Universal behavior of four-boson systems from a functional-renormalization-group analysis

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We apply a functional renormalization group to systems of four bosonic atoms close to the unitary limit. We work with a local effective action that includes a dynamical trimer field, and we use this field to eliminate structures that do not correspond to the Faddeev-Yakubovsky equations. In the physical limit, we find three four-body bound states below the shallowest three-body state. The values of the scattering lengths at which two of these states become bound are in good agreement with exact solutions of the four-body equations and experimental observations. The third state is extremely shallow. During the evolution we find an infinite number of four-body states based on each three-body state which follow a double-exponential pattern in the running scale. None of the four-body states shows any evidence of dependence on a four-body parameter.

I. INTRODUCTION

Systems where two-body scattering lengths are much longer than ranges of the forces between the particles are important in various areas of physics. Their low-energy properties display universal scaling behavior, controlled by the "unitary limit" in which the scattering length tends to infinity. In nuclear physics, the large scattering lengths are large enough that low-energy aspects of few-nucleon systems can be described in this framework [1,2]. In atomic physics, the shallow dimer of ^{4}He atoms leads to a scattering length that is about 100 times larger than the size of the atoms [3]. Even better examples are provided by ultra-cold atoms in traps, where Feshbach resonances can be used to tune the scattering lengths to values very close to the unitary limit [4].

In the unitary limit, three-boson systems display a remarkable effect, first predicted by Efimov in 1970 [5,6]. They possess an infinite tower of three-body bound states, with energies in a constant ratio of ∼515.0. This breaks the expected scale invariance to a discrete symmetry, with one three-body parameter needed to fix the energies of all these states. In real systems, the sequence of deeply bound states is cut off by the range of the forces, and the shallowest ones by the finite scattering length. Three-fermion systems can also show Efimov behavior, provided there are enough species to allow spatially symmetric states. Although there were suggestions that the $A = 3$ nuclei ^{3}H and ^{3}He could be interpreted as Efimov states [7], the first clear observation of such states was in an ultra-cold gas of caesium atoms [8]. Reviews of the field can be found in Refs. [9–11].

This behavior in the three-body sector feeds through to four-body systems, where most numerical calculations find two bound states in each Efimov cycle [12–14] whose energies are fixed ratios to the nearest three-body state. However, in contrast, Hadizadeh *et al.* find up to three four-body states per cycle, with energies that depend on an additional four-body parameter [15,16], supporting their earlier results of Ref. [17]. Experimental evidence for two four-body states based on an Efimov three-body state has been seen in the recombination rates of trapped ^{133}Cs atoms [18], with resonances that are consistent with the results of Refs. [12–14].

Renormalization-group methods have been applied to elucidate scaling behavior in few-body systems [7,11,19–21]

and hence to determine their relevant parameters. Here we apply a functional renormalization group (FRG) [22,23] to the four-boson problem. During the evolution we observe a double-exponential pattern of four-body states built on each three-body state, similar to the "super-Efimov" behavior found by Nishida, Moroz, and Son in a two-dimensional three-body system [24]. These have energies that can be expressed in terms of a universal scaling function, similar to that in Refs. [15,16], but they show no evidence of dependence on an additional four-body parameter. The states in our "super-Efimov" pattern are not necessarily physical and, away from the unitary limit, we find that only three of them are present in the last Efimov cycle and so can appear as physical bound states. The two deepest of these states appear for scattering lengths that are in reasonable agreement with those found in studies of four-body equations [13,14] and experimental observations [18].

This paper is structured as follows. In Sec. II we present the FRG and running action that we use to study four-atom systems. Previous results on the three-body sector are summarized in Sec. III, as they provide key input into our four-body equations. Those equations are presented in Sec. IV together with our results for the four-body sector. We summarize and conclude in Sec. V.

II. FRG AND RUNNING ACTION

The FRG we use is based on a running version of effective action that generates the one-particle irreducible Green's functions [22,23]. A regulator is added to the theory to suppress fluctuations with momenta below some scale k. For large k, we start with a suitably parametrized "bare" action. The method works by evolving from this bare action to the limit $k \to 0$, where all quantum fluctuations are included and the action becomes physical. Away from this limit, that is for $k > 0$, the running action is not physical because of the partial suppression of fluctuations. Even though it is fully nonperturbative, the driving term in the FRG equation for the action has the form of a one-loop integral. Instead of diagrammatic expansions, practical approximation schemes are obtained by truncating the effective action to a finite number of terms.

This FRG is being applied to systems of nonrelativistic particles, in order to study, in particular, dense matter [25–27].

In that context, it provides an alternative to traditional many-body methods. As part of this program, studies of few-body systems in the same framework are needed to fix the input parameters. These studies are also proving interesting in their own right [28–31].

A key ingredient of our approach is a trimer field. Such fields have been introduced before, in Refs. [28–30]. However in the previous application to the four-boson problem [30], this field was used to explore the dependence of amplitudes on an external energy. In contrast, our approach emphasises its dynamical role. This allows us to describe the atom-trimer channel of the four-body system and hence to obtain equations with a structure like that of the Faddeev-Yakubovsky equations [32].

In this work we study systems of up to four nonrelativistic bosonic "atoms." We represent the atoms by the field $\psi(x)$ and we also introduce dimer and trimer fields, $\phi(x)$ and $\chi(x)$, in order to include energy-dependent propagators for two- and three-body subsystems. The evolution equation for the effective action $\Gamma_k[\psi,\psi^*,\phi,\phi^*,\chi,\chi^*]$ takes the form [23]

$$\partial_k \Gamma = -\frac{i}{2}\,\mathrm{Tr}[(\partial_k \mathbf{R})\,((\mathbf{\Gamma}^{(2)} - \mathbf{R})^{-1})] + \frac{\delta\Gamma}{\delta\Phi}\cdot\partial_k\Phi, \qquad (1)$$

where $\mathbf{\Gamma}^{(2)}$ denotes the matrix of second derivatives of the action with respect to the fields and $\mathbf{R}$ the regulator that is added to suppress low-momentum modes. The trace Tr and the scalar

product in the final term include integrals over energy and three momenta as well as sums over the different types of field. The final term in the equation appears when we include fields that depend explicitly on the scale k, as in Refs. [28,30,33].

For our regulators $\mathbf{R}$, we use the form suggested by Litim [34], which is optimized for local interactions. This suppresses the contributions of modes with momenta $q < k$ by replacing the kinetic energy in the inverse propagator for each field with a constant. For the atom field it has the form

$$R_a(q,k) = \frac{k^2 - q^2}{2m}\,\theta(k - q). \qquad (2)$$

The dimer and trimer regulators have similar forms but also contain the wave-function renormalization factors defined below.

The key ingredient in any practical application of the FRG is the choice of truncation for the running action. Here we work with only local interactions. This reduces the functional differential equation for the action to a set of coupled ordinary differential equations for renormalization factors and coupling constants multiplying the terms that appear in the action, as defined below. Large numbers of diagrams contribute to the driving terms, as in the versions without trimer fields studied in Refs. [30,31].

The running action we use is

$$
\begin{aligned}
\Gamma_k[\psi,\psi^*,\phi,\phi^*,\chi,\chi^*] = \int d^4x \Bigg[&\psi^* \left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{2m}\right)\psi + Z_d\,\phi^*\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{4m}\right)\phi + Z_t\,\chi^*\left(\mathrm{i}\,\partial_0 + \frac{\nabla^2}{6m}\right)\chi - u_d\phi^*\phi - u_t\chi^*\chi \\
&- \frac{g}{2}(\phi^*\psi\psi + \psi^*\psi^*\phi) - h(\chi^*\phi\psi + \phi^*\psi^*\chi) - \lambda\,\phi^*\psi^*\phi\psi - \frac{u_{dd}}{2}(\phi^*\phi)^2 \\
&- \frac{v_d}{4}(\phi^*\phi^*\phi\psi\psi + \phi^*\psi^*\psi^*\phi\phi) - \frac{w}{4}\phi^*\psi^*\psi^*\phi\psi\psi - u_{tt}\,\chi^*\psi^*\chi\psi \\
&- \frac{u_{dt}}{2}(\phi^*\phi^*\chi\psi + \chi^*\psi^*\phi\phi) - \frac{v_t}{2}(\phi^*\psi^*\psi^*\chi\psi + \chi^*\psi^*\phi\psi\psi) \Bigg].
\end{aligned}
\qquad (3)
$$

This contains kinetic terms for atom, dimer, and trimer fields with wave-function renormalization factors and interaction terms with up to four underlying atoms. This action was also used by Schmidt and Moroz [30] (see in particular the Appendix to that paper) but they chose to eliminate the four-atom couplings with trimer fields (u_{tt}, u_{dt}, and v_t) so that channels with dynamic trimers are not needed. The analogous fermionic couplings without trimers were studied in Ref. [31].

The inverse propagators for the fields in Eq. (3) are expanded up to first order in the energy, which implies first-order time derivatives in the action. In each channel, the zero-energy point for this expansion is taken to be the threshold for breakup of an n-atom state into n free atoms. Spatial derivatives appear at second order as required by Galilean invariance, which follows from our choice of regulator [34].

The wave-function renormalization factors $Z_{d,t}$, self-energies $u_{d,t}$, and couplings h, λ etc. all run with the regulator scale k. In vacuum, there is no renormalization factor for the atom field ψ and the coupling g remains constant during the evolution.

Even though atom-atom scattering near the unitary limit can be described by an atom-atom contact interaction, the running action in Eq. (3) does not contain such a term. This is because it can be eliminated through a Hubbard-Stratonovich transformation at some large starting scale K. The atom-atom term is not regenerated by the evolution and so atom-atom scattering is mediated only by the coupling g to dimers. At zero energy, the scattering is given by $g^2/u_d(k)$ where $u_d(k)$ evolves linearly with k. We choose its initial value $u_d(K)$ such that, in the physical limit, $u_d(0)$ gives the desired scattering length a [25,26].

In contrast, the atom-dimer interaction λ is regenerated even if we set it to zero initially. By introducing fields that depend explicitly on the scale k, as in Refs. [28,30,33], we can cancel the evolution of this and some other couplings. If we set their initial values to zero at the starting scale K, then these couplings are effectively eliminated from the problem. Here we take the trimer to run as

$$\partial_k \chi = \zeta_1\,\phi\psi + \zeta_2\,\psi^\dagger\chi\psi + \zeta_3\,\psi^\dagger\phi\phi + \zeta_4\,\psi^\dagger\phi\psi\psi, \qquad (4)$$

where the $\zeta_i(k)$ are

$$\zeta_1 = -\frac{\partial_k \lambda}{2\,h}, \tag{5a}$$

$$\zeta_2 = -\frac{\partial_k v_t}{h} + \frac{u_{tt}\,\partial_k \lambda}{2\,h^2} - \frac{u_t\,v_t\,\partial_k \lambda}{2\,h^3} + \frac{u_t\,\partial_k w}{8\,h^2}, \tag{5b}$$

$$\zeta_3 = -\frac{\partial_k v_d}{4\,h} + \frac{u_{dt}\,\partial_k \lambda}{2\,h^2}, \tag{5c}$$

$$\zeta_4 = -\frac{\partial_k w}{8\,h} + \frac{v_t\,\partial_k \lambda}{2\,h^2}. \tag{5d}$$

The first term in Eq. (4) cancels the running of λ, and the others do the same for the four-atom couplings v_d, w, and v_t.

Once we have eliminated these couplings, the physical processes that give rise to their evolution are implicitly present in the flows of the remaining couplings through contributions to their flows from the final term in Eq. (1). For example, if the contact interaction λ is eliminated, atom-dimer scattering only occurs through coupling to the trimer. The effects responsible for the evolution of λ are now codified in a term proportional to $u_t \zeta_1$ in the flow of $h(k)$, arising from the first term of Eq. (4).

III. THREE-BODY SECTOR

The three-body sector, described by the couplings $h(k)$, $u_t(k)$, and $Z_t(k)$, has been studied using this action by Floerchinger *et al.* [28]. We summarize its main features here to provide some "landmarks" for our four-body results. In the unitary limit, the flow equations for the three-atom couplings have the forms

$$\partial_k(h^2) = -\frac{312}{125\,k}\,h^2(k) - \frac{256}{125\,k^3}\,u_t(k), \tag{6a}$$

$$\partial_k u_t = \frac{56\,k}{125}\,h^2(k), \tag{6b}$$

$$\partial_k Z_t = -\frac{448}{625\,k}\,h^2(k), \tag{6c}$$

where, to simplify the expressions, u_t and Z_t have been redefined to absorb constant factors of $g^2\,m$ and $1/g^2$, respectively.

These equations describe the flows for regulator scales $k \gg 1/a$, where a is the two-body scattering length. In this limit, the equations are scale invariant and so we expect their solutions to scale as powers of k. Indeed this system of differential equations is satisfied if $h^2(k)$ and $Z_t(k)$ behave as k^d and $u_t(k)$ as k^{2+d} where d has two possible values,

$$d_\pm = -281/125 \pm i\,\sqrt{535}/25. \tag{7}$$

Since these are a complex-conjugate pair, we can form real solutions and define rescaled quantities that oscillate periodically in $t = \ln(k/K)$:

$$\hat{H}(k) = k^{281/125}\,h^2(k), \tag{8a}$$

$$\hat{u}_t(k) = k^{31/125}\,u_t(k), \tag{8b}$$

$$\hat{Z}_t(k) = k^{281/125}\,Z_t(k). \tag{8c}$$

This periodic behavior is a consequence of the Emimov effect [5,6] which breaks the scale invariance of theory to

a discrete symmetry. It follows from the complex scaling exponents in Eq. (7). For the truncated action and regulator used here, the scaling factor in momentum is $\sim$29.8 [28,30], which yields longer cycles than the true value of $\sim$22.7.

In this framework, atom-dimer scattering at zero energy is given by the combination $h(k)^2/u_t(k)$, which evolves in the same way as $\lambda(k)$ in the theory without the trimer [28,30]. It displays a sequence of poles that are equally spaced in t, reflecting the discrete scaling symmetry of the Efimov effect. Each of these poles corresponds to the passage of a three-body bound state through the three-atom threshold as k is lowered. In the physical limit they build up the infinite tower of Efimov states.

Although the flow equations in the three-body sector require three initial conditions, only one of these defines a physical parameter. This fixes the initial phase of the periodic functions or, equivalently, the scale at which the first Efimov pole appears. Physical quantities are independent of the magnitudes of the couplings since they depend only on the ratios $h(k)^2/u_t(k)$ and $h(k)^2/Z_t(k)$.

IV. FOUR-BODY SECTOR

In the four-atom sector, we use the scale dependence of the trimer to eliminate the couplings v_d, w, and v_t that include the dimer-atom-atom channel. This leaves only the ones involving the dimer-dimer and atom-trimer channels, u_{dd}, u_{dt}, and u_{tt}. The first of these, u_{dd}, describes dimer-dimer scattering at zero energy (the four-atom threshold). Similarly u_{tt} describes atom-trimer scattering and u_{dt} the coupling between the two channels. This choice reflects the structure of the Faddeev-Yakubovsky equations used in most direct calculations of four-body systems [32]. In contrast, Schmidt and Moroz [30] also introduced a trimer field to treat energy dependence but kept only the couplings u_{dd}, v_d, and w.

The evolution of the four-atom couplings, u_{dd}, u_{dt}, and u_{tt}, is governed by a system of three coupled nonlinear differential equations. We define regulated energies for atoms, dimers, and trimers,

$$E_a(q,k) = \frac{q^2}{2\,m} + R_a(q,k), \tag{9a}$$

$$E_d(q,k) = \frac{q^2}{4\,m} + \frac{R_d(q,k)}{Z_d(k)} + \frac{u_d(k)}{Z_d(k)}, \tag{9b}$$

$$E_t(q,k) = \frac{q^2}{6\,m} + \frac{R_t(q,k)}{Z_t(k)} + \frac{u_t(k)}{Z_t(k)}, \tag{9c}$$

where the single-atom self-energy contains

$$u_d(k,a) = \frac{M g^2}{\pi^2}\left(\frac{k}{6} - \frac{\pi}{4\,a}\right). \tag{10}$$

From these we construct the quantities

$$T^X_{\alpha,\beta,\gamma,\delta} = \frac{\partial_k R_X\,(Z_d)^{-\beta-\gamma}\,(Z_t)^{-\delta}}{(E_a)^\alpha\,(E_d)^\beta\,(E_a+E_d)^\gamma\,(E_a+E_t)^\delta}, \tag{11}$$

for $X = a,d,t$. In terms of these, the system of equations can be written

$$\partial_k u_{dd} = \int \frac{d^3q}{(2\pi)^3}\left[\frac{3\,g^4}{8}T^a_{4,0,0,0} + \frac{g^2h^2}{2}\left(2\,T^a_{3,0,0,1} + T^a_{2,0,0,2}Z_t + T^t_{2,0,0,2}\right)\right.$$
$$\left. + \frac{(u_{dd})^2}{2}T^d_{0,2,0,0}\,(Z_d)^{-1} - 2\,g\,h\,u_{dt}\left(T^a_{2,0,0,1} + T^a_{1,0,0,2}Z_t + T^t_{1,0,0,2}\right) + 2\,(u_{dt})^2\left(T^a_{0,0,0,2}Z_t + T^t_{0,0,0,2}\right)\right], \quad (12a)$$

$$\partial_k u_{dt} = \int \frac{d^3q}{(2\pi)^3}\left[-\frac{g^3 u_t\, u_{dd}}{4\,h}\left(T^a_{2,1,1,0} + T^a_{1,1,2,0}Z_d + T^d_{1,2,1,0} + T^d_{1,1,2,0}\right) - \frac{g\,h\,u_{tt}}{2}\left(T^a_{2,0,0,1} + T^a_{1,0,0,2}Z_t + T^t_{1,0,0,2}\right)\right.$$
$$- \frac{g\,h\,u_{dd}}{2}\left(T^a_{0,1,2,0}Z_d + T^d_{0,2,1,0} + T^d_{0,1,2,0}\right) + \frac{g^4 u_t\, u_{dt}}{8\,h^2}\left(2\,T^a_{3,0,1,0} + T^a_{2,0,2,0}Z_d + T^d_{2,0,2,0}\right)$$
$$- \frac{g^2 u_t\, u_{dt}}{2}\left(T^a_{2,0,1,1} + T^a_{1,0,2,1}Z_d + T^a_{1,0,1,2}Z_t + T^d_{1,0,2,1} + T^t_{1,0,1,2}\right) + \frac{u_{dd}\, u_{dt}}{2}T^d_{0,2,0,0}\,(Z_d)^{-1}$$
$$- h^2 u_{dt}\left(T^a_{0,0,2,1}Z_d + T^a_{0,0,1,2}Z_t + T^d_{0,0,2,1} + T^t_{0,0,1,2}\right) + u_{dt}\, u_{tt}\left(T^a_{0,0,0,2}Z_t + T^t_{0,0,0,2}\right)$$
$$+ \frac{g^3 h\, u_t}{4}\left(2\,T^a_{3,0,1,1} + T^a_{2,0,2,1}Z_d + T^a_{2,0,1,2}Z_t + T^d_{2,0,2,1} + T^t_{2,0,1,2}\right)$$
$$\left. + \frac{g\,h^3}{2}\left(T^a_{2,0,1,1} + T^a_{1,0,2,1}Z_d + T^a_{1,0,1,2}Z_t + T^d_{1,0,2,1} + T^t_{1,0,1,2}\right)\right], \quad (12b)$$

$$\partial_k u_{tt} = \int \frac{d^3q}{(2\pi)^3}\left[g^2 h^2\left(T^a_{2,1,1,0} + T^a_{1,1,2,0}Z_d + T^d_{1,2,1,0} + T^d_{1,1,2,0}\right) + \frac{g^4 u_t\, u_{tt}}{4\,h^2}\left(2\,T^a_{3,0,1,0} + T^a_{2,0,2,0}Z_d + T^d_{2,0,2,0}\right)\right.$$
$$+ g^4 u_t\left(2\,T^a_{3,1,1,0} + T^a_{2,1,2,0}Z_d + T^d_{2,2,1,0} + T^d_{2,1,2,0}\right) - 2\,g\,h\,u_{dt}\left(T^a_{0,1,2,0}Z_d + T^d_{0,2,1,0} + T^d_{0,1,2,0}\right)$$
$$+ \frac{g^4 (u_t)^2}{4}\left(2\,T^a_{3,0,2,1} + 2\,T^a_{2,0,3,1}Z_d + T^a_{2,0,2,2}Z_t + 2\,T^d_{2,0,3,1} + T^t_{2,0,2,2}\right) + (u_{dt})^2\, T^d_{0,2,0,0}\,(Z_d)^{-1}$$
$$- 2\,h^2 u_{tt}\left(T^a_{0,0,2,1}Z_d + T^a_{0,0,1,2}Z_t + T^d_{0,0,2,1} + T^t_{0,0,1,2}\right) + (u_{tt})^2\left(T^a_{0,0,0,2}Z_t + T^t_{0,0,0,2}\right)$$
$$+ g^2 h^2 u_t\left(T^a_{2,0,2,1} + 2\,T^a_{1,0,3,1}Z_d + T^a_{1,0,2,2}Z_t + 2\,T^t_{1,0,3,1} + T^t_{1,0,2,2}\right) - g^2 u_t\, u_{tt}\left(T^a_{2,0,1,1} + T^a_{1,0,2,1}Z_d\right)$$
$$- g^2 u_t\, u_{tt}\left(T^a_{1,0,1,2}Z_t + T^d_{1,0,2,1} + T^t_{1,0,1,2}\right) + \frac{g^6 (u_t)^2}{4\,h^2}\left(3\,T^a_{4,1,1,0} + T^a_{3,1,2,0}Z_d + T^d_{3,2,1,0} + T^d_{3,1,2,0}\right)$$
$$\left. - \frac{g^3 u_t\, u_{dt}}{h}\left(T^a_{2,1,1,0} + T^a_{1,1,2,0}Z_d + T^d_{1,2,1,0} + T^d_{1,1,2,0}\right) + h^4\left(2\,T^a_{0,0,3,1}Z_d + T^a_{0,0,2,2}Z_t + 2\,T^d_{0,0,3,1} + T^t_{0,0,2,2}\right)\right]. \quad (12c)$$

The appearance of $h^2(k)$, $u_t(k)$, and $Z_t(k)$ in the four-body flow equations, Eqs. (12), means that they inherit the Efimov periodicity of the three-body sector. This also leads to two types of singularity in the equations. One arises from terms with denominators containing either one or two powers of the regulated energy of an atom plus a trimer, $E_a(k) + E_t(k)$. This passes through zero energy once in every Efimov cycle, at the point where a regulated atom-trimer threshold drops below the four-atom threshold as we lower k. At each crossing we expect additional contributions to the imaginary parts of the four-body couplings, as a channel with a new Efimov state becomes open.

The other type of divergent term has a factor of $1/[h(k)]^2$. These lead to unphysical singularities in the four-body couplings, which mark the start of a short region within each Efimov cycle where $h^2(k)$ and $Z_t(k)$ have opposite signs. In these regions, the trimer field has a ghostlike character, with a propagator $h^2(k)/[Z_t\,p_0 - u_t(k)]$ that has a negative residue at

its pole. This is a warning that not all features of the effective action are physical for nonzero values of k. Fortunately these regions are well separated from the threshold regions where the phenomena of interest occur.

In the scaling regime the four-atom couplings display Efimov periodicity. This can be seen most clearly if they are multiplied by appropriate powers of k, analogously to the rescaling of the three-body sector in Eqs. (8). Here we define the couplings,

$$\hat{u}_{dd}(k) = k^3\, u_{dd}(k), \quad (13a)$$
$$\hat{u}_{dt}(k) = k^{781/250}\, u_{dt}(k), \quad (13b)$$
$$\hat{u}_{tt}(k) = k^{406/125}\, u_{tt}(k), \quad (13c)$$

where the powers of k are determined from dimensional analysis of the running action and the scalings in the three-body sector, Eqs. (6).

The flow equations for these rescaled couplings can be written

$$\partial_t \hat{u}_{dd} = \frac{1}{\pi^2} + 3\,\hat{u}_{dd} + \frac{8\pi^2\,\hat{u}_{dd}^2}{15} + \frac{\hat{H}\,\partial_t \hat{Z}_t}{45\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{2\,\hat{H}\,\hat{u}_t}{3\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{1573\,\hat{H}}{1875\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t} + \frac{2\,\hat{H}}{3\pi^2\,\hat{E}_{at}\,\hat{Z}_t} - \frac{2\,\hat{U}_{dt}\,\partial_t \hat{Z}_t}{45\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{2\,\hat{u}_t\,\hat{U}_{dt}}{3\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2}$$
$$- \frac{6938\,\hat{U}_{dt}}{5625\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{2\,\hat{U}_{dt}}{3\pi^2\,\hat{E}_{at}\,\hat{Z}_t} + \frac{\hat{U}_{dt}^2\,\partial_t \hat{Z}_t}{45\pi^2\,\hat{H}\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{2219\,\hat{U}_{dt}^2}{5625\pi^2\,\hat{H}\,\hat{E}_{at}^2\,\hat{Z}_t}, \tag{14a}$$

$$\partial_t \hat{U}_{dt} = -\frac{496\pi^2\,\hat{H}\,\hat{u}_{dd}}{375} - \frac{1096\pi^2\,\hat{u}_{dd}\,\hat{u}_t}{375} + \frac{2\,\hat{H}^2\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{2\,\hat{H}\,\hat{u}_t\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{18\,\hat{H}^2\,\hat{u}_t}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{28\,\hat{H}\,\hat{u}_t^2}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{8938\,\hat{H}^2}{9375\,\hat{E}_{at}^2\,\hat{Z}_t}$$
$$+ \frac{11438\,\hat{H}\,\hat{u}_t}{9375\,\hat{E}_{at}^2\,\hat{Z}_t} + \frac{66\,\hat{H}^2}{125\,\hat{E}_{at}\,\hat{Z}_t} + \frac{116\,\hat{H}\,\hat{u}_t}{125\,\hat{E}_{at}\,\hat{Z}_t} - \frac{\hat{H}\,\hat{u}_{tt}\,\partial_t \hat{Z}_t}{90\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{\hat{H}\,\hat{u}_t\,\hat{u}_{tt}}{6\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{3469\,\hat{H}\,\hat{u}_{tt}}{11250\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{\hat{H}\,\hat{u}_{tt}}{6\pi^2\,\hat{E}_{at}\,\hat{Z}_t} + 3\,\hat{U}_{dt}$$
$$+ \frac{8\pi^2\,\hat{u}_{dd}\,\hat{U}_{dt}}{15} - \frac{2\,\hat{H}\,\hat{U}_{dt}\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{2\,\hat{u}_t\,\hat{U}_{dt}\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{8\,\hat{H}\,\hat{u}_t\,\hat{U}_{dt}}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{18\,\hat{u}_t^2\,\hat{U}_{dt}}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{2146\,\hat{H}\,\hat{U}_{dt}}{3125\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{8938\,\hat{u}_t\,\hat{U}_{dt}}{9375\,\hat{E}_{at}^2\,\hat{Z}_t}$$
$$- \frac{16\,\hat{H}\,\hat{U}_{dt}}{125\,\hat{E}_{at}\,\hat{Z}_t} - \frac{66\,\hat{u}_t\,\hat{U}_{dt}}{125\,\hat{E}_{at}\,\hat{Z}_t} + \frac{\hat{u}_{tt}\,\hat{U}_{dt}\,\partial_t \hat{Z}_t}{90\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{2219\,\hat{u}_{tt}\,\hat{U}_{dt}}{11250\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t}, \tag{14b}$$

$$\partial_t \hat{u}_{tt} = \frac{4384\pi^2\,\hat{H}}{375} + \frac{13568\pi^2\,\hat{u}_t}{375} + \frac{9184\pi^2\,\hat{u}_t^2}{375\,\hat{H}} + \frac{8\pi^2\,\hat{H}^2\,\partial_t \hat{Z}_t}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{16\pi^2\,\hat{H}\,\hat{u}_t\,\partial_t \hat{Z}_t}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2}$$
$$+ \frac{192\pi^2\,\hat{H}^2\,\hat{u}_t}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{8\pi^2\,\hat{u}_t^2\,\partial_t \hat{Z}_t}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{624\pi^2\,\hat{H}\,\hat{u}_t^2}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{432\pi^2\,\hat{u}_t^3}{125\,\hat{E}_{at}^2\,\hat{Z}_t^2} + \frac{33752\pi^2\,\hat{H}^2}{15625\,\hat{E}_{at}^2\,\hat{Z}_t}$$
$$+ \frac{87504\pi^2\,\hat{H}\,\hat{u}_t}{15625\,\hat{E}_{at}^2\,\hat{Z}_t} + \frac{53752\pi^2\,\hat{u}_t^2}{15625\,\hat{E}_{at}^2\,\hat{Z}_t} + \frac{384\pi^2\,\hat{H}^2}{625\,\hat{E}_{at}\,\hat{Z}_t} + \frac{1968\pi^2\,\hat{H}\,\hat{u}_t}{625\,\hat{E}_{at}\,\hat{Z}_t} + \frac{1584\pi^2\,\hat{u}_t^2}{625\,\hat{E}_{at}\,\hat{Z}_t}$$
$$+ \frac{406\,\hat{u}_{tt}}{125} + \frac{256\,\hat{u}_t\,\hat{u}_{tt}}{125\,\hat{H}} - \frac{4\,\hat{H}\,\hat{u}_{tt}\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{4\,\hat{u}_t\,\hat{u}_{tt}\,\partial_t \hat{Z}_t}{75\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{16\,\hat{H}\,\hat{u}_t\,\hat{u}_{tt}}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2} - \frac{36\,\hat{u}_t^2\,\hat{u}_{tt}}{25\,\hat{E}_{at}^2\,\hat{Z}_t^2}$$
$$- \frac{4292\,\hat{H}\,\hat{u}_{tt}}{3125\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{17876\,\hat{u}_t\,\hat{u}_{tt}}{9375\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{32\,\hat{H}\,\hat{u}_{tt}}{125\,\hat{E}_{at}\,\hat{Z}_t} - \frac{132\,\hat{u}_t\,\hat{u}_{tt}}{125\,\hat{E}_{at}\,\hat{Z}_t} + \frac{\hat{u}_{tt}^2\,\partial_t \hat{Z}_t}{90\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t^2}$$
$$+ \frac{2219\,\hat{u}_{tt}^2}{11250\pi^2\,\hat{E}_{at}^2\,\hat{Z}_t} - \frac{1984\pi^2\,\hat{U}_{dt}}{375} - \frac{4384\pi^2\,\hat{u}_t\,\hat{U}_{dt}}{375\,\hat{H}} + \frac{16\pi^2\,\hat{U}_{dt}^2}{15\,\hat{H}}, \tag{14c}$$

where we have defined the rescaled atom-trimer energy $\hat{E}_{at} = 2/3 + \hat{u}_\chi / \hat{Z}_\chi$ and the modified coupling $\hat{U}_{dt} = \hat{H}^{1/2}\,\hat{u}_{dt}$. As in Eqs. (6) we have absorbed powers of the constants g^2 and m into the couplings to try to simplify the expressions.

We have numerically integrated the coupled equations for $u_{dd}(k)$, $u_{dt}(k)$, and $u_{tt}(k)$ through several Efimov cycles, and we have checked that any transients caused by our choice of initial conditions die out within the first cycle. All three couplings show similar structures but they are most clearly visible in $u_{tt}(k)$ and so we present only results for its flow. One cycle of the rescaled coupling $\hat{u}_{tt}(k)$ in the unitary limit is shown in Fig. 1. At the value of $t = \ln(k/K)$ where the atom-trimer threshold passes through zero energy, $t = t_3 \simeq -4.85$, we see the expected discontinuity in the slope of the imaginary part signaling the opening of a new channel. The unphysical singularity arising from the zero of $h^2(k)$ is the structure that can be seen at $t \simeq -3.0$.

Several simple poles can also be seen in Fig. 1, at $t \simeq -3.83$, -4.67, and just below the threshold. When we look more closely at the region close to an atom-trimer threshold, as in Fig. 2, we find an infinite sequence of these poles.
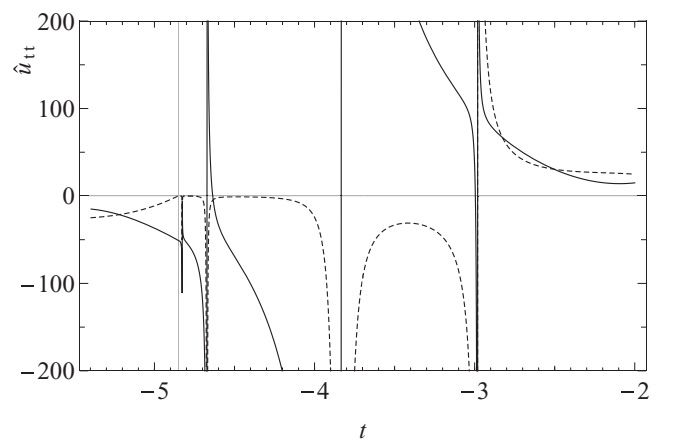

FIG. 1. One Efimov cycle of the flow of the rescaled coupling $\hat{u}_{tt}(k)$ in the unitary limit, plotted against $t = \ln(k/K)$. The real part is shown by the solid curve and the imaginary part by the dashed one. The atom-trimer threshold corresponding to the vanishing of $E_a(k) + E_t(k)$ is marked by the gray vertical line at $t = t_3 \simeq -4.85$.
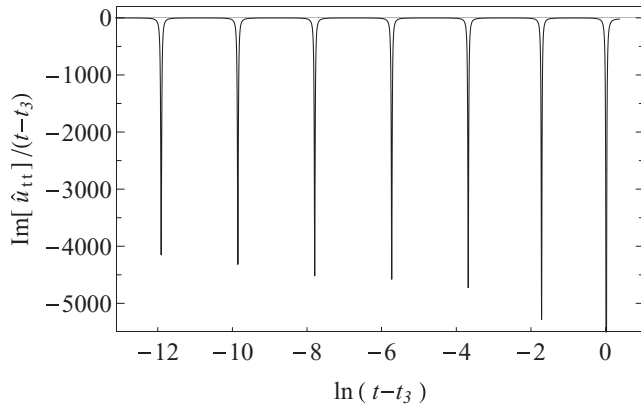
FIG. 2. The imaginary part of $\hat{u}_{tt}(k)/(t-t_3)$ just before the threshold $t_3 \simeq -4.85$ shown in Fig. 1, plotted against $x = \ln(t-t_3)$. Apart from the rightmost one, corresponding to the deepest four-body state, the poles are approximately equally spaced.

These become equally spaced in the variable $x = \ln(t-t_3)$. These poles do not correspond to singularities in the equations but are generated by the evolution of the couplings. Like the singularities that appear in the three-body sector, we interpret them as bound states or, rather, narrow resonances since they have finite imaginary parts as a result of coupling to open channels with more deeply bound trimers. However, as we discuss below, not all of these poles may appear as physical states.

The introduction of the trimer field to describe energy dependence in the three-body sector is essential for generating these poles as they do not appear in the FRG equations for the couplings without trimer fields [35].[1] The scales at which these poles appear follow a double-exponential, "super-Efimov" pattern, similar to that observed in the two-dimensional three-body system studied by Nishida *et al.* [24].

Mathematically this structure arises from the forms of our differential equations which are analogous to that of the RG equation of Ref. [24]. The key terms that lead to the "super-Efimov" behavior are the ones that are singular at the atom-trimer threshold. These arise from diagrams that are similar to those in Fig. 2 of that paper. However we should stress these states appear for nonzero values of k, where the action is not physical. Moreover the four-body flow equations depend on a scale as a result of the breaking of scale invariance by the Efimov effect. These states can therefore move relative to the atom-trimer threshold during the evolution to the physical limit. In particular, they may pass through the nearby atom-trimer threshold to become virtual states. If so, only a finite number of bound states may persist in that limit. Furthermore, a theorem of Amado and Greenwood forbids an infinite number of four-body bound states based on a zero-energy trimer state [37]. Nonetheless, the presence of these virtual states might be relevant to the rich structure of states being found in four-body systems away from the unitary limit. For example, Deltuva [38] has recently described a tower of four-body bound states lying

just below the atom-atom-dimer threshold in systems with finite dimer binding energy.

The local form of the action, Eq. (3), does not allow us to study the full energy dependence in the four-body channels and so we cannot directly determine the spectrum in the physical limit. Instead, we can examine where these states cross zero energy as we move away from the unitary limit by taking a nonzero atom-atom scattering length, $a < 0$. Such zero-energy states are the ones observed in experiments on ultra-cold atoms in traps, as they lead to resonant enhancements of the loss of atoms at particular values of the scattering length [10,18].

With a finite scattering length, the final Efimov cycle no longer has the same form as in the unitary limit. An example is shown in Fig. 3. For $t \gtrsim -2.3$ the flow of the four-atom coupling matches Fig. 1, but beyond this point differences become increasingly visible. The example shown has the scattering length tuned so that the shallowest trimer state has exactly zero binding energy at $k = 0$. In this case, we find three four-body states appearing in the final Efimov cycle (the poles close to $t = -4.1, -5.6,$ and -7.1). There is thus no conflict with the theorem of Amado and Greenwood [37] that there are only a finite number of these four-body states. We denote the corresponding scattering length by a_3. When we further decrease a, we find that the values $a_4^{(n)}$ at which these states cross the four-atom threshold are related to a_3 by

$$a_4^{(0)}/a_3 \simeq 0.438, \quad a_4^{(1)}/a_3 \simeq 0.877, \quad a_4^{(2)}/a_3 \simeq 0.9967. \tag{15}$$

For the two lowest states, these ratios are within 5% of the results of exact solutions to the four-body equations [13,14], and hence they are also in reasonable agreement with the experimental numbers [18]. The third state lies extremely close to the atom-trimer threshold. If it is real, then it will be a challenge to observe both numerically and experimentally. However this state may just be an artifact of our truncation since improvements to the action which shorten the Efimov cycle might make it unbound.
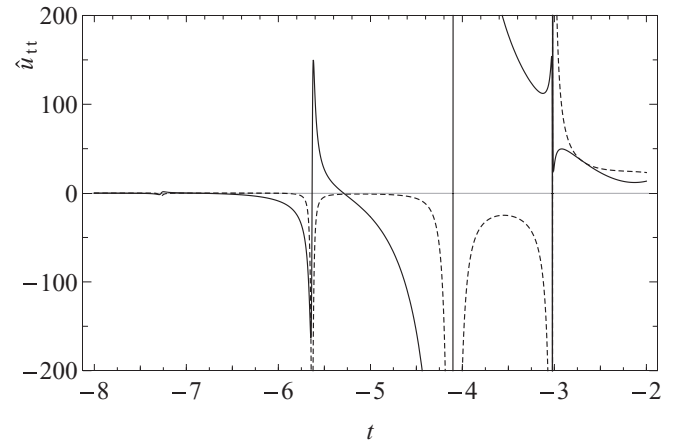

FIG. 3. The final cycle of the flow of the rescaled coupling $\hat{u}_{tt}(k)$ plotted against $t = \ln(k/K)$. The solid line corresponds to the real part and the imaginary to the dashed one. The atom-atom scattering length has been tuned so that the last three-body state appears at $k = 0$ ($t = -\infty$).

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[1]The four-body states seen in Ref. [30] have been found to be numerical artifacts [35,36].

Returning to the double-exponential behavior observed during the evolution, the scale $k_4^{(n)}$ at which the nth excited four-body state appears can be written in the form

$$k_4^{(n)} = k_3 \exp[\alpha \, e^{-\beta n}], \tag{16}$$

where $\alpha \simeq 1.53$, $\beta \simeq 2.06$, and k_3 denotes the scale corresponding to the atom-trimer threshold for the next three-body Efimov state. This describes the energies of all states except the lowest ($n = 0$) to a very good approximation. The ratios between scales for subsequent states can be expressed in the form of a universal scaling function,

$$k_4^{(n+1)}/k_4^{(n)} = \left(k_3/k_4^{(n)}\right)^{1-\exp(-\beta)}. \tag{17}$$

A similar scaling relation between the binding energies has also been found by Hadizadeh *et al.* [15,16], although its functional form is quite different and it predicts at most three four-body states in an Efimov cycle. More importantly, and in contrast to the results of those authors, the scales at which our states appear do not depend on any new four-body scale: The parameter α in Eq. (16) has a fixed value which is independent of the initial conditions we impose on the four-body couplings. The independence of any four-body parameter also applies to the physical states discussed above.

V. CONCLUSIONS

In summary, we have used the FRG to study systems of four bosons close to the unitary limit. In contrast to previous approaches, we introduce a dynamical trimer field and use this to match the channel structure of the Faddeev-Yakubovsky equations. In the physical limit, where the cutoff scale tends to zero, we examine the points at which three- and four-body states pass through zero energy as we vary the atom-atom scattering length. We find three four-body states in the last Efimov cycle. The lowest two of these pass through zero for scattering lengths that are in good agreement with the results of exact solutions of the Faddeev-Yakubovsky equations [13,14] and with experimental observations [18]. The third state is extremely weakly bound and so may be an artifact of our truncated action.

In the unitary limit, the evolution generates an infinite number of four-body resonant states during each Efimov cycle, although it seems unlikely that all of these persist to the physical limit. These states lie just below each atom-trimer threshold and follow a double-logarithmic, or "super-Efimov" pattern [24]. They obey a universal scaling relation analogous to that of Ref. [15]. However the scales at which they appear are independent of the initial conditions on the four-body couplings. This supports the conclusion of Refs. [12,39] that there is no additional relevant parameter in four-boson systems with contact interactions.

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