Optimal probabilistic measurement of phase

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When measuring the phase of quantum states of light, the optimal single-shot measurement implements a projection on the unphysical phase states. If we want to improve the precision further we need to accept a reduced probability of success, either by implementing a probabilistic measurement or by probabilistically manipulating the measured quantum state by means of noiseless amplification. We analyze the limits of this approach by finding the optimal probabilistic measurement that, for a given rate of success, maximizes the precision with which the phase can be measured.

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Phase is a central concept in both classical and quantum optics. It was, however, a matter of lengthy dialogue before the quantum description of phase was established. The initial attempts of Dirac to treat phase as a canonical conjugate to photon number failed because it is impossible to represent phase by a quantum mechanical observable [1]. As a consequence, phase cannot be projectively measured; it can only be estimated (or guessed) by analyzing the results of other measurements. Despite this, phase states do exist [2] (even if they are not orthogonal) and they were eventually used to construct a well-behaved phase operator [3]. Other attempts to describe phase properties of quantum states relied on the measurement-related phase distribution [4]. Both approaches were later reconciled with the fundamental canonical phase distribution [5].

The canonical phase distribution characterizes phase properties of a quantum state and it is completely independent of its photon number distribution. It can be used to obtain a wide range of quantities related to phase estimation, but it also determines how much information about the phase of the state can be obtained by performing a measurement only on a single copy of it. True, the ideal canonical phase measurement does not and cannot exist, but several approximative approaches have been suggested [6,7].

Aside from improving the actual detector scheme, the overall performance of phase measurement can be enhanced also by specific alteration of the measured quantum state. A highly nonclassical quantum state can in principle lead to an unparalleled precision [8], while weakly nonclassical states are both beneficial and experimentally feasible [9]. However, if the state is unaccessible prior to phase encoding, we need to rely on operations that can enhance the amount of phase information already carried by the scrutinized state. Such operations are commonly referred to as noiseless amplifiers and a great deal of attention was recently devoted both to the concept [10] and to the experimental realizations [11]. The cost of this improvement comes in the reduced success rate of the operation. The amplification is therefore not very practical when the measurements can be repeated, but it may be useful when the event to be detected is rare and we need to be certain that the only measurement outcome obtained corresponds to the theoretical value as closely as possible.

However, even in the scenarios in which the probabilistic approach is worth considering, it would be more prudent to design an actual probabilistic measurement of phase. Such a measurement would be conceptually similar to methods of unambiguous discrimination of quantum states [12], except that a truly errorless detection would be possible only in the limit of zero probability. Rather than this regime of limited interest, the question is how does reducing the success rate of the measurement help us measure the phase more precisely. Maybe even more importantly, we ask what the theoretical limits of this approach are. In this paper we attempt to answer these questions.

Let us start by reviewing what we actually mean by the term "phase measurement." Phase has a well-defined meaning only in the context of an interferometric setup, where it expresses the relative length difference between the two optical paths. In the context of continuous-variable (CV) quantum optics [13], phase is often considered a stand-alone property. However, this is only because the other path in the interferometer, represented by the local oscillator, is taken for granted. In a sense this is justified, as the local oscillator is intense enough to be, for all intents and purposes, just a classical reference framing the associated quantum system. Measuring the phase of the quantum system is then equivalent to discerning a value of the parameter ϕ, which is encoded into the quantum state by means of an operator $\exp(i\phi\hat{n})$, where $\hat{n}$ is the photon number operator. Apart from special cases it is impossible to determine the parameter ϕ perfectly. Rather than complete knowledge, the result of the measurement provides the observer just with the best guess of the parameter, where the quality of the guess depends on both the state of the measured system and the phase measurement employed.

The simplest single-shot measurement of the phase of optical signals relies on simultaneous measurement of quadrature operators X and P, corresponding to the Hermitian and the anti-Hermitian part of the annihilation operator. The phase can be then deduced from the measurement results x' and p' by taking $\phi = \tan^{-1}(p'/x')$. Of course, in addition to knowledge of the phase, this particular measurement also provides us with knowledge of the energy of the state. Therefore, the obtained phase information is not as complete as it could be.

The best possible measurement that can be imagined is the so-called canonical measurement of phase. It can be mathematically described as a projection on idealized phase states $|\theta\rangle = \sum_{k=0}^{\infty} e^{i\theta k}|k\rangle$. These phase states are not normalized, which makes them similar to eigenstates of continuous operators (such as position and momentum), but they are also not orthogonal. The nonorthogonality is actually responsible for the impossibility of measuring phase completely because a single measured value of θ is not exclusive just to a single phase state. For any quantum state $\hat{\rho}$ the results of the canonical phase measurement can be characterized by the probability distribution $P(\theta) = \mathrm{Tr}[\hat{\rho}|\theta\rangle\langle\theta|]$, i.e., the canonical phase distribution. The shape of the distribution is solely given by the employed quantum state; the encoded phase value is represented only as a linear displacement. For a particular measured value θ the value $P(\theta)$ is related to the probability that the measured value is the encoded value. Simplistically, we can say that for any quantum state, the quality of phase encoding is given by the width of the canonical distribution. This can be formally expressed by evaluating the variance of the phase distribution, but it is actually more convenient to use a different quantity that takes into account the periodicity of the phase in the interval $|0,2\pi\rangle$ [14]. The new quantity is the phase variance $V = |\mu|^{-2} - 1$, where $\mu = \langle \exp i\theta\rangle$ [15]. The phase variance is completely independent of displacement in θ and therefore is completely determined by the state $\hat{\rho}$. We can also see that the phase variance solely depends on the value of the parameter μ, which we are going to use from now on.

For an arbitrary pure quantum state

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{1}$$

the value of μ can be found as

$$\mu = \sum_{n=0}^{\infty} c_n c_{n+1}^{*}. \tag{2}$$

If we fix the magnitudes of the individual coefficients, μ will be maximized when all the coefficients are real and positive. For quantum states from a limited-dimensional Hilbert space, the parameter μ can be straightforwardly maximized and optimal states for phase encoding can be found [6,16,17]. The existence of such ideal states tells us that there are limits to how well the phase can be encoded in a limited-dimensional Hilbert space. In contrast, if the Hilbert space is infinite, which is the case in CV quantum optics communication, it is in principle possible to encode the phase perfectly, in such the way that $\mu = 1$ and consequently the phase variance is zero. As this is obviously the case in classical communication, where phase can be encoded and decoded with arbitrary precision, the inability to measure phase in quantum physics stems from employing quantum states that are so weak their Hilbert space is effectively limited. However, there is a key difference between these states and states from a Hilbert space with factually limited dimension. The difference is that the infinite-dimensional Hilbert space offers a possibility of measuring the state arbitrarily well if we accept a reduced probability of success.

The idea that measurement can be improved when we accept a reduced probability of success is not a new one. When discriminating quantum states drawn from a finite ensemble, one can accept the existence of inconclusive results (reduced success rate) in order to reduce the probability of erroneous result to zero [12]. Similarly, when measuring a continuous parameter such as phase, it is possible to conditionally transform the quantum states in such a way that the subsequent measurement leads to more precise results [10,11]. Taken as whole, the combination of probabilistic operation and measurement is essentially a probabilistic measurement. In the following we develop a unified picture describing the probabilistic measurement of the phase of a quantum state and derive bounds for the optimal one. Namely, we will look for such a measurement that, for a given probability of success, yields the best possible result.

The extension of the canonical measurement of phase into the probabilistic regime can be represented by a set of operators Π_ϕ, each of them corresponding to a positive detection event of value ϕ and a single operator Π_0 representing the inconclusive results. Together these operators form a positive-operator-valued measure (POVM). For the canonical deterministic measurement of phase these operators are $\Pi_\phi^{(D)} = \frac{1}{2\pi}|\phi\rangle\langle\phi|$. Keeping the pure-state projector structure intact, we can express the probabilistic POVMs as

$$\Pi_\phi^{(P)} = \frac{1}{2\pi} F|\phi\rangle\langle\phi|F^{\dagger}, \quad \Pi_0^{(P)} = 1 - \int \Pi_\phi^{(P)} d\phi. \tag{3}$$

Here $F = \mathrm{diag}(f_0, f_1, \ldots)$, where $|f_j| \leqslant 1$ for all $j = 0,1,\ldots$, is an operator diagonal in Fock space. It is practical to represent the probabilistic measurement by a filter, transmitting and modifying the quantum state with some limited probability, followed by the deterministic canonical phase measurement. The operator F then plays the role of the probabilistic filter and the task of finding the optimal measurement is reduced to finding the optimal operator F.

After the first glance at the problem, one issue immediately becomes apparent. For any quantum state ρ, the probability of successful measurement $P = 1 - \mathrm{Tr}[\rho\Pi_0^{(P)}]$ is dependent on the choice of the measured state. The optimal measurement therefore needs to be tailored to a specific state or to a class of states. However, let us first approach the task in a general way. Suppose we have an input quantum state (1). For phase encoding it is best when all the coefficients c_n are real and positive, so we will assume this is the case [18]. The act of the filter transforms this state into a new one

$$|\psi_f\rangle = \frac{1}{\sqrt{P}} \sum_{n=0}^{\infty} f_n c_n |n\rangle, \tag{4}$$

where $P = \sum_{n=0}^{\infty} f_n^2 c_n^2$ is the probability of success and the filter parameters f_n are also considered real and positive. For any given probability P, the act of finding the optimal filter relies on maximization of

$$\mu = \sum_{n=0}^{\infty} f_n f_{n+1} c_n c_{n+1} \tag{5}$$

under the condition $\sum_{n=0}^{\infty} c_n^2 f_n^2 = P$. The problem can be reduced to solving the system of equations

$$\begin{aligned} f_{n-1}a_{n-1} + f_{n+1}a_n &= \lambda f_n x_n, \\ n &= 0,1,\ldots, \\ \sum_{n=0}^{\infty} x_n f_n^2 &= P, \end{aligned} \tag{6}$$

where $a_n = c_n c_{n+1}$, $x_n = c_n^2$, $f_{-1} = 0$ by convention, and λ is the Lagrange multiplier. Finding the solution under the most general conditions is not an easy task. Fortunately, there are some simplifications that can be made, provided we are applying the filtration to the practically significant coherent states.

A coherent state $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{k=1}^{\infty}\frac{\alpha^k}{\sqrt{k!}}|k\rangle$ can be considered a quantum version of a classical complex amplitude of light. It can be used to describe the state of light produced by a well-stabilized laser and it has a place both in classical communication [19] and in quantum cryptography [20], both of which can employ phase encoding. Coherent states are fairly well localized in the Fock space: For any coherent state there always exists a finite N-dimensional Fock subspace such that the probability of the state manifesting outside of it can be made arbitrarily small. As a consequence, those higher Fock dimensions do not significantly contribute to the state's properties and the values of the respective filters can be set to one, i.e., $f_n = 1$ for all $n \geqslant N$. Of course, with severe filtering leading to extremely low success rates, some previously dismissable Fock numbers can start being relevant, but this can be remedied by choosing an even higher photon number N' as the new threshold of significance.

This dramatically simplifies the process of finding the optimal filter. All the filter coefficients for $n = 0,\ldots,N$ can be now expressed in the form

$$f_n = f_0\mathcal{P}_n(\lambda), \tag{7}$$

where $\mathcal{P}_n(\lambda)$ is a polynomial of λ defined by the recursive relation

$$P_{n+1}(\lambda) = \frac{\lambda x_n\mathcal{P}_n(\lambda) - a_{n-1}\mathcal{P}_{n-1}(\lambda)}{a_n}, \tag{8}$$

with $\mathcal{P}_0(\lambda) \equiv 1$ and $\mathcal{P}_1(\lambda) = x_0/a_0$. Since f_0 can be obtained from the condition $f_N = f_0\mathcal{P}_N(\lambda) = 1$, getting the full solution is reduced to finding the correct value of the Lagrange multiplier λ, which is one of the roots of the polynomial equation

$$\sum_{n=0}^{N} x_n\mathcal{P}_n(\lambda)^2 = \left(P - 1 + \sum_{n=0}^{N} x_n\right)\mathcal{P}_N(\lambda)^2. \tag{9}$$

To be of physical relevance, the obtained λ needs to be real and it has to lead to a filter with parameters, which are all positive and bounded by one. Among the values of λ satisfying those condition, the one corresponding to the global extreme, rather than just a local one, needs to be selected by directly checking the respective value of μ.

Interestingly enough, not all combinations of α, P, and N lead to physical filters. In fact, for any specific pair of values of α and P, there are only a handful of values of N providing physically relevant filters. This is illustrated in Fig. 1, where it can be seen that for $\log_{10} P = -1.3$ both $N = 2$ and 3 provide a physical filter ($N = 3$ is optimal). There is no filter for $N = 1$ because it is just impossible to reach such a low probability by damping only a single coefficient. There are also no physical filters for $N \geqslant 4$: All the obtained values of λ are either complex or lead to filters that are not bounded by one. This could be resolved by adding additional boundary conditions for the set of equations, but it turns out that it is not necessary, as in these cases the optimal filter can be obtained for a different value of N. The particular optimal N needs to be found numerically. Fortunately this is a simple matter of checking a range of values of N and finding the one that leads to positive results. For illustration, several values of N optimal for some range of α and P are depicted in Fig. 2. As another illustration, Fig. 3 shows improvement of the probabilistic measurement for several coherent states with different amplitudes. Finally, the optimal filters for a specific coherent state and a range of success probabilities are depicted in Fig. 4.

We have introduced the concept of optimal probabilistic measurement of quantum phase and shown how such a measurement can be constructed. The approach can be used for any quantum state, but we have mainly focused on practically relevant coherent states, for which we have managed to obtain

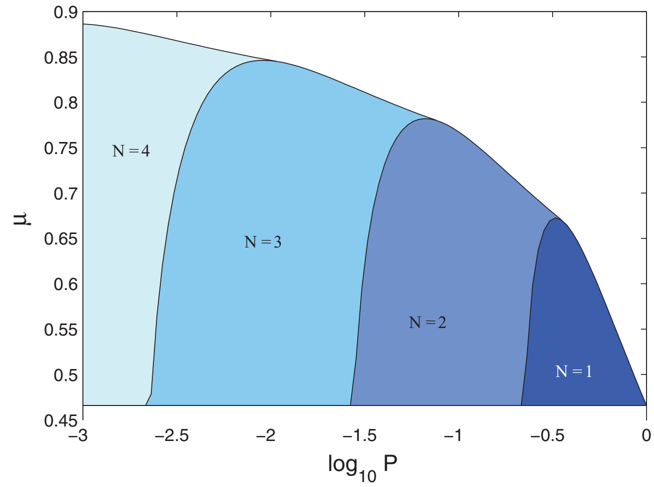

FIG. 1. (Color online) Value of μ for the optimal probabilistic measurement of the phase of the coherent state with $\alpha = 0.5$ dependent on the probability of success. Differently colored areas correspond to filters with different filter parameters N.

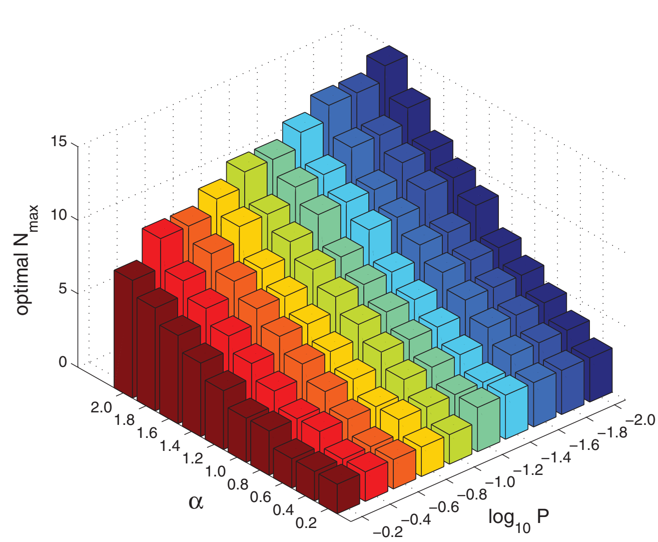

FIG. 2. (Color online) Optimal filter parameters N dependent on the coherent amplitude of the coherent state α and the probability of the successful measurement P.

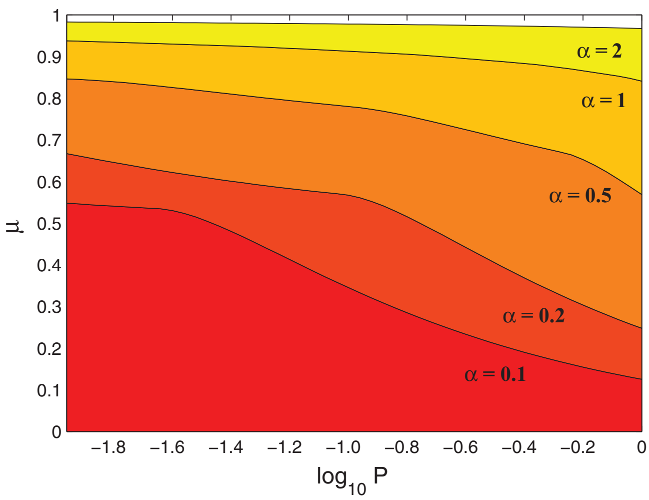

FIG. 3. (Color online) Value of μ for the optimal probabilistic measurement of the phase for various coherent states.

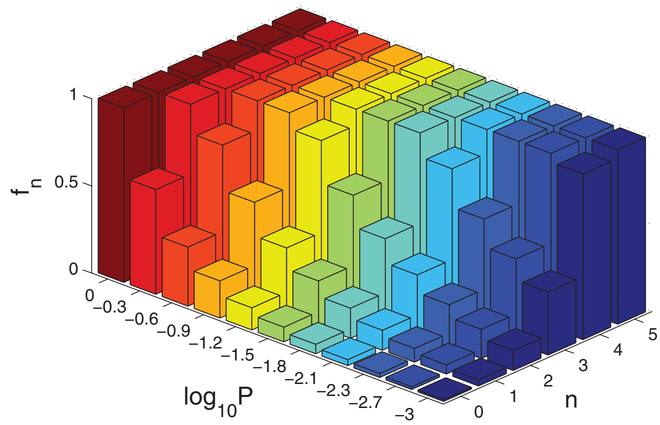

FIG. 4. (Color online) Optimal filters for the coherent state with $\alpha = 0.5$ and a range of success probabilities.

the form of the optimal measurement in a semianalytic form. The probabilistic aspect of the measurement can be represented by a filter transmitting various Fock space elements with different amplitudes. The derived optimal measurement sets an upper bound on the trade-off between the quality and the probability of success of phase measurements. The filter required for such a measurement is a highly nonlinear operation, but in light of the recent advent of manipulating light on the individual photon level [11], it might be within experimental reach.

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[1] P. A. M. Dirac, Proc. R. Soc. London Ser. A **114**, 243 (1927); W. H. Louisell, Phys. Lett. **7**, 60 (1963); L. Susskind and J. Glogower, Physics **1**, 49 (1964); P. Carruthers and M. M. Nieto, Rev. Mod. Phys. **40**, 411 (1968).

[2] R. Loudon, *The Quantum Theory of Light* (Oxford University Press, New York, 2000).

[3] D. T. Pegg and S. M. Barnett, Phys. Rev. A **39**, 1665 (1989).

[4] C. W. Helsrom, Int. J. Theor. Phys. **11**, 357 (1974); J. H. Shapiro and S. R. Shepard, Phys. Rev. A **43**, 3795 (1991).

[5] U. Leonhardt, J. A. Vaccaro, B. Böhmer, and H. Paul, Phys. Rev. A **51**, 84 (1995).

[6] H. M. Wiseman and R. B. Killip, Phys. Rev. A **56**, 944 (1997); **57**, 2169 (1998).

[7] K. L. Pregnell and D. T. Pegg, Phys. Rev. Lett. **89**, 173601 (2002).

[8] P. Kok, A. N. Boto, D. S. Abrams, C. P. Williams, S. L. Braunstein, and J. P. Dowling, Phys. Rev. A **63**, 063407 (2001).

[9] H. Yonezawa, D. Nakane, T. A. Wheatley, K. Iwasawa, S. Takeda, H. Arao, K. Ohki, K. Tsumura, D. W. Berry, T. C. Ralph, H. M. Wiseman, E. H. Huntington, and A. Furusawa, Science **337**, 1514 (2012).

[10] T. C. Ralph and A. B. Lund, in *Proceedings of the Ninth International Conference on Quantum Communications Measurement and Computing*, edited by A. Lvovsky, AIP Conf. Proc. No. 155 (AIP, New York, 2009); P. Marek and R. Filip, Phys. Rev. A **81**, 022302 (2010); J. Fiurášek, *ibid.* **80**, 053822 (2009).

[11] G. Y. Xiang, T. C. Ralph, A. P. Lund, N. Walk, and G. J. Pryde, Nat. Photon. **4**, 316 (2010); F. Ferreyrol, M. Barbieri, R. Blandino, S. Fossier, R. Tualle-Brouri, and P. Grangier, Phys. Rev. Lett. **104**, 123603 (2010); M. A. Usuga, C. R. Müller, C. Wittmann, P. Marek, R. Filip, C. Marquardt, G. Leuchs, and U. L. Andersen, Nat. Phys. **6**, 767 (2010); A. Zavatta, J. Fiurášek, and M. Bellini, Nat. Photon. **5**, 52 (2011); C. R. Müller, C. Wittmann, P. Marek, R. Filip, C. Marquardt, G. Leuchs, and U. L. Andersen, Phys. Rev. A **86**, 010305(R) (2012).

[12] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987); D. Dieks, *ibid.* **126**, 303 (1988); A. Peres, *ibid.* **128**, 19 (1988); S. J. van Enk, Phys. Rev. A **66**, 042313 (2002).

[13] U. Leonhardt, *Essential Quantum Optics* (Cambridge University Press, Cambridge, 2010).

[14] Z. Hradil, Phys. Rev. A **46**, R2217 (1992).

[15] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1974).

[16] Expression (2) can also be maximized with the help of Lagrange multipliers and the symmetry condition $c_n = c_{N-n}$.

[17] G. S. Summy and D. T. Pegg, Opt. Commun. **77**, 75 (1990).

[18] If this were not the case and some of the coefficients were negative or complex, the situation could be remedied by using complex filter coefficients with $|f_n| = 1$, which could improve phase properties at no cost in success rate.

[19] V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, J. H. Shapiro, and H. P. Yuen, Phys. Rev. Lett. **92**, 027902 (2004).

[20] F. Grosshans and P. Grangier, Phys. Rev. Lett. **88**, 057902 (2002); V. Scarani, H. Bechmann-Pasquinucci, N. J. Cerf, M. Dušek, N. Lütkenhaus, and M. Peev, Rev. Mod. Phys. **81**, 1301 (2009).