Spectral invariance and the scaling law with random electromagnetic fields

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We consider the far-zone spectrum generated by finite, statistically homogeneous, planar electromagnetic sources having three electric field components. We introduce an electromagnetic scaling law which ensures spectral invariance, i.e., that the normalized far-zone spectrum is the same in any paraxial or nonparaxial direction and equal to that of the source. The scaling law for electromagnetic fields presented in this work is more general than that put forward in a previous publication. In particular, we show paraxially that even though the individual field components do not obey the scalar scaling law, spectral invariance may be achieved, and conversely, that spectral invariance may not hold even if the field components separately follow the scalar scaling law.

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I. INTRODUCTION

Spectroscopy is a standard tool in the characterization of light sources, atomic particles, molecular clusters, material media, and the like. In the 1980s a new aspect to that topic was introduced when it was found that the normalized spectrum of random light may change on propagation even in free space [1] (for a review on the subject, see [2]). The spectral changes originate from the partial spatial coherence of the source. This fundamental finding indicates that the measured far-zone spectrum may differ in shape from that of the source, or it may vary as a function of direction. Remarkable demonstrations of these effects were provided with thermal sources supporting surface plasmon and phonon polaritons [3], and containing surface gratings [4]. It was shown theoretically that for finite, planar secondary sources which are statistically homogeneous, the normalized far-zone spectrum is the same in any direction and equals the source spectrum (corresponding to spectral invariance) if the so-called scaling law is satisfied [1]. Experimental demonstrations, using both acoustic [5] and light waves [6], prove that the source correlations may induce redshifts and blueshifts of spectral lines. This phenomenon is known as the Wolf effect, and it has been suggested to provide a possible, alternative explanation for the spectral redshifts of stellar objects [7,8].

In the above treatments dealing with the scaling law, the optical field is analyzed in terms of the scalar theory of light, thereby excluding the situations of wide-angle emission from partially polarized sources which necessitate three-component electric fields. Recently, an electromagnetic extension was considered restricting the analysis to the paraxial regime [9], and the same scaling law was established for both transverse components of the electric field. In this work, we reassess the scaling law in the electromagnetic context by considering nonparaxial random fields generated by finite, statistically homogeneous, planar sources containing all three electric field components. The spectral invariance and the associated scaling law for such sources hold for the total spectral density, i.e., the sum of the spectral densities pertaining to the individual components. Our results imply that spectral invariance may generally be achieved, even though none of the three electric field components obey the scalar scaling law.

We demonstrate the electromagnetic scaling law established here by considering the electric field emanating from an aperture in a blackbody cavity. Further, we show two explicit examples of converse nature in the paraxial regime. First, we describe a practical source whose individual field components violate the scaling law but the total normalized radiation spectrum and source spectrum coincide. And second, we analyze an electromagnetic planar source whose components separately obey the scaling law but the spectrum is not invariant in the far zone.

This work is organized as follows. In Sec. II the scalar scaling law is recalled, and in Sec. III its electromagnetic counterpart is established. Sections IV–VI illustrate the results with the specific examples, and the main conclusions are summarized in Sec. VII.

II. SPECTRAL INVARIANCE OF SCALAR FIELDS

In his pioneering paper [1] Wolf analyzed the invariance of far-field spectra within the framework of scalar coherence theory. In scalar analysis the electric field at a point $\mathbf{r}$ and frequency ω is represented by $E(\mathbf{r},\omega)$, and the correlations between the fields at two points in space are described by the cross-spectral density (CSD) function $W(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E^*(\mathbf{r}_1,\omega)E(\mathbf{r}_2,\omega)\rangle$, where the asterisk denotes complex conjugation and the angle brackets stand for ensemble averaging. The spectrum, or spectral density, of the field is $S_{\mathrm{sc}}(\mathbf{r},\omega) = W(\mathbf{r},\mathbf{r},\omega)$, where the subscript sc is used to distinguish the scalar spectral density from its electromagnetic counterpart.

In [1] a finite planar source of area D radiating into the half space $z > 0$ was considered (see Fig. 1). The source was taken to have the same spectrum $S_{\mathrm{sc}}^{(0)}(\omega)$ at each point and the complex degree of spectral coherence, defined as [10],

$$\mu^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \frac{W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{\sqrt{W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_1,\omega)W^{(0)}(\boldsymbol{\rho}_2,\boldsymbol{\rho}_2,\omega)}}, \quad (1)$$

was assumed to be of the form $\mu^{(0)}(\Delta\boldsymbol{\rho},\omega)$, where $\Delta\boldsymbol{\rho} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ is the difference of spatial coordinates, within the source area. Here $\boldsymbol{\rho}_i = (x_i, y_i)$, with $i \in (1,2)$, are two positions in

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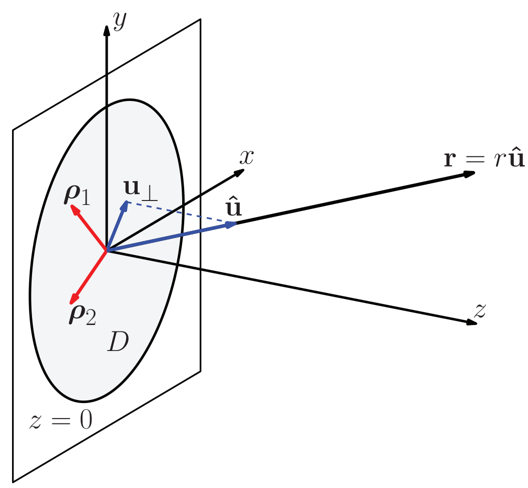

FIG. 1. (Color online) Geometry and notations. The homogeneous source of area D in plane $z = 0$ generates a field in half space $z > 0$. Of interest is the frequency dependence of the optical intensity at an arbitrary far-zone point $\mathbf{r}$, i.e., the radiant intensity in direction $\hat{\mathbf{u}}$.

D and the superscript (0) is used to emphasize functions evaluated in the source plane $z = 0$. Moreover, the dimensions of the source were assumed to be much larger than the spectral correlation width. A source of this kind then has the CSD of the form

$$W^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = S^{(0)}_{\rm sc}(\omega)\mu^{(0)}(\Delta\boldsymbol{\rho},\omega), \tag{2}$$

and it is an example of a simple quasihomogeneous, planar, scalar source [11].

The spectral radiant intensity $J_{\rm sc}(\hat{\mathbf{u}},\omega)$ is the power at frequency ω radiated by the source into the far zone per unit solid angle. It can be shown that the normalized far-field spectrum, i.e., the spectral radiant intensity divided by the total radiant intensity,

$$s^{(\infty)}_{\rm sc}(\hat{\mathbf{u}},\omega) = \frac{J_{\rm sc}(\hat{\mathbf{u}},\omega)}{\int_0^\infty J_{\rm sc}(\hat{\mathbf{u}},\omega)\,d\omega}, \tag{3}$$

with $\hat{\mathbf{u}}$ being the directional unit vector, generated by a source obeying Eq. (2), is [1]

$$s^{(\infty)}_{\rm sc}(\hat{\mathbf{u}},\omega) = \frac{k^2 S^{(0)}_{\rm sc}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)}{\int_0^\infty k^2 S^{(0)}_{\rm sc}(\omega)\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega)\,d\omega}, \tag{4}$$

where $k = \omega/c$ is the wave number and c is the speed of light. Moreover, $\tilde{\mu}^{(0)}$ denotes the two-dimensional (2D) spatial Fourier transform of $\mu^{(0)}$, and $\mathbf{u}_\perp = (u_x,u_y)$ is the transverse part of $\hat{\mathbf{u}}$. For spectral invariance one demands that the normalized spectrum in the far zone is uniform in all directions. This is achieved when $\tilde{\mu}^{(0)}$ factors as

$$\tilde{\mu}^{(0)}(k\mathbf{u}_\perp,\omega) = F(\omega)\tilde{H}(\mathbf{u}_\perp), \tag{5}$$

where F and H are arbitrary functions, resulting in directional independence of the right-hand side of Eq. (4). Furthermore, the normalized far-field spectrum must also be equal to the normalized source spectrum, i.e., $s^{(\infty)}_{\rm sc}(\omega) = s^{(0)}_{\rm sc}(\omega)$. Taking the inverse Fourier transform of Eq. (5) and using the fact that the complex degree of spectral coherence is a correlation function, whereby $\mu^{(0)}(0,\omega) = 1$, one ends up with

$$F(\omega) = [k^2 H(0)]^{-1}, \tag{6}$$

where H is the 2D inverse Fourier transform of $\tilde{H}$. Thus, Eq. (4) can now be written as

$$s^{(\infty)}_{\rm sc}(\hat{\mathbf{u}},\omega) = s^{(\infty)}_{\rm sc}(\omega) = \frac{S^{(0)}_{\rm sc}(\omega)}{\int_0^\infty S^{(0)}_{\rm sc}(\omega)\,d\omega} = s^{(0)}_{\rm sc}(\omega), \tag{7}$$

and the condition for the invariance of the spectrum on propagation is attained. By combining Eqs. (5) and (6) one gets

$$\mu^{(0)}(\Delta\boldsymbol{\rho},\omega) = \frac{H(k\Delta\boldsymbol{\rho})}{H(0)}, \tag{8}$$

which is called the scaling law by Wolf [1]. According to this law, the complex degree of spectral coherence of the source must depend only on the quantity $k\Delta\boldsymbol{\rho}$ for the normalized far-field spectrum to be independent of direction and equal to the normalized source spectrum. Of course, this scaling law necessarily holds only for sources of the type considered in the analysis, i.e., homogeneous, planar, scalar sources. An important class of sources that obey Eq. (8) are planar Lambertian sources [12].

III. SPECTRAL INVARIANCE OF ELECTROMAGNETIC FIELDS

We next extend Wolf's scalar scaling law to electromagnetic sources. In electromagnetic coherence theory, a realization of the random electric field can be represented by the three-component vector $\mathbf{E}(\mathbf{r},\omega) = [E_x(\mathbf{r},\omega),E_y(\mathbf{r},\omega),E_z(\mathbf{r},\omega)]^{\rm T}$, where the superscript T denotes the transpose. Correlations between the electric field components at two spatial points are given by the 3×3 CSD matrix [13,14] $\mathbf{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle\mathbf{E}^*(\mathbf{r}_1,\omega)\mathbf{E}^{\rm T}(\mathbf{r}_2,\omega)\rangle$. As there now are multiple field components, the spectral density of the field is a sum of the spectral densities of the individual components, i.e., $S_{\rm em}(\mathbf{r},\omega) = {\rm tr}\,\mathbf{W}(\mathbf{r},\mathbf{r},\omega)$, where the subscript em refers to an electromagnetic quantity and tr denotes the trace.

We emphasize that in the following analysis we consider all three Cartesian field components of the source and the far field. Thus, we are not restricted to paraxial beams but can deal with arbitrary wide-angle radiation. In line with the scalar case, we take the source to be finite (area D), planar, located in the plane $z = 0$, and radiating into the half space $z > 0$ (see Fig. 1). We also assume that the source correlations are statistically homogeneous, i.e., $\mathbf{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \mathbf{W}^{(0)}(\Delta\boldsymbol{\rho},\omega)$ in D, which implies that the spectral density is uniform across the source, $S^{(0)}_{\rm em}(\boldsymbol{\rho},\omega) = S^{(0)}_{\rm em}(\omega)$. We can then write the CSD matrix in the source plane as

$$\mathbf{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = S^{(0)}_{\rm em}(\omega)\mathbf{w}^{(0)}(\Delta\boldsymbol{\rho},\omega), \tag{9}$$

where

$$\mathbf{w}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = \frac{\mathbf{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)}{\left[S^{(0)}_{\rm em}(\boldsymbol{\rho}_1,\omega)S^{(0)}_{\rm em}(\boldsymbol{\rho}_2,\omega)\right]^{1/2}} \tag{10}$$

is the CSD matrix normalized with the spectral densities.

The question now is whether it is possible to acquire a scaling law on the basis of these assumptions also for random three-component electromagnetic sources. In other words, can one find physical conditions for the normalized far-zone spectrum to be independent of direction and equal to the normalized source spectrum? As we show below, the answer is affirmative.

We begin by considering the CSD matrix in the far field. By using the method of stationary phase [13], the far-zone CSD matrix at a pair of points $\mathbf{r}_i = r_i\hat{\mathbf{u}}_i$, with $i \in (1,2)$, can be written as [15]

$$\mathbf{W}^{(\infty)}(r_1\hat{\mathbf{u}}_1,r_2\hat{\mathbf{u}}_2,\omega) = (2\pi k)^2 u_{z1}u_{z2}\frac{e^{ik(r_2-r_1)}}{r_1r_2} \times \mathbf{T}(k\mathbf{u}_{\perp 1},k\mathbf{u}_{\perp 2},\omega), \tag{11}$$

where $u_{zi} = (1-\mathbf{u}_{\perp i}^2)^{1/2}$ and

$$\mathbf{T}(k\mathbf{u}_{\perp 1},k\mathbf{u}_{\perp 2},\omega) = \frac{1}{(2\pi)^4}\iint_{-\infty}^{\infty} \mathbf{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \times e^{-i(\mathbf{k}_{\perp 2}\cdot\boldsymbol{\rho}_2-\mathbf{k}_{\perp 1}\cdot\boldsymbol{\rho}_1)}\, d^2\boldsymbol{\rho}_1\, d^2\boldsymbol{\rho}_2 \tag{12}$$

is the angular correlation matrix. Here $\mathbf{k}_{\perp i} = k\mathbf{u}_{\perp i} = (k_{xi},k_{yi})$, with $i \in (1,2)$, are the transverse components (spatial-frequency vectors) of the wave vectors corresponding to propagating plane waves. The spectral radiant intensity of the electric field characterizing the angular distribution of power flow is given by

$$J_{\mathrm{em}}(\hat{\mathbf{u}},\omega) = \lim_{r\to\infty}[r^2 S^{(\infty)}(r\hat{\mathbf{u}},\omega)] = (2\pi k u_z)^2 \operatorname{tr}\mathbf{T}(k\mathbf{u}_\perp,k\mathbf{u}_\perp,\omega). \tag{13}$$

Inserting Eqs. (9) and (12) into Eq. (13) and expressing it in terms of the average and difference spatial coordinates yields

$$J_{\mathrm{em}}(\hat{\mathbf{u}},\omega) = (ku_z)^2 D S_{\mathrm{em}}^{(0)}(\omega)\operatorname{tr}\tilde{\mathbf{w}}^{(0)}(k\mathbf{u}_\perp,\omega), \tag{14}$$

where $\tilde{\mathbf{w}}^{(0)}$ is the 2D spatial Fourier transform of $\mathbf{w}^{(0)}$. The normalized spectrum in the far field then takes on the form

$$s_{\mathrm{em}}^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{J_{\mathrm{em}}(\hat{\mathbf{u}},\omega)}{\int_0^\infty J_{\mathrm{em}}(\hat{\mathbf{u}},\omega)\,d\omega} = \frac{k^2 S_{\mathrm{em}}^{(0)}(\omega)\operatorname{tr}\tilde{\mathbf{w}}^{(0)}(k\mathbf{u}_\perp,\omega)}{\int_0^\infty k^2 S_{\mathrm{em}}^{(0)}(\omega)\operatorname{tr}\tilde{\mathbf{w}}^{(0)}(k\mathbf{u}_\perp,\omega)\,d\omega}, \tag{15}$$

much as in the scalar case.

Equation (15) indicates that the normalized far-zone spectrum will be direction independent if the trace of the Fourier transform of source's normalized CSD matrix factors into spectral and directional parts as

$$\operatorname{tr}\tilde{\mathbf{w}}^{(0)}(k\mathbf{u}_\perp,\omega) = F(\omega)\tilde{H}(\mathbf{u}_\perp). \tag{16}$$

By taking the Fourier inverse of Eq. (16) and using the fact that $\operatorname{tr}\mathbf{w}^{(0)}(0,\omega) = 1$, which follows from Eq. (10), we see that the frequency dependence of F is of the form

$$F(\omega) = [k^2 H(0)]^{-1}, \tag{17}$$

and the normalized far-zone spectrum is also equal to the normalized source spectrum, i.e.,

$$s_{\mathrm{em}}^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S_{\mathrm{em}}^{(0)}(\omega)}{\int_0^\infty S_{\mathrm{em}}^{(0)}(\omega)\,d\omega} = s_{\mathrm{em}}^{(0)}(\omega). \tag{18}$$

Inserting Eq. (17) back to the Fourier inverse of Eq. (16) then yields

$$\operatorname{tr}\mathbf{w}^{(0)}(\Delta\boldsymbol{\rho},\omega) = \frac{H(k\Delta\boldsymbol{\rho})}{H(0)}. \tag{19}$$

This equation constitutes the scaling law for electromagnetic sources. It states that for a homogeneous source obeying Eq. (19), the normalized far-field spectrum is independent of (paraxial or nonparaxial) direction and equals that of the source. In fact, within the accuracy of Fresnel's diffraction formula, the normalized spectrum remains invariant on propagation from the source to the far field. The scaling law also implies that the knowledge of the elements of the normalized CSD matrix at a single frequency is sufficient to characterize the normalized CSD matrix at all frequencies.

Although the electromagnetic scaling law of Eq. (19), which holds for homogeneous, three-component, planar, secondary sources and the electric fields they produce, appears as a straightforward generalization of the scalar scaling law, there is one fundamental difference. In the electromagnetic case it is the trace of $\mathbf{w}^{(0)}$ that satisfies the scaling law; hence it is not necessary that individual components of $\mathbf{w}^{(0)}$ separately obey the scalar scaling law. We point out that Eq. (19) is thus more general than Eq. (24) in [9], established for paraxial fields. In essence, the scaling law of [9] requires the normalized diagonal components of the CSD matrix [16] to satisfy the scalar scaling law, and so it is a special case of the three-component scaling law derived in this work. Further, in the paraxial case Eq. (19) can be expressed in the form

$$\eta_0^{(0)}(\Delta\boldsymbol{\rho},\omega) = \frac{H(k\Delta\boldsymbol{\rho})}{H(0)}, \tag{20}$$

where $\eta_0(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is a two-point Stokes parameter that corresponds to the visibility of the intensity-interference fringes in Young's experiment [17,18]. Thus, even in the paraxial case, the individual field components do not need to obey the scalar scaling law, Eq. (8). On the other hand, even if the individual components obey the scalar scaling law, it is not guaranteed that the normalized far-zone spectrum is invariant on direction.

In the following three sections, we illustrate the physical nature of the electromagnetic scaling law and its differences to the scalar case. Section IV presents an example of a highly nonparaxial field whose normalized far-zone spectrum is invariant and equals that of the source. Furthermore, the same holds for each Cartesian component individually. Section V discusses a situation in which the spectrum of a paraxial field is invariant and equals that of the source, but the x and y field components do not obey the scalar law, Eq. (8). In Sec. VI we discuss the opposite case in which the two components of a paraxial field both obey the scalar scaling law, but their superposition leads to a normalized far-zone spectrum that varies upon the observation direction.

IV. EXAMPLE I: RADIATION EMANATING FROM BLACKBODY CAVITY

As an example, we first consider the field emanating from a circular opening in a blackbody cavity wall. The CSD matrix of the aperture field has been analyzed previously [19,20], and quite recently reassessed [21]. The aperture field has three electric field components and the diagonal elements of its CSD matrix (in the $z = 0$ plane) are of the form

$$W_{xx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = 2\pi a_0(\omega)\left[j_0(k|\Delta\boldsymbol{\rho}|) - \frac{j_1(k|\Delta\boldsymbol{\rho}|)}{k|\Delta\boldsymbol{\rho}|} + j_2(k|\Delta\boldsymbol{\rho}|)\frac{(x_2-x_1)^2}{|\Delta\boldsymbol{\rho}|^2}\right], \quad (21a)$$

$$W_{yy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = 2\pi a_0(\omega)\left[j_0(k|\Delta\boldsymbol{\rho}|) - \frac{j_1(k|\Delta\boldsymbol{\rho}|)}{k|\Delta\boldsymbol{\rho}|} + j_2(k|\Delta\boldsymbol{\rho}|)\frac{(y_2-y_1)^2}{|\Delta\boldsymbol{\rho}|^2}\right], \quad (21b)$$

$$W_{zz}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = 2\pi a_0(\omega)\left[j_0(k|\Delta\boldsymbol{\rho}|) - \frac{j_1(k|\Delta\boldsymbol{\rho}|)}{k|\Delta\boldsymbol{\rho}|}\right]. \quad (21c)$$

Here $a_0(\omega)$ equals Planck's spectrum divided by 4 [21], and the functions j_i, $i = 0,1,2$, are spherical Bessel functions of order i. It is straightforward to show, using the identities of spherical Bessel functions, that for the electric field in the opening

$$\operatorname{tr}\mathbf{w}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) = j_0(k|\Delta\boldsymbol{\rho}|), \quad (22)$$

which clearly satisfies the electromagnetic scaling law of Eq. (19). The same result holds also for a true blackbody field in a plane within the cavity [12]. Therefore, the normalized far-field spectrum emanating from an aperture in a blackbody cavity is in all directions the same as the normalized spectrum of the aperture field. The normalized spectrum, in turn, is specified by Planck's law and is explicitly given by

$$s_{\mathrm{em}}^{(\infty)}(\omega) = s_{\mathrm{em}}^{(0)}(\omega) = 15\left(\frac{\hbar}{\pi k_B T}\right)^4 \frac{\omega^3}{e^{\hbar\omega/k_B T} - 1}, \quad (23)$$

where $\hbar$ is the reduced Planck constant, k_B is the Boltzmann constant, and T is the absolute temperature.

Comparing Eqs. (21) to the scalar scaling law, Eq. (8), reveals that not only does the total field obey the electromagnetic scaling law, but also the individual field components have spectra that are invariant in the far zone.

V. EXAMPLE II: SPECTRAL COMPENSATION BETWEEN THE FIELD COMPONENTS

Consider next the following example, shown in Fig. 2. A paraxial electric field that is finite but statistically homogeneous within its cross section (area D) is split into the two arms of a Mach-Zehnder interferometer. On recombination of the two paths the last mirror is positioned such that the beam from the first arm is shifted by distance a in the $-x$ direction, as illustrated in the figure. The shift a is taken to be much larger than the transverse correlation length but much smaller than the characteristic dimension of the field itself. In the second arm, the y component of the electric field is delayed by $\Delta\phi = \pi$ radians with respect to the x component, for example, with an achromatic half-wave plate (H in the figure). The path-length difference due to a and the phase shift of the x component in the half-wave plate is compensated, e.g., by an additional delay line (C in the figure). The compensation is positioned in one of the two arms, depending on the actual values of a and the phase shift in H.

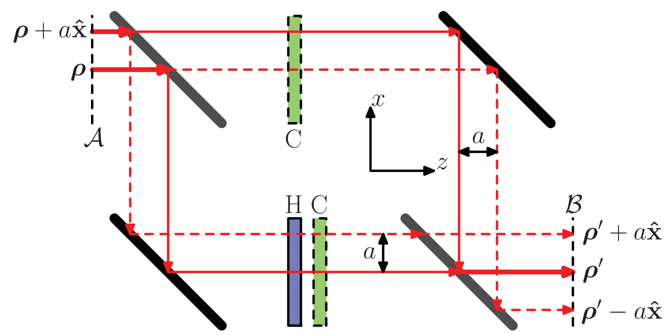

FIG. 2. (Color online) Modified Mach-Zehnder interferometer. The axial displacement (by distance a) of the last mirror creates a transverse shift a between the two replicas of the beam entering at plane $\mathcal{A}$. The shift produces an additional spatial correlation into the recombined beam at plane $\mathcal{B}$. Elements H (half-wave plate) and C (compensator) are described in the text.

We assume that the input field is unpolarized, and that the x and y components are uncorrelated at any two positions $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ in the input plane $\boldsymbol{\mathcal{A}}$. In addition, we take the correlation properties of both components to be equal and obey the scalar scaling law, Eq. (8). Hence

$$W_{xx}(\Delta\boldsymbol{\rho},\omega) = W_{yy}(\Delta\boldsymbol{\rho},\omega) = S_{\mathrm{em}}(\omega)\frac{H(k\Delta\boldsymbol{\rho})}{2H(0)}, \quad (24)$$

where $S_{\mathrm{em}}(\omega)$ is the spectrum in the plane $\boldsymbol{\mathcal{A}}$.

It follows from the geometry that the electric field realizations at any position $\boldsymbol{\rho}'$ in the output plane $\boldsymbol{\mathcal{B}}$ are obtained from

$$E_i(\boldsymbol{\rho}',\omega) = \frac{\beta(\omega)}{\sqrt{2}}[E_i(\boldsymbol{\rho} + a\hat{\mathbf{x}},\omega) \pm E_i(\boldsymbol{\rho},\omega)], \quad (25)$$

where $i \in (x,y)$, $\beta(\omega)$ represents the phase shifts and losses, assumed to be equal along the two paths, and the $+$ or $-$ sign is chosen for the x or y component, respectively. The $-$ sign associated with the y component is due to the π phase shift in the second arm. The diagonal elements of the CSD matrix in the output plane can now be expressed in terms of the corresponding elements in the input plane:

$$W_{ii}(\Delta\boldsymbol{\rho}',\omega) = \frac{S'_{\mathrm{em}}(\omega)}{4H(0)}[2H(k\Delta\boldsymbol{\rho}',\omega) \pm H(k\Delta\boldsymbol{\rho}' + ak\hat{\mathbf{x}},\omega) \pm H(k\Delta\boldsymbol{\rho}' - ak\hat{\mathbf{x}},\omega)], \quad (26)$$

where $S'_{\mathrm{em}}(\omega) = |\beta(\omega)|^2 S_{\mathrm{em}}(\omega)$ is the spectral density in plane $\mathcal{B}$, and the $\pm$ sign is chosen as above for $i \in (x,y)$. Here we have made use of the homogeneity of the input CSD matrix and the fact that the shift a is small compared to the transverse beam dimension.

If the output plane $\mathcal{B}$ of the Mach-Zehnder interferometer acts as a secondary source and the field is diffracted from the plane freely, we find that the far-zone radiation patterns of the x and y components are characterized by the radiant intensities

$$J_x(\hat{\mathbf{u}},\omega) = u_z^2 D S'_{\mathrm{em}}(\omega)\frac{\tilde{H}(\mathbf{u}_\perp)}{H(0)}\cos^2(k_x a/2), \quad (27a)$$

$$J_y(\hat{\mathbf{u}},\omega) = u_z^2 D S'_{\mathrm{em}}(\omega)\frac{\tilde{H}(\mathbf{u}_\perp)}{H(0)}\sin^2(k_x a/2). \quad (27b)$$

The sine and cosine terms in these expressions are due to the interference between the beams from the two arms. The total radiant intensity is of the form

$$J_{\mathrm{em}}(\hat{\mathbf{u}},\omega) = u_z^2 D S'_{\mathrm{em}}(\omega)\frac{\tilde{H}(\mathbf{u}_\perp)}{H(0)}, \qquad (28)$$

i.e., because of the phase difference created in the second arm, the interference patterns of x and y components compensate each other and no intensity interference is observed.

We see at once that the normalized far-field spectra of the x and y components are

$$s_x^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S'_{\mathrm{em}}(\omega)\cos^2(k_x a/2)}{\int_0^\infty S'_{\mathrm{em}}(\omega)\cos^2(k_x a/2)\,d\omega}, \qquad (29a)$$

$$s_y^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S'_{\mathrm{em}}(\omega)\sin^2(k_x a/2)}{\int_0^\infty S'_{\mathrm{em}}(\omega)\sin^2(k_x a/2)\,d\omega}. \qquad (29b)$$

The total normalized electric field spectrum in the far zone is

$$s_{\mathrm{em}}^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S'_{\mathrm{em}}(\omega)}{\int_0^\infty S'_{\mathrm{em}}(\omega)\,d\omega}. \qquad (30)$$

Thus, the normalized far-zone spectrum of the whole field is independent of direction, even though the same is not generally true for the individual field components.

This example is analogous to a Young's interferometer with a coherent field, but with orthogonal polarization states at the two pinholes [17,22,23]. If, for example, the field in the first pinhole is linearly polarized at 45 degrees from the x direction towards the y direction and the field in the second pinhole is linearly polarized at −45 degrees, and if the intensities at the two pinholes are equal, the x component creates a cosinusoidal intensity pattern on the observation screen, whereas the y component creates a sinusoidal one. Also, in that case the two patterns compensate each other so that the total intensity is not modulated at all, even though the correlation at the pinholes is perfect.

VI. EXAMPLE III: FIELD WHOSE INDIVIDUAL COMPONENTS OBEY THE SCALING LAW, BUT THE SPECTRUM IS NOT INVARIANT IN THE FAR ZONE

Assume again that we are dealing with a paraxial field whose uncorrelated x and y components obey the scalar scaling law, Eq. (8). However, now we do not demand that the components are identical, but just require that

$$W_{xx}^{(0)}(\Delta\boldsymbol{\rho},\omega) = S_x^{(0)}(\omega)\frac{H_x(k\Delta\boldsymbol{\rho})}{H_x(0)}, \qquad (31a)$$

$$W_{yy}^{(0)}(\Delta\boldsymbol{\rho},\omega) = S_y^{(0)}(\omega)\frac{H_y(k\Delta\boldsymbol{\rho})}{H_y(0)}, \qquad (31b)$$

where $S_x^{(0)}(\omega)$ and $S_y^{(0)}(\omega)$ are the spectral densities of x and y components within the cross section of the field at the source plane. Further, $H_x(k\Delta\boldsymbol{\rho})$ and $H_y(k\Delta\boldsymbol{\rho})$ are, in general, different functions. Now

$$\eta_0^{(0)}(\Delta\boldsymbol{\rho},\omega) = \frac{W_{xx}^{(0)}(\Delta\boldsymbol{\rho},\omega) + W_{yy}^{(0)}(\Delta\boldsymbol{\rho},\omega)}{S_x^{(0)}(\omega) + S_y^{(0)}(\omega)} = \frac{1}{S_x^{(0)}(\omega) + S_y^{(0)}(\omega)}\left[S_x^{(0)}(\omega)\frac{H_x(k\Delta\boldsymbol{\rho})}{H_x(0)} + S_y^{(0)}(\omega)\frac{H_y(k\Delta\boldsymbol{\rho})}{H_y(0)}\right], \qquad (32)$$

which is not of the form of Eq. (20), unless $H_x(k\Delta\boldsymbol{\rho}) = H_y(k\Delta\boldsymbol{\rho})$ or $S_x^{(0)}(\omega) = S_y^{(0)}(\omega)$.

Consider, for example, an electric field with

$$S_x^{(0)}(\omega) = S_0 \exp\left[-(\omega-\omega_{0x})^2/\Omega_x^2\right], \qquad (33a)$$

$$S_y^{(0)}(\omega) = S_0 \exp\left[-(\omega-\omega_{0y})^2/\Omega_y^2\right], \qquad (33b)$$

$$H_x(k\Delta\boldsymbol{\rho}) = \exp\left[-k^2(\Delta\rho)^2/\sigma_x^2\right], \qquad (33c)$$

$$H_y(k\Delta\boldsymbol{\rho}) = \exp\left[-k^2(\Delta\rho)^2/\sigma_y^2\right], \qquad (33d)$$

where S_0 is a real positive constant, $\omega_{0i} > 0$, $i \in (x,y)$, are the central frequencies of the components, and Ω_i and σ_i/k are their spectral widths and transverse correlation lengths, respectively. To ensure that we are dealing with a valid complex-analytic signal, we demand that $\Omega_i \ll \omega_{0i}$. Further, to ensure that the field is paraxial, we also require that σ_i is much greater than the wavelength. Making use of Eqs. (12) and (13), and the identity [24]

$$\int_{-\infty}^{\infty} \exp(-p^2x^2 \pm qx)\,dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p}, \quad p > 0, \qquad (34)$$

we find that the radiant intensities of the components are

$$J_x(\hat{\mathbf{u}},\omega) = u_z^2 D S_x^{(0)}(\omega)\frac{\sigma_x^2}{4\pi}\exp\left[-\left(u_x^2+u_y^2\right)\sigma_x^2/4\right], \qquad (35a)$$

$$J_y(\hat{\mathbf{u}},\omega) = u_z^2 D S_y^{(0)}(\omega)\frac{\sigma_y^2}{4\pi}\exp\left[-\left(u_x^2+u_y^2\right)\sigma_y^2/4\right], \qquad (35b)$$

and, as expected, the normalized spectral densities of the components are invariant in the far zone:

$$s_x^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S_x^{(0)}(\omega)}{\int_0^\infty S_x^{(0)}(\omega)\,d\omega} = s_x^{(0)}(\omega), \qquad (36a)$$

$$s_y^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{S_y^{(0)}(\omega)}{\int_0^\infty S_y^{(0)}(\omega)\,d\omega} = s_y^{(0)}(\omega). \qquad (36b)$$

However, the normalized spectral density of the total field is now given by

$$s_{\mathrm{em}}^{(\infty)}(\hat{\mathbf{u}},\omega) = \frac{\sigma_x^2\exp\left[-(\omega-\omega_{0x})^2/\Omega_x^2\right]\exp\left[-\left(u_x^2+u_y^2\right)\sigma_x^2/4\right] + \sigma_y^2\exp\left[-(\omega-\omega_{0y})^2/\Omega_y^2\right]\exp\left[-\left(u_x^2+u_y^2\right)\sigma_y^2/4\right]}{\sqrt{\pi}\left\{\Omega_x\sigma_x^2\exp\left[-\left(u_x^2+u_y^2\right)\sigma_x^2/4\right] + \Omega_y\sigma_y^2\exp\left[-\left(u_x^2+u_y^2\right)\sigma_y^2/4\right]\right\}}, \qquad (37)$$

where we have again made use of Eq. (34) and our assumption $\Omega_i \ll \omega_0$. We observe at once from Eq. (37) that the normalized far-zone spectrum varies upon the direction of observation, unless either $\Omega_x = \Omega_y$ and $\omega_{0x} = \omega_{0y}$ or $\sigma_x = \sigma_y$.

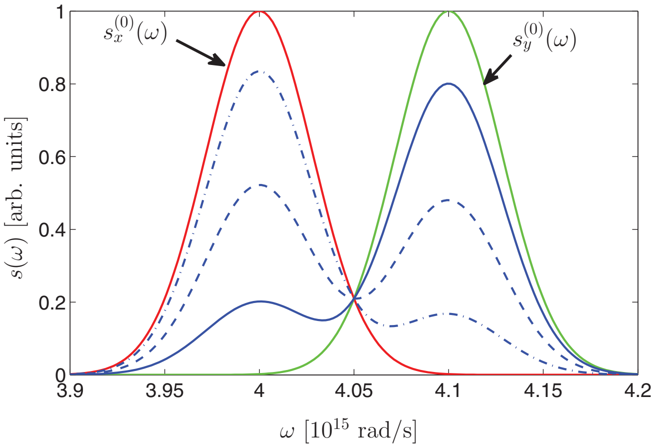

FIG. 3. (Color online) Normalized spectra of the x (red) and y (green) source components, as well as the normalized far-zone spectrum of the total field for $u_x = 0$ (solid blue), $u_x = 0.014$ (dashed blue), and $u_x = 0.02$ (dash-dotted blue). The other parameters are $u_y = 0$, $\omega_{0x} = 4 \times 10^{15}$ rad/s, $\omega_{0y} = 4.1 \times 10^{15}$ rad/s, $\Omega_x = \Omega_y = 4 \times 10^{13}$ rad/s, $\sigma_x = 100$, and $\sigma_y = 200$.

Figure 3 illustrates an example of the normalized spectral densities as a function of ω for three different values of u_x. We see from the figure that on the z axis the contribution to the normalized spectrum from the y component is dominant, and the significance of the x component increases when u_x becomes larger. This is due to the smaller transverse correlation length ($\sim$7.5 μm) of the x component compared to that of y ($\sim$15 μm).

The physical reason for this result, that may appear counterintuitive at first, is after all quite simple: Since the far-zone diffraction patterns of the individual field components are different, as are their spectra, the total spectrum and its normalized version vary as a function of the direction in the far zone. The fact that the normalized spectra of the components are invariant does not affect the situation, since the normalized spectrum of the total field is not a direct sum of the normalized spectra of the components.

VII. CONCLUSIONS

In this work we considered spectral invariance on radiation and formulated an electromagnetic scaling law that is valid for finite planar, statistically homogeneous sources. Such sources may, in general, contain all three electric field components, as they must in the case of wide-angle radiation. When the source obeys the scaling law, the normalized far-zone spectrum is the same in every direction, paraxial or nonparaxial, and equal to the normalized source spectrum. The scaling law in its three-component form was assessed for wide-angle radiation emanating from an opening in a blackbody cavity wall. Furthermore, we found in general that the individual source-field components do not need to satisfy the scalar scaling law in order to achieve spectral invariance for the total electric field. On the other hand, we also showed that the normalized far-zone spectrum may not remain invariant even if the source-field components individually obey the scaling law. Explicit examples of both of these circumstances were given in the paraxial electromagnetic regime. The results of this research can find applications in the spectroscopy of light sources, characterization of material objects, and in modeling the transport of radiation from sources.

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