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Stability of N-soliton molecules in dispersion-managed optical fibers

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We investigate the stability of N-soliton molecules in dispersion-managed optical fibers with focus on the recently realized 2- and 3-soliton molecules. We calculate their binding energy using an averaged nonlinear Schrödinger equation. A combination of variational and numerical solutions to this equation shows that it describes well the intensity profiles and relative separations of the experimental molecules. Extending the calculation to larger values of N, the binding energy per soliton is found to saturate at $N \geqslant 7$.

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In the last two decades, developments in fiber-optic communications have demonstrated that dispersion management (DM) presents a novel attractive type of nonlinear carrier of information in optical fiber links. A few years ago, a stable bound state of two DM solitons in optical fibers was realized experimentally [1] and most recently 3-soliton molecules in DM optical fibers were also realized by the same group [2]. The main motivation behind creating such molecules is to increase the bit rate of data transfer in optical fibers. Coding with two or more solitons per clock period increases the alphabet beyond the binary scheme of a single soliton. In this manner, the Shannon limit [3], which soon will be reached, may be exceeded [4].

The main concern in soliton molecules being data carriers is their stability against disintegration. Hence, intensive interest in their stability has emerged [5–11]. The existence of a nonzero binding energy of the soliton molecules is an indication of its stability. The energy of a stable soliton molecule should have a minimum for a finite separation between the solitons. In Ref. [12] it was shown that the energy of a 2-soliton molecule indeed exhibits such a minimum, and the potential of interaction was also shown to be of molecular type.

The main aim of the present work is to provide a theoretical framework that explains the stability of 2- and 3-soliton molecules as observed by Mitschke and co-workers [2,4]. Specifically, we will show variationally that there is indeed a nonzero binding energy for 3-soliton molecules in DM fibers. The calculation provides an estimate for the strength of the bond between the solitons and shows regions in the parameter space where the molecule becomes unstable. Here, we address the problem of calculating the binding energy of the soliton molecule using an averaged nonlinear Schrödinger equation (NLSE). It was shown in Ref. [13] that solitons in DM fiber can be described by an effective nonlinear Schrödinger equation with constant coefficients and a quadratic potential. The averaged equation is more appealing for capturing the main features of the binding mechanism. At first, we show that the averaged equation is not integrable, hence variational and numerical approaches will be followed. For both cases we compare the intensity profiles with the experimental ones and obtain a good agreement. Finally, the calculation is then extended to the larger values of N up to $N = 12$.

We first show that the evolution of solitons in dispersion-managed dissipative optical fibers obeys a NLSE with a quadratic potential [13] which is integrable [14]. It turns out, however, that integrability restricts the time dependence of the dispersion to the nonrealistic case of exponential form. Therefore, an effective NLSE will be derived by averaging over one period of the dispersion map [13]. The effective equation will then be used to calculate the binding energy of the soliton molecule.

Solitons in dispersion-managed dissipative optical fibers are described by the following NLSE:

$$i\,q_z + \frac{d(z)}{2}\,q_{tt} + |q|^2\,q = -i\,\Gamma(z)\,q, \tag{1}$$

where $q(t,z)$ is the envelope function of the soliton and the subscripts denote partial derivatives. Here z and t are normalized distance and time, and $d(z)$ corresponds to the dispersion management map defined by

$$d(z) = \begin{cases} d^+, & 0 \leqslant z \leqslant L^+, \\ d^-, & L^+ < z \leqslant L^+ + L^-, \end{cases} \tag{2}$$

where $d^{+,-}$ are constant group velocity dispersions of the fiber segments $L^{+,-}$, respectively. The loss (gain) corresponds to positive (negative) $\Gamma(z)$.

The transformation $q(t,z) = \exp[-\int \Gamma(z)\,dz]\,u(t,z)$ moves the loss term to the coefficient of the nonlinear term. A quadratic phase chirp develops due to the propagation of the soliton in the fiber which corresponds to the transformation $u(t,z) = w(t,z)\,a(z)\,\exp[i\,C(z)\,t^2/2]$, where $C(z)$ is a real function. With the scaling transformations $\tau = p(z)\,t$, $w(t,z) = W(t,z)\,a(z)$, where $p(z) = \exp[-\int C(z)\,dz]$ and $a(z)$ is a real function, the last equation takes the form

$$i\,W_z + \frac{d\,p^2}{2}\,W_{\tau\tau} + a^2\,e^{-2\int \Gamma\,dz}\,|W|^2\,W - \frac{1}{2}\,\kappa\,\tau^2\,W = 0, \tag{3}$$

where

$$\dot{a}/a + C\,d/2 = 0, \tag{4}$$

$$\dot{p}/p + C\,d = 0, \tag{5}$$

which give $a = c_1\,\sqrt{p}$, where c_1 is a real arbitrary constant, and defining

$$\kappa = \frac{\dot{C} + d\,C^2}{p^2}, \tag{6}$$

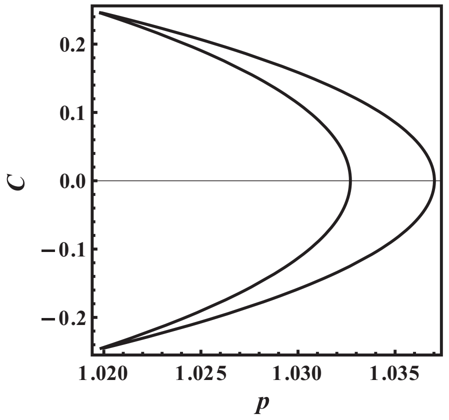

FIG. 1. A closed orbit in the p-C plane, which solves Eqs. (4)–(6), with the experimental parameters $d^- = -4.259$ ps^2/km, $d^+ = 5.159$ ps^2/km, $L^+ = 22$ m, and $L^- = 24$ m of Ref. [4] and the assignments $\kappa^+ = 0.009$ and $\kappa^- = -0.01$.

Note that Eq. (3) can be easily put into integrable form [15]. However, integrability restricts $d(z)$ to be exponential, which is not a realistic option. Typically, Eqs. (4) and (5) are solved using Nijhof's method [16], where a close orbit in the p-C plane guarantees a unique solution for appropriate values of κ^+ and κ^- defined as

$$\kappa(z) = \begin{cases} \kappa^+, & 0 \leqslant z \leqslant L^+, \\ \kappa^-, & L^+ < z \leqslant L^+ + L^-. \end{cases} \tag{7}$$

A closed orbit was indeed found using the experimental parameters of Ref. [2] (see below), as shown in Fig. 1. Using the solutions for $p(z)$ and $C(z)$, Eq. (3) can be averaged over one dispersion period to give

$$iW_z + \frac{\beta}{2} W_{\tau\tau} + A_0 |W|^2 W - \frac{1}{2} K_0 \tau^2\, W = 0, \tag{8}$$

where $\langle \cdots \rangle \equiv \int_0^L (\cdots) dz/L$, $\beta = \langle dp^2 \rangle$, $A_0 = c_1^2 \langle p \rangle$, and $K_0 = \langle \kappa \rangle$, and a dissipationless fiber $\Gamma(z) = 0$ was considered. Thus, solitons in a dispersion-managed fiber are described by this averaged equation. It does indeed describe the core and oscillatory tails of the dispersion-managed solitons [13]. This equation will be used in the following to calculate the binding energy and equilibrium size of soliton molecules.

For numerical purposes, it is useful to reduce Eq. (8) to a dimensionless form. First we introduce the parameters $z' = A_0 z$, $\beta' = \beta/A_0$, and $K_0' = K_0/A_0$. Then Eq. (8) becomes

$$iW_{z'} + \frac{\beta'}{2} W_{\tau\tau} + |W|^2 W - \frac{1}{2} K_0' \tau^2 W = 0. \tag{9}$$

We introduce the dimensionless variables $Z = z'/L'$, $T = \tau/\tau_m$, and $\Psi = W\sqrt{L'}$, where τ_m is the characteristic time scale equal to the pulse duration of the laser source and $L' = (L^+ + L^-)A_0$ is the length of the dispersion map period. In terms of these parameters, the dimensionless NLSE takes the form

$$i\Psi_Z + \frac{D}{2} \Psi_{TT} + |\Psi|^2 \Psi - \frac{1}{2} B T^2 \Psi = 0, \tag{10}$$

where $D = \beta' L'/\tau_m^2$ and $B = K_0' L' \tau_m^2$. We use the experimental parameters for the DM map corresponding to the setup of [2]. The pulse duration $\tau_m = 0.25$ ps, $d^- = -4.259$ ps^2/km, $d^+ = 5.159$ ps^2/km, $A_0 = 1.7$ W^{-1}km^{-1}, $L^+ = 24$ m, $L^- = 22$ m, and $L' = 0.078$ W^{-1}. Note that $d(z)$ here is the negative of that in Refs. [2,4]. Using these experimental values in Nijhof's method, as described above, we get $\beta = \langle dp^2 \rangle = 0.71$ and $K_0 = \langle \kappa \rangle = -0.0156$. Notice that $A_0 = c_1^2 \langle p \rangle = 1.7$ is given as an experimental parameter which accounts for a specific selection of the arbitrary constant c_1. Thus, the scaled coefficients D and B take the values $D = 0.521$ and $B = -4.5 \times 10^{-5}$.

We use a variational calculation to show that 2- and 3-soliton molecules have indeed nonzero binding energy. This will be evident from the minimum of the energy in terms of a finite separation between solitons. The depth of the minimum will give an estimate to the strength of the bond in the molecule.

We employ the following 3-soliton trial wave function

$$\Psi(Z,T) = A \sum_{j=1}^{3} A_j \exp\left[-\frac{(T - \eta_j)^2}{q^2} + i\varphi_j \right] \tag{11}$$

where A guarantees the normalization of Ψ to the number of solitons in the molecule, namely, $N = 3$. The variational parameters $q(Z)$, $\varphi(Z)$, and $\eta(Z)$ correspond respectively to the width, phase, and peak position of the soliton. The energy functional corresponding to Eq. (10) reads

$$E = \int_{-\infty}^{\infty} \left[\frac{D}{2} |\nabla \Psi|^2 + B Z^2 |\Psi|^2 - \frac{1}{2} |\Psi|^4 \right] dT. \tag{12}$$

In real units, this energy can be expressed as $E[J] = (\tau_m/L')E$.

Plotting the energy as a function of the soliton width q using the experimental values of all other parameters, as in Ref. [4], a minimum is obtained at $q = 0.32$ ps which is close to the experimental value for the 3-soliton molecule. In terms of the separation between solitons, $\Delta_1 = \eta_1 - \eta_2$ and $\Delta_2 = \eta_2 - \eta_3$, Fig. 2 shows that the energy has a local minimum for finite values of separations, namely, at $(\Delta_1, \Delta_2) = (-1.0, -1.2)$. This agrees also with the experimental values for equilibrium separations, as will be shown more clearly below in Fig. 3. The fact that there is also a minimum at $\Delta_1 = \Delta_2 = 0$ indicates

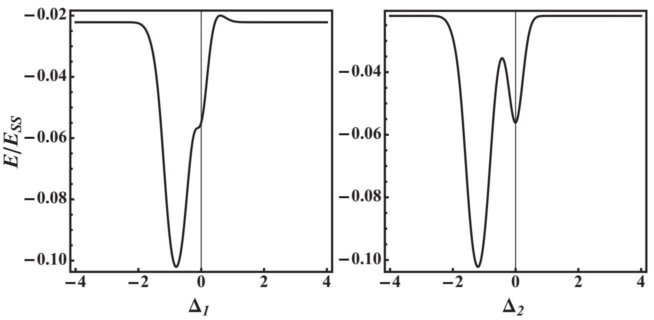

FIG. 2. Binding energy of three solitons relative to that of the single-soliton energy E_{ss} as function of the separation Δ_1 and Δ_2 with same experimental parameters of Refs. [2,4] and for $D = 0.521$ and $B = -4.5 \times 10^{-5}$.

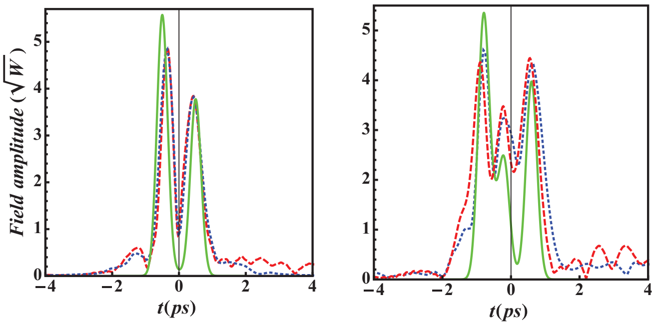

FIG. 3. (Color online) Field amplitude envelopes along dispersion-managed fiber of 2-soliton molecule (left), 3-soliton molecule (right). Green solid line: our variational calculation; blue dotted line: experimental data of [2]; and red dashed line: simulation of [2].

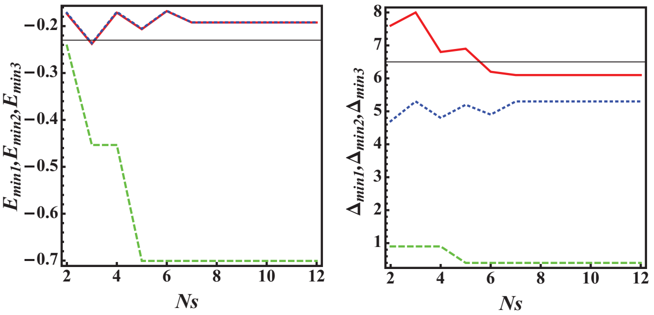

FIG. 4. (Color online) Left panel: Energy minimum vs the number of solitons for $A_0 = 1$, $\gamma = 0.5$, $\delta = 0.5$, $D = 0.521$, and $B = -4.5 \times 10^{-5}$. Red solid line: $E_{\mathrm{min}1}$; blue dotted line: $E_{\mathrm{min}2}$; and green dashed line: $E_{\mathrm{min}3}$. Right panel: Solitons separations vs the number of solitons for the same parameters. Red solid line: $\Delta_{\mathrm{min}1}$; blue dotted line: $\Delta_{\mathrm{min}2}$; and green dashed line: $\Delta_{\mathrm{min}3}$. $E_{\mathrm{min}1,2,3}$ and $\Delta_{\mathrm{min}1,2,3}$ correspond, respectively, to the energy and separation at the first, second, and the third minima of the energy.

that the stability of the soliton molecule is reduced due to the possibility of tunneling from the off-centered minima to the one at the origin, signifying the merging of the three solitons into one.

The value of the energy at the local minimum represents a measure of the strength of the bond in the molecule. The binding energy of the 3-soliton molecule relative to that of the single-soliton energy, denoted by E_{ss}, is taken from Eq. (12) and equals $E/E_{ss} = 0.9$.

In Fig. 3 we compare the intensity profile obtained by our variational calculation with the experimental and simulation results of Ref. [2] for both 2-soliton and 3-soliton molecules. The figure shows a good agreement between the variational calculation on the one hand and the experimental data and the direct numerical solution of the NLSE on the other. Our curves were calculated using the experimental values of all parameters apart from the separations between solitons which were left as variational parameters. The intensity profile was then calculated using the solitons separations obtained by minimizing the energy functional.

To study the equilibrium properties of N-soliton molecules, we extend the variational formalism of the previous section to larger values of N. Therefore, we consider the generalized Gaussian trial function

$$\Psi(T,Z) = A \sum_{j=1}^{N_s} A_j \exp\left[-\frac{(T-\eta_j)^2}{q^2} + i\varphi_j\right], \qquad (13)$$

where $A_j = A_0[1+\gamma(-1)^j]$, $\eta_j = -(N_s-1)/2 + (j-1)\Delta$, and $\varphi_j = \pi(-1)^j$ with N_s being the number of solitons in the molecule. This represents a string of N solitons each of width q, amplitude A_j, center-of-mass position η_j, and phase ϕ_j. For simplicity, the amplitudes were taken to alternate between $A_0(1+\gamma)$ and $A_0(1-\gamma)$, the separation between adjacent solitons is Δ, and phases alternating between π and $-\pi$. In this manner, the parameter space is reduced to two parameters only, γ and Δ. Using the trial function, Eq. (13), in the energy functional, Eq. (12), it is straightforward to calculate the energy of the N-solitons molecule. The energy expressions turn out to be lengthy and hence will not be shown here for convenience. One can see from Fig. 4 that the binding energy of three solitons is the largest (in magnitude) among all other molecules. The binding energy starts to stabilize when the number of solitons reaches 7. Another important remark is that the odd number of soliton molecules is more stable than the even ones. A similar behavior is obtained for the separation where it decreases with increasing number of solitons to saturate at $N_s \geqslant 7$. Also, a careful observation of the same figure shows that the separation between the second and third solitons converges at $N_s > 4$ to zero, which means that all solitons will eventually collapse into one and therefore the molecule does not exist for this case.

To verify that the equilibrium positions in Fig. 4 correspond to stable molecules, we have solved numerically the NLSE for three solitons. Our numerical solution was performed by the split-step fast Fourier transform method [17], with 2048 Fourier modes and the step size was $\delta z = 5 \times 10^{-4}$. In one case, we started the simulation with solitons located at the equilibrium initial positions and in another case with initial positions that were deviated from the equilibrium ones. Figure 5(a) shows that a stable soliton exists and the oscillations of the solitons are almost absent, while in Fig. 5(b), larger amplitude oscillations are caused by the deviation from the equilibrium positions. This clearly indicates that the

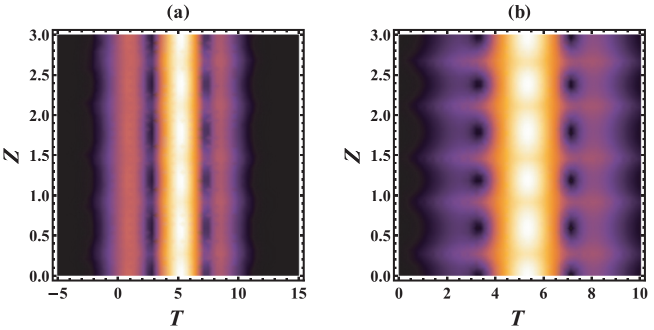

FIG. 5. (Color online) Spatiotemporal plot of the field amplitude for a 3-soliton molecule with (a) the equilibrium positions that minimize the energy, as in Fig. 4, and (b) compressed initial positions: $\eta_1 = 4.0$, $\eta_2 = 5.3$, $\eta_3 = 7.0$. The parameters are the same as in Fig. 4.

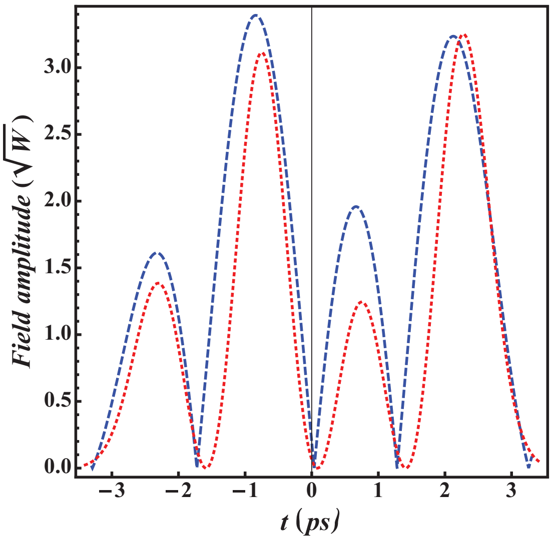

FIG. 6. (Color online) Field amplitude envelopes along dispersion-managed fiber of 4-soliton molecule for the same parameters as in Fig. 4. Blue dashed line: our numerical simulation of NLSE; red dotted line: our variational calculation.

equilibrium positions in Fig. 4 indeed correspond to stable molecules.

To check the validity of our variational calculations for many soliton molecules, we solve again numerically our NLSE. Figure 6 shows that our numerical simulation agrees reasonably well with our variational calculation for four soliton molecule.

In conclusion, we have considered N-soliton molecules propagating in dispersion-managed optical fibers. The well-known effective nonlinear Schrödinger equation was first rederived and then shown to be not integrable for the realistic situation. Using a variational calculation, the binding energy of the soliton molecules was calculated with emphasis on the 2- and 3-soliton molecules. From the locations and depths of the local minima in the equilibrium energy of the molecule, the bond length and strength were calculated. The calculated sizes of the 2- and 3-soliton molecules agreed favorably with the experimental values and the numerical simulation as shown in Fig. 3. Furthermore, we have solved numerically the NLSE and verified the stability of the 3-soliton molecules, as shown in Fig. 5. In addition, the field amplitudes of our trial function agreed favorably with the numerical solution, as shown in Fig. 6.

It should be noted that Raman scattering and higher order dispersion terms ought to affect the equilibrium properties of the molecule. These effects will be investigated in a future work. The calculation was also extended for larger molecules to show a nonmonotonic dependence of the molecule's binding energy and size in terms of the number of its constituent solitons.

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[1] M. Stratmann, T. Pagel, and F. Mitschke, Phys. Rev. Lett. **95**, 143902 (2005).
[2] P. Rohrmann, A. Hause, and F. Mitschke, Sci. Rep. **2**, 866 (2012).
[3] C. E. Shannon, Bell Syst. Tech. J. **27**, 379 (1948); **27**, 623 (1948).
[4] P. Rohrmann, A. Hause, and F. Mitschke, Phys. Rev. A **87**, 043834 (2013).
[5] B. A. Malomed, Prog. Opt. **43**, 69 (2002).
[6] B. Feng and B. A. Malomed, Opt. Commun. **229**, 173 (2004).
[7] U. Al Khawaja and Abdelâali Boudjemâa, Phys. Rev. E **86**, 036606 (2012).
[8] U. Al Khawaja, Phys. Rev. E **81**, 056603 (2010).
[9] A. Hause, H. Hartwig, and F. Mitschke, Phys. Rev. A **82**, 053833 (2010).
[10] S. K. Turitsyn, B. G. Bale, and M. P. Fedoruk, Phys. Rep. **521**, 135 (2012).
[11] V. N. Serkin, A. Hasegawa, and T. L. Belyaeva, Phys. Rev. Lett. **98**, 074102 (2007).
[12] A. Hause, H. Hartwig, M. Böhm, and F. Mitschke, Phys. Rev. A **78**, 063817 (2008).
[13] A. Hasegawa, Y. Kodama, and A. Maruta, Opt. Fiber Technol. **3**, 197 (1997).
[14] Z. Xu, L. Li, Z. Li, G. Zhou, and K. Nakkeeran, Phys. Rev. E **68**, 046605 (2003).
[15] U. Al Khawaja, J. Phys. A: Theor. Gen. **39**, 9679 (2006).
[16] J. H. B. Nijhof, N. J. Doran, W. Forysiak, and F. M. Knox, Electron. Lett. **33**, 1726 (1997); J. H. B. Nijhof, W. Forysiak, and N. J. Doran, IEEE J. Sel. Top. Quantum Electron. **6**, 330 (2000).
[17] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. (Academic, New York, 1995).