity of the potential, and the initially confined state at $t = 0$,

$$\Psi(r,0) = \sqrt{\frac{2}{a}}\, \sin\left(\frac{q\,\pi}{a}\right) r, \tag{29}$$

where $q = 1,2,3,\ldots$. We choose the parameters $\lambda = 100$ and $a = 1$. The above model is known to many authors as the Winter model [40].

Resonant states obey the Schrödinger equation of the problem with the complex energy eigenvalues $\kappa_n^2 = \mathcal{E}_n - i\Gamma_n/2$, namely

$$\frac{d^2}{dr^2} u_n(r) + \left[\kappa_n^2 - V(r)\right] u_n(r) = 0. \tag{30}$$

The solutions to Eq. (30) with the potential given by (28) read

$$u_n(r) = \begin{cases} A_n\, \sin(\kappa_n r), & r \leqslant a, \\ B_n\, e^{i\kappa_n r}, & r \geqslant a. \end{cases} \tag{31}$$

From the continuity of the above solutions and the discontinuity of its derivatives with respect to r (due to the δ-function interaction) at the boundary value $r = a$, it follows that the κ_n's satisfy the equation

$$2i\kappa_n + \lambda(e^{2i\kappa_n a} - 1) = 0. \tag{32}$$

For $\lambda > 1$ one may write the approximate analytical solutions to Eq. (32) as [16,17]

$$\kappa_n \approx \frac{n\pi}{a}\left(1 - \frac{1}{\lambda a}\right) - i\,\frac{1}{a}\left(\frac{n\pi}{\lambda a}\right)^2. \tag{33}$$

Using the above expression as an initial value in the expression of the Newton-Rapshon method,

$$\kappa_n^{r+1} = \kappa_n^r - F\left(\kappa_n^r\right)\big/\dot{F}\left(\kappa_n^r\right), \tag{34}$$

where $\dot{F} = [dF/dk]_{k=\kappa_n}$, one may obtain κ_n with the desired degree of approximation by iteration.

The normalization coefficient for resonant states may be evaluated by substitution of Eq. (31), for $r \leqslant a$, into Eq. (7), to obtain the analytical expression

$$A_n = \left[\frac{2\lambda}{\lambda a + e^{-2i\kappa_n a}}\right]^{1/2}. \tag{35}$$

Similarly, using Eqs. (29), (31), and (35) into Eq. (13) allows us to write the expansion coefficient C_n as

$$C_n = \left[\frac{\lambda a}{\lambda a + e^{-2i\kappa_n a}}\right]^{1/2}\left[\frac{2q\pi\, \sin(\kappa_n a)\,(-1)^q}{\kappa_n^2 a^2 - q^2\pi^2}\right]. \tag{36}$$

Using Eqs. (31) and (35) it is straightforward to obtain an analytical expression for the integral I_n appearing in Eq. (23) for $n = \ell$.

It is worth noticing that as the intensity of the potential $\lambda \to \infty$, the complex poles $\kappa_n = \alpha_n - i\beta_n$ tend to the real infinite box eigenvalues, and similarly, the resonant eigenfunctions

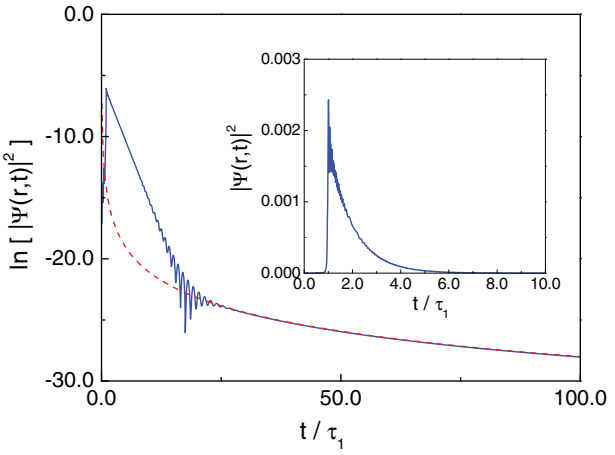

FIG. 1. (Color online) Transition from the exponential to the nonexponential regime shown in a plot of $\ln[|\Psi(r,t)|^2]$ obtained from Eq. (10) as a function of time (in lifetime units), for the δ potential with parameters $\lambda = 100$, $q = 1$, $a = 1$, at the position $r = 500a$. The inverse power-law long-time asymptotic behavior (dashed line) calculated using the second term of Eq. (15) is included for comparison. The inset shows the formation of a main wave front in the probability density. Other important system parameters are the lifetime $\tau_1 = 84.0585$, and the pole corresponding to the first state, $k_1 = 3.11052 - i\,9.5614 \times 10^{-4}$.

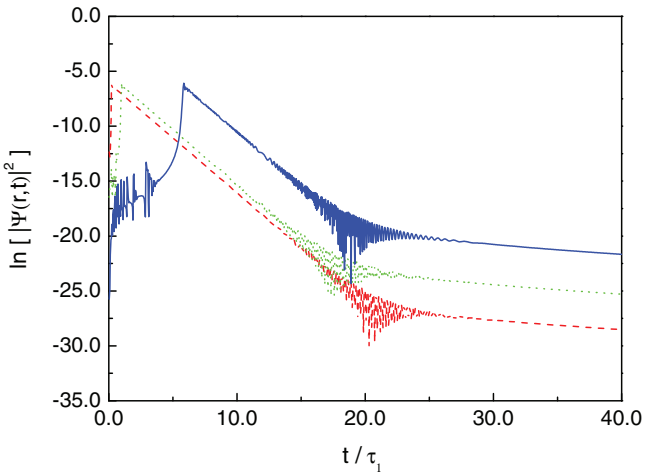

FIG. 2. (Color online) For the system depicted in Fig. 1 we illustrate the r dependence of the *exponential to nonexponential* transition by plotting $\ln[|\Psi(r,t)|^2]$ obtained from Eq. (10) as a function of time, at the fixed positions $r = 100a$ (dotted line), $r = 500a$ (dashed line), and $r = 3000a$ (solid line). Notice that the probabilities associated to the transitions increase as the distance r from the interaction region becomes larger.

$u_n(r)$ tend to the infinite box model eigenfunctions. This means that for a finite value of the intensity λ, an initial infinite box state $\Psi(r,0)$ with $q = m$, has a larger overlap with the resonant state $u_m(r)$ than with any other resonant state. One sees, therefore, that for a given finite value of the intensity λ and a radius a of the δ potential one may evaluate the corresponding set of complex poles $\{\kappa_n\}$ and expansion coefficients $\{C_n\}$ of the problem.

Figure 1 exhibits a plot of $\ln[|\Psi(r,t)|^2]$, given by (10), as a function of time in lifetime units, given by $\tau = \Gamma_1^{-1}$, calculated at the distance $r = 500a$. Here we choose $q = 1$, and hence the real part of the overlap of the initial state with the resonant state with $n = 1$ is the order of unity, i.e., $\mathrm{Re}\, C_1^2 \approx 1$. A well-known consequence of this is that the exponential and long-time nonexponential behavior can be described accurately with just the complex pole $n = 1$ [27]. However, the description of the buildup of the probability density occurring at short times requires a few tens of poles. We may observe a regime where the decay is purely exponential, followed by a transient where the transition from exponential to nonexponential decay occurs, that is characterized by dense oscillations. From then on, the decay follows an inverse power-law behavior as t^{-3}, which dominates the process as dictated by Eq. (10) (dashed line). For completeness, we have included in the inset of Fig. 1 the formation of the main wave front in the probability density calculated with Eq. (10).

Now let us now analyze the behavior of the transition from the exponential to nonexponential decay as a function of the distance r from the decaying source. This is illustrated in Fig. 2, where we plot $\ln[|\Psi(r,t)|^2]$ using Eq. (10) as a function of time calculated at different values of the position: $r = 100a$, $r = 500a$, and $r = 3000a$. One sees that as r increases, there is a buildup regime where the probability density exhibits a succession of peaks that correspond to the high resonance levels of the system. At longer times the probability density exhibits an exponentially decaying regime that is followed by a transition to a nonexponential behavior. In order to describe accurately such behaviors more poles are required. Here and in the rest of our calculations we consider 1000 poles. As may be seen, as the value of the position r increases, the transition to the long-time nonexponential behavior corresponds to larger values of the probability density (increased observability). This behavior is consistent with the results obtained in Ref. [33], where the transition from the exponential to the postexponential regime is explored by using the simple model of an exponential decaying source with a single resonance.

If we increase further the distance r at which the probability density is measured, we enter a regime where exponential decay is no longer present, and where the decay process is dominated by an inverse power-law behavior of t^{-3}. We illustrate this in Fig. 3, where we plot the $\ln[|\Psi(r,t)|^2]$ using Eq. (10) (solid line) as a function of time at a fixed position $r = 1 \times 10^8 a$ using the same system parameters as given previously. Figure 3 exhibits a peculiar peaked structure that appears prior to the situation where the quantum state has reached the inverse power-law behavior of t^{-3} [second term of Eq. (15)] (dashed line). We have found that this structure of the decaying wave function, Eq. (10), corresponds to a regime where both the distance r and the time t attain very large values. In the inset to Fig. 3 we appreciate that this peaked structure (solid line) may be accurately described by Eq. (24) (solid line) for the state $\ell = 1$. Furthermore, it is worth mentioning that the above calculations are indistinguishable from the numerical integration using continuum wave solutions as given by Eq. (25) (dotted line).

In Fig. 4 we can appreciate that if the point of detection of the probability density is further increased (larger r), the peak value of the time domain resonance (solid line) decreases as we

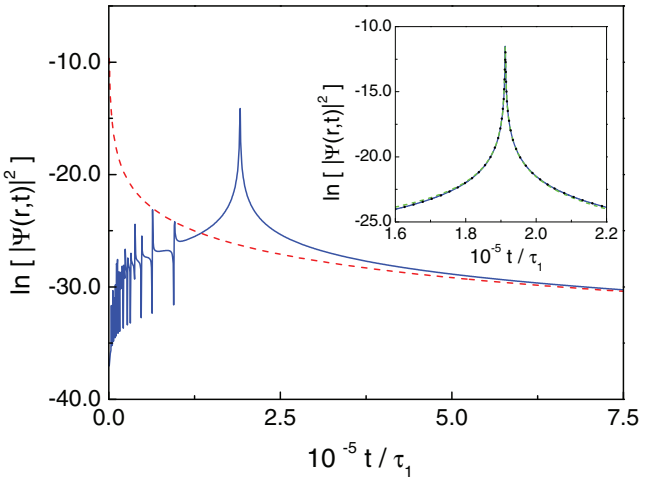

FIG. 3. (Color online) Plot of $\ln[|\Psi(r,t)|^2]$ obtained from Eq. (10) (solid line), as a function of time, at a fixed position $r = 1 \times 10^8 a$, for the system depicted in Fig. 1. We can appreciate the formation of a sharp-peaked structure that we named *time domain resonance*. Also shown is the inverse power-law long-time asymptotic behavior (dashed line) calculated with the second term of Eq. (15). In the inset we compare the results for $\ln[|\Psi(r,t)|^2]$ using Eq. (10) with one pole (solid line), the expression given by Eq. (24) using the state with $\ell = 1$ (dashed line), and a numerical calculation using the continuum states as given by Eq. (25) (dotted line). As can be appreciated all these plots are indistinguishable.

move away from the source. Notice also that the corresponding width increases over several lifetimes. In Fig. 4 we have included plots of the peaked expression given by Eq. (24) (dotted line) and an excellent agreement is obtained. Moreover, from Eq. (24), we can see that the amplitude of the *time domain resonance* decreases inversely with time, i.e., $|\Psi(r,t)|^2 \sim t^{-1}$, which is consistent with the behavior observed in Fig. 4.

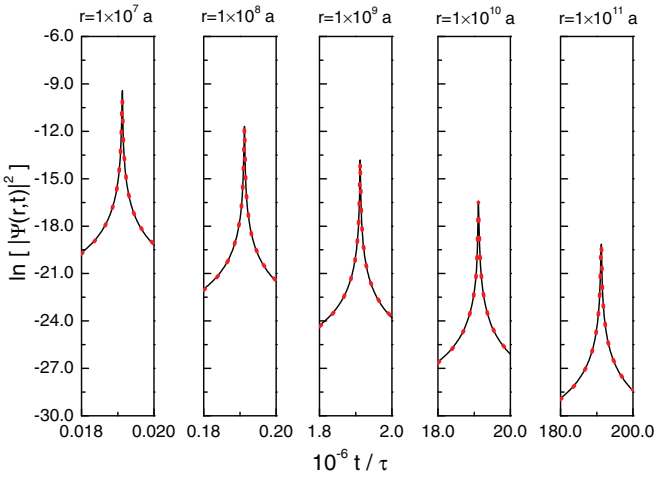

FIG. 4. (Color online) Dynamical behavior of the time domain resonance [Eq. (10)] (solid line), for the system depicted in Fig. 1, for different values of the position (indicated on the graphs) along the decaying region. The panels illustrate that the amplitude (observability) of this transient structure diminishes as the parameter r is increased. We have also included the time domain resonance computed with Eq. (24) (dotted line) and an excellent agreement is obtained.

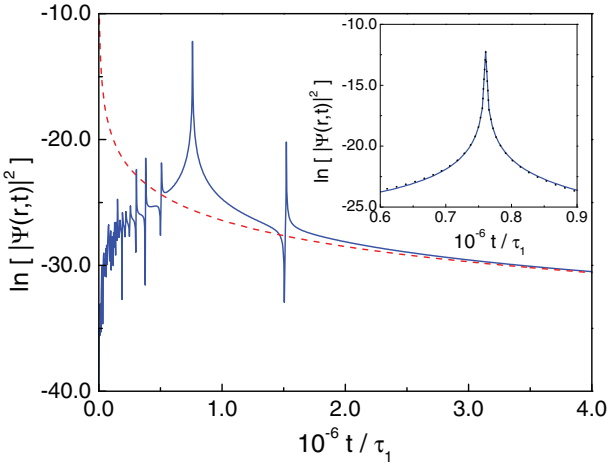

FIG. 5. (Color online) Plot of $\ln[|\Psi(r,t)|^2]$ obtained from Eq. (10) (solid line), as a function of time, at a fixed position $r = 1 \times 10^8 a$ for the initial state given by Eq. (29) with $q = 2$. In the inset we show the time domain resonance calculated using Eq. (10) with one pole (solid line), and the steepest descent approximation given by Eq. (24) (dotted line) using $\ell = 2$. A very good agreement is obtained. We also included in the main graph, the inverse power-law t^{-3} long-time asymptotic behavior (dashed line) calculated with the second term of Eq. (15).

In order to explore the effect of the initial state on $\ln[|\Psi(r,t)|^2]$, in Fig. 5 we plot Eq. (10) (solid line) as a function of time for the initial state given by Eq. (29) with $q = 2$. This choice favors a maximal overlap between the initial state and the corresponding resonant state $u_2(r)$, i.e., $\mathrm{Re}\{C_2^2\} \approx 1$. Notice that as a consequence, a *time domain resonance* with $n = 2$ clearly emerges from the background of the other overlapping neighboring resonances. In the inset, we zoom out the time domain resonance, and compare the results for $\ln[|\Psi(r,t)|^2]$ using Eq. (10) with the single pole approximation $n = 1$ (solid line), with the steepest descent approximation given by Eq. (24) (dotted line) for the second state ($\ell = 2$), and again, a very good agreement is obtained.

It might be of interest to study under what conditions it is possible to increase the observability of the time domain resonance. We can infer from Eq. (24) that both the intensity and position of this transient structure strongly depends on the value of α_ℓ and β_ℓ. A useful parameter to characterize this situation is the dimensionless ratio R, defined as $R \equiv \mathcal{E}_\ell / \Gamma_\ell$ [41,42], which by using (16) may be written as

$$R = \frac{1}{4}\left[\frac{\alpha_\ell}{\beta_\ell} - \frac{\beta_\ell}{\alpha_\ell}\right]. \tag{37}$$

For the decaying system discussed in Fig. 1, $\alpha_1 = 3.110\,52$ and $\beta_1 = 9.5614 \times 10^{-4}$. It follows then, using Eq. (37), that $R = 813.29$. Clearly, since $R \gg 1$, this situation corresponds to the case of a very narrow resonance, as may be easily seen by inspection of Eq. (22). This suggests to analyze a decaying system with a smaller value of R, i.e., a system with a broader resonance. In Fig. 6 we show the time domain resonance [using Eq. (10) (solid line)] for a system with $\lambda = 6$ and $a = 1$, at $r = 500a$. In this case $\alpha_1 = 2.75$ and $\beta_1 = 0.404$, which, using (37) yields $R = 4.92$. We note that for this particular value of R, the time domain resonance maximum occurs at $\sim$140 lifetimes,

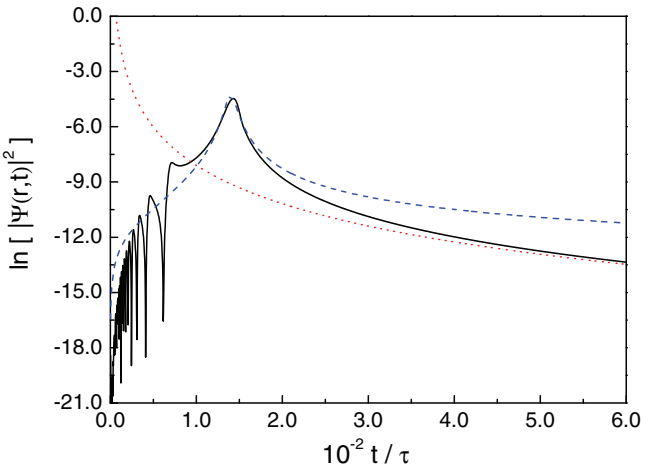

FIG. 6. (Color online) Plot of $\ln[|\Psi(r,t)|^2]$ obtained from Eq. (10) (solid line) as a function of time, for the δ potential with parameters $\lambda = 6$, and $a = 1$ at the position $r = 500a$. The inverse power-law t^{-3} long-time asymptotic behavior (dotted line) calculated using the second term of Eq. (15) is included for comparison. This decaying system is characterized by a very broad resonance, which yields $R = 4.92$. Note that the time domain resonance can be approximated by Eq. (22) (dashed line), and that its magnitude is significantly increased in comparison with the one observed in Fig. 2. Other important system parameters are the lifetime $\tau_1 = 0.6455$, and the pole corresponding to the first state, $k_1 = 2.75 - i\,0.1404$.

and its intensity is larger (by $\sim$5 orders of magnitude) than the one observed in the case of Fig. 2. We have included in Fig. 6 a plot of the peaked-shape function given by Eq. (22). The above implies that the observability of the time domain resonance strongly depends on the value of the parameter R, and hence one may expect larger values of probability associated to this transient structure when R is smaller.

V. CONCLUSIONS

In this work we consider an exact analytical expression for the decaying wave function along the external interaction region, using a resonant-state formulation that takes into account the full energy spectrum of the system. We discuss a regime, occurring at very long distances and times, where the probability density exhibits a time domain resonance. We demonstrate that this dynamical feature may be described by a time-dependent function possessing a peaked shape that depends on the resonance parameters corresponding to the largest expansion coefficient of the decaying state, i.e., Eq. (24). From a physical point of view, that coefficient measures the overlap strength between the initial decaying wave function and the corresponding resonance state of the system. We also demonstrate that the observability of the time domain resonance can be enhanced by choosing quantum systems with small values of the quantity R, defined by (37), which refers to systems with broad resonances. Our findings indicate that the resonance parameters α_ℓ and β_ℓ of the time domain resonance allow us to infer, provided that $R > 1$ (for $R < 1$, the system never decays exponentially [42]), that in the remote past the corresponding system decayed exponentially with a decay rate given by $\Gamma_\ell = 4\alpha_\ell\beta_\ell$. It would be of interest to explore the consequences of the results obtained here with the decay of identical particles [43,44] to find out how entanglement may affect the corresponding dynamics.

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