PHYSICAL REVIEW A **88**, 042337 (2013)

Classical and quantum capacities of a fully correlated amplitude damping channel

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(Received 10 September 2013; published 31 October 2013)

We study information transmission over a fully correlated amplitude damping channel acting on two qubits. We derive the single-shot classical channel capacity and show that entanglement is needed to achieve the channel best performance. We discuss the degradability properties of the channel and evaluate the quantum capacity for any value of the noise parameter. We finally compute the entanglement-assisted classical channel capacity.

DOI: 10.1103/PhysRevA.88.042337 PACS number(s): 03.67.Hk, 03.65.Yz

I. INTRODUCTION

Physical processes can be viewed, in terms of information theory, as channels mapping the input (initial) state onto the final (output) state, the transmission being in space (as in communication channels) or in time (as in the run of a computer). The performance of a noisy classical channel can be characterized by a single number, i.e., its *capacity*, defined as the maximum rate at which information can be reliably transmitted down the channel [1]. On the other hand, noisy quantum communication channels [2,3] can use quantum systems as carriers of both classical or quantum information, by encoding classical bits by means of quantum states or by transferring (unknown) quantum states between, say, subunits of a quantum computer. Therefore, different capacities must be defined. The *classical capacity* C [4–6] and the *quantum capacity* Q [7–9] of a noisy quantum channel are defined as the maximum number of, respectively, bits and qubits that can be reliably transmitted per channel use. The *entanglement-assisted classical capacity* C_E gives the capacity of transmitting classical information, provided the sender and the receiver share unlimited prior entanglement [10–12]. This quantity upper bounds the other capacities: We have $Q \leqslant C \leqslant C_E$ [13].

Noise effects can be conveniently described in the quantum operation formalism [2,3]: Any input state ρ is mapped onto the output state $\rho' = \mathcal{E}(\rho)$ by a linear, completely positive, trace-preserving (CPT) map $\mathcal{E}$. The simplest models for quantum channels are memoryless; that is, the quantum operation describing n channel uses is $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. On the other hand, real systems exhibit *memory*—or *correlation*—effects among consecutive uses. Such effects become unavoidable when increasing the transmission rate in quantum channels, as it can be explored experimentally in optical fibers [14] or in solid-state implementations of quantum hardware suffering from low-frequency noise [15]. Quantum memory channels [16], i.e., $\mathcal{E}_n \neq \mathcal{E}^{\otimes n}$, attracted increasing attention in the last years. Interesting new features emerge in several models, including depolarizing channels [17,18], Pauli channels [19–21], dephasing channels [22–26], Gaussian channels [27], lossy bosonic channels [28,29], spin chains [30], collision models [31], complex network dynamics [32], and a micromaser model [33]. In particular, phenomenological models with Markovian correlated noise (see, e.g., [17,19,20,22–24,34–36]) show that the transmission of classical information can be enhanced by employing maximally entangled rather than separable states as information carriers [17,19,20]. Furthermore, memory can enhance the quantum capacity of a channel, as shown for a Markov-chain dephasing channel, whose quantum capacity can be analytically computed [23,24]. The main difficulty in the calculation of quantum channel capacities resides in the fact that, due to the superadditivity property of the related entropic quantities [8,37], maximization is requested over all possible n-use input states in the limit $n \to \infty$. For this reason, so far only a few memory-channel models have been fully solved in terms of their capacities [23,24,29].

In this paper, we extend the class of solved quantum channels to systems with damping, by considering a two-qubit amplitude damping channel $\mathcal{E}_m$ with memory, in which the relaxation processes from a qubit excited state towards the ground state only occur simultaneously for the two qubits. The channel is parametrized by η, which is the conditional probability that the system, once it is found with the two qubits both in their excited state, does not decay. This channel is the fully correlated limit of the amplitude damping channel with memory introduced in Ref. [38] and recently investigated in Ref. [39]. For channel $\mathcal{E}_m$ we compute the single-shot capacity C_1, that is, the classical capacity optimized over single uses of the two-qubit channel, the quantum capacity Q, and the entanglement-assisted classical capacity C_E. In particular, we show that the ensemble optimizing the capacity C_1 must contain entangled two-qubit input states.

The paper is organized as follows. In Sec. II we first introduce the channel model and describe the channel covariance properties. In Sec. III, we discuss how to find the quantum ensemble which maximizes the *Holevo quantity*, showing the explicit form of such optimal ensemble. We derive the form of the product state capacity C_1 of $\mathcal{E}_m$ and prove that entangled states are necessary to achieve the capacity. We finally give an analytical expression for C_1. In Sec. IV we show that

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the channel is degradable when η is inside a given range; we find the system density operator which maximizes the *coherent information*, and we determine the quantum capacity of $\mathcal{E}_m$ for all possible values of η. In Sec. V we derive the entanglement-assisted channel capacity and we finally summarize the main results in Sec. VI.

II. THE MODEL

We first briefly review the memoryless amplitude damping channel (ad) [2,3], which acts on a generic single-qubit state ρ as

$$\rho \rightarrow \rho' = \mathcal{E}_1(\rho) = \sum_{i\in\{0,1\}} E_i\,\rho\,E_i^\dagger, \tag{1}$$

where the Kraus operators E_i are given by

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\eta} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{1-\eta} \\ 0 & 0 \end{pmatrix}. \tag{2}$$

Here we are using the orthonormal basis $\{|0\rangle,|1\rangle\}$ ($\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|$). This channel describes relaxation processes, such as spontaneous emission of an atom, in which the system decays from the excited state $|1\rangle$ to the ground state $|0\rangle$. The channel acts as follows on a generic single-qubit state:

$$\begin{aligned}\rho &= \begin{pmatrix} 1-p & \gamma \\ \gamma* & p \end{pmatrix} \rightarrow \rho' \\ &= \mathcal{E}(\rho) = \begin{pmatrix} 1-\eta\,p & \sqrt{\eta}\,\gamma \\ \sqrt{\eta}\,\gamma* & \eta\,p \end{pmatrix}.\end{aligned} \tag{3}$$

Note that the noise parameter η ($0 \leqslant \eta \leqslant 1$) plays the role of channel transmissivity. Indeed, for $\eta = 1$ we have a noiseless channel, whereas for $\eta = 0$ the channel cannot carry any information since for any possible input we always obtain the same output state $|0\rangle$.

For two channel uses, a *memory* amplitude damping channel was introduced in Ref. [38]:

$$\rho \rightarrow \rho' = \mathcal{E}(\rho) = (1-\mu)\mathcal{E}_1^{\otimes 2}(\rho) + \mu\mathcal{E}_m(\rho). \tag{4}$$

Here ρ is a generic two-qubit input state, and μ ($0 \leqslant \mu \leqslant 1$) is the memory parameter: The memoryless channel is recovered when $\mu = 0$, while for $\mu = 1$ we obtain the "full memory" amplitude damping channel $\mathcal{E}_m$. In $\mathcal{E}_m$ the relaxation phenomena are fully correlated. In other words, when a qubit undergoes a relaxation process, the other qubit does the same; see Fig. 1. In this way only the state $|11\rangle \equiv |1\rangle \otimes |1\rangle$ can decay, while the other states $|ij\rangle \equiv |i\rangle \otimes |j\rangle$, $i,j \in \{0,1\}$, $ij \neq 11$,

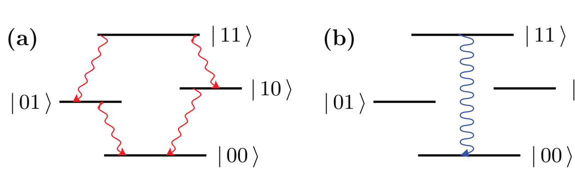

FIG. 1. (Color online) Simple sketch of the relaxation mechanisms in the channels $\mathcal{E}_1^{\otimes 2}$ (a) and $\mathcal{E}_m$ (b). In the memoryless setting $\mathcal{E}_1^{\otimes 2}$ relaxation is allowed from any level. In the full memory, relaxation phenomena in the two qubits are fully correlated, and relaxation is allowed only from $|11\rangle$.

are noiseless. In the Kraus formalism we have that

$$\rho \rightarrow \rho' = \mathcal{E}_m(\rho) = \sum_i B_i\,\rho\,B_i^\dagger, \tag{5}$$

with the Kraus operators

$$B_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{\eta} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{1-\eta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

In this paper we focus on the fully correlated channel $\mathcal{E}_m$, for which we will compute analytically the single-shot classical capacity C_1, the quantum capacity Q, and the entanglement-assisted classical capacity C_E.

A. Channel properties

In this section, we discuss the covariance properties of channel $\mathcal{E}_m$, from which the properties of general channel can be derived, with respect to some unitary transformations. We first consider the following ones:

$$\mathcal{R}_1 = \sigma_z \otimes \mathbb{1}, \quad \mathcal{R}_2 = \mathbb{1} \otimes \sigma_z, \quad \mathcal{R}_3 = \sigma_z \otimes \sigma_z. \tag{7}$$

The Kraus operator B_0 commutes with each $\mathcal{R}_i$, since B_0 and $\mathcal{R}_i$ have a diagonal form: $B_0\mathcal{R}_i = \mathcal{R}_i B_0$. The operator B_1 commutes with $\mathcal{R}_3$ and anticommutes with $\mathcal{R}_1$ and $\mathcal{R}_2$:

$$\begin{aligned}&\mathcal{R}_1 B_1 = -B_1\mathcal{R}_1, \quad \mathcal{R}_2 B_1 = -B_1\mathcal{R}_2, \\ &\mathcal{R}_3 B_1 = B_1\mathcal{R}_3.\end{aligned} \tag{8}$$

The channel is *covariant* with respect to the unitaries $\mathcal{R}_i$, namely,

$$\mathcal{E}_m(\mathcal{R}_i\,\rho\,\mathcal{R}_i) = \mathcal{R}_i\,\mathcal{E}_m(\rho)\,\mathcal{R}_i. \tag{9}$$

Actually, we can see that

$$\begin{aligned}\mathcal{E}_m(\mathcal{R}_1\,\rho\,\mathcal{R}_1) &= B_0\mathcal{R}_1\rho\mathcal{R}_1 B_0^\dagger + B_1\mathcal{R}_1\rho\mathcal{R}_1 B_1^\dagger \\ &= \mathcal{R}_1 B_0\rho B_0^\dagger\mathcal{R}_1 + (-\mathcal{R}_1 B_1)\rho(-B_1^\dagger\mathcal{R}_1) \\ &= \mathcal{R}_1(B_0\rho B_0^\dagger + B_1\rho B_1^\dagger)\mathcal{R}_1 \\ &= \mathcal{R}_1\,\mathcal{E}_m(\rho)\,\mathcal{R}_1,\end{aligned} \tag{10}$$

where we used $B_0^\dagger = B_0$ and $\mathcal{R}_1 B_1^\dagger = (B_1\mathcal{R}_1)^\dagger = (-\mathcal{R}_1 B_1)^\dagger = -B_1^\dagger\mathcal{R}_1$. Covariance under $\mathcal{R}_2$ can be proved in the same way. Since $\mathcal{R}_3$ commutes with both B_0 and B_1, covariance of the channel under $\mathcal{R}_3$ follows trivially.

Finally, we consider the SWAP operation,

$$\mathcal{S}_w \equiv |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|. \tag{11}$$

The action of this gate is to transform the state $|01\rangle$ into $|10\rangle$, and vice versa, while it leaves unchanged the states $|00\rangle$ and $|11\rangle$. From the structure of the operators (6), it is immediate to verify that $\mathcal{S}_w$ commutes with B_0 and B_1. Therefore, the channel $\mathcal{E}_m$ is covariant with respect to $\mathcal{S}_w$, namely,

$$\mathcal{E}_m(\mathcal{S}_w\,\rho\,\mathcal{S}_w) = \mathcal{S}_w\,\mathcal{E}_m(\rho)\,\mathcal{S}_w. \tag{12}$$

III. CLASSICAL CAPACITY

The classical capacity C of a quantum channel concerns the ability of the channel to convey classical information. It measures the maximum amount of classical information that can be reliably transmitted down the channel per channel use. In computing the classical capacity, the full optimization over all entangled uses is generally required. In this section, we address the problem of finding the capacity C_1 [2] of the fully correlated channel $\mathcal{E}_m$. To do this we have to to maximize the so-called Holevo quantity χ [2,3,40] with respect to one use of the channel $\mathcal{E}_m$. Given a quantum source $\{p_\alpha,\rho_\alpha\}$, which is described by the density operator $\rho = \sum_\alpha p_\alpha \rho_\alpha$, we are dealing with the optimization problem [4–6]

$$C_1(\mathcal{E}_m) = \max_{\{p_\alpha,\rho_\alpha\}} \chi(\mathcal{E}_m,\{p_\alpha,\rho_\alpha\}), \tag{13}$$

where the quantity to be optimized is the Holevo quantity, which is defined as

$$\chi(\mathcal{E}_m,\{p_\alpha,\rho_\alpha\}) \equiv S(\mathcal{E}_m(\rho)) - \sum_\alpha p_\alpha S(\mathcal{E}_m(\rho_\alpha)), \tag{14}$$

where $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy. The first term in (14) is the channel output entropy of the quantum source described by ρ, whereas the second term is the channel average output entropy. Since for any ensemble of mixed states one can find an ensemble of pure states described by same density operator, and whose Holevo quantity (14) is at least as large [5], in the following we only consider ensembles of pure states $\{p_k,|\psi_k\rangle\}$:

$$C_1(\mathcal{E}_m) = \max_{\{p_k,|\psi_k\rangle\}} \chi(\mathcal{E}_m,\{p_k,|\psi_k\rangle\}), \tag{15}$$

$$\chi(\mathcal{E}_m,\{p_k,|\psi_k\rangle\}) = S(\mathcal{E}_m(\rho)) - \sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)), \tag{16}$$

where now $\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k|$.

A. Searching for ensembles that maximize χ

In this section, we use the channel covariance properties discussed in Sec. II to find the form of the ensembles $\{p_k,|\psi_k\rangle\}$ solving the maximization problems (15) and (16). We proceed along three steps: steps I and II exploit the covariance properties of channel $\mathcal{E}_m$, while step III uses the specific structure of the eigenvalues of the output states. Finally in step IV, we give the form of the optimal ensemble and the expression of the corresponding Holevo quantity.

1. Step I: Exploiting channel covariance with respect to the operations $\mathcal{R}_i$

Given a generic ensemble $\{p_k,|\psi_k\rangle\}$, we build a new ensemble by replacing each state $|\psi_k\rangle$ in $\{p_k,|\psi_k\rangle\}$ with the set

$$\{|\psi_k\rangle, \mathcal{R}_1|\psi_k\rangle, \mathcal{R}_2|\psi_k\rangle, \mathcal{R}_3|\psi_k\rangle\},$$

each state occurring with probability $\tilde{p}_k = p_k/4$ [41]. We refer to this new ensemble as $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$, and call $\tilde{\rho} = \sum_k \tilde{p}_k|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|$ the associated density operator:

$$\tilde{\rho} = \sum_k \frac{p_k}{4}\left(|\psi_k\rangle\langle\psi_k| + \sum_{i=1}^{3}\mathcal{R}_i|\psi_k\rangle\langle\psi_k|\mathcal{R}_i\right) = \frac{1}{4}\left(\rho + \sum_{i=1}^{3}\mathcal{R}_i\rho\mathcal{R}_i\right). \tag{17}$$

It can be verified that $\tilde{\rho}$ has the same diagonal elements of ρ, with all vanishing off-diagonal entries.

We now show that

$$\chi(\mathcal{E}_m,\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}) \geqslant \chi(\mathcal{E}_m,\{p_k,|\psi_k\rangle\}). \tag{18}$$

To this end we first notice that

$$\begin{aligned} S(\mathcal{E}_m(|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|)) &= S(\mathcal{E}_m(\mathcal{R}_i|\psi_k\rangle\langle\psi_k|\mathcal{R}_i)) \\ &= S(\mathcal{R}_i\,\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)\,\mathcal{R}_i) \\ &= S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)), \end{aligned} \tag{19}$$

where we used Eq. (9) and the fact that a unitary operation does not change the von Neumann entropy. Therefore, by replacing the old ensemble with the new one, the second term in the Holevo quantity (16) does not change:

$$\begin{aligned} \sum_k \tilde{p}_k S(\mathcal{E}_m(|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|)) &= 4\sum_k \frac{p_k}{4} S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)) \\ &= \sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)). \end{aligned} \tag{20}$$

For the output entropy related to $\tilde{\rho}$ we find

$$\begin{aligned} S(\mathcal{E}_m(\tilde{\rho})) &= S\left(\mathcal{E}_m\left(\frac{1}{4}\rho + \frac{1}{4}\sum_{i=1}^{3}\mathcal{R}_i\rho\mathcal{R}_i\right)\right) \\ &= S\left(\frac{1}{4}\mathcal{E}_m(\rho) + \frac{1}{4}\sum_{i=1}^{3}\mathcal{E}_m(\mathcal{R}_i\rho\mathcal{R}_i)\right) \\ &\geqslant \frac{1}{4}S\left(\mathcal{E}_m(\rho)\right) + \frac{1}{4}\sum_{i=1}^{3}S\left(\mathcal{E}_m(\mathcal{R}_i\rho\mathcal{R}_i)\right) \\ &= S(\mathcal{E}_m(\rho)), \end{aligned} \tag{21}$$

where we have used the linearity of $\mathcal{E}_m$, the concavity of von Neumann entropy [2], and Eq. (19).

Relations (20) and (21) prove the inequality (18), and we can summarize the conclusions as follows: For any ensemble of pure states we can find another ensemble, whose density matrix has the same diagonal, with zero off-diagonal entries, and whose Holevo quantity is at least as large. In the following, we work with ensembles $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$; we omit the tilde hereafter.

To fix the notation, we introduce the expression of the generic input state in $\{p_k,|\psi_k\rangle\}$,

$$|\psi_k\rangle = a_k|00\rangle + b_k|01\rangle + c_k|10\rangle + d_k|11\rangle, \tag{22}$$

where $a_k, b_k, c_k, d_k \in \mathbb{C}$ and $|a_k|^2+|b_k|^2+|c_k|^2+|d_k|^2 = 1$. The corresponding density matrix is given by

$$\rho = \sum_k p_k|\psi_k\rangle\langle\psi_k| = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{23}$$

where

$$\alpha = \sum_k p_k |a_k|^2, \quad \beta = \sum_k p_k |b_k|^2, \quad \gamma = \sum_k p_k |c_k|^2, \\ \delta = \sum_k p_k |d_k|^2 = 1 - \alpha - \beta - \gamma. \tag{24}$$

2. Step II: Exploiting channel covariance with respect to the SWAP operation

Starting from any ensemble $\{p_k,|\psi_k\rangle\}$ defined in Eqs. (22) and (23), we can generate another ensemble by replacing each state $|\psi_k\rangle$ with the couple of states $\{|\psi_k\rangle, \mathcal{S}_w|\psi_k\rangle\}$, each one occurring with probability $p_k/2$. The state $\mathcal{S}_w|\psi_k\rangle$ is obtained from $|\psi_k\rangle$ by exchanging the coefficients b_k and c_k in Eq. (22). We call this new ensemble $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ and $\tilde{\rho}$ the corresponding density operator:

$$\tilde{\rho} = \frac{1}{2}\sum_k p_k\, |\psi_k\rangle\langle\psi_k| + \frac{1}{2}\sum_k p_k\, \mathcal{S}_w|\psi_k\rangle\langle\psi_k|\mathcal{S}_w \\ = \frac{1}{2}\rho + \frac{1}{2}\mathcal{S}_w\rho\mathcal{S}_w = \frac{1}{2}\rho + \frac{1}{2}\rho(\beta \leftrightarrow \gamma). \tag{25}$$

Again, the ensemble $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ has a Holevo quantity χ at least as large as that of the parent ensemble $\{p_k,|\psi_k\rangle\}$.

To prove this we first observe that the second term of χ Eq. (16) does not change,

$$\sum_k \tilde{p}_k S(\mathcal{E}_m(|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|)) = \frac{1}{2}\sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)) \\ + \frac{1}{2}\sum_k p_k S(\mathcal{E}_m(\mathcal{S}_w|\psi_k\rangle\langle\psi_k|\mathcal{S}_w)) \\ = \sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)), \tag{26}$$

where we have used (12). Then for the first term we find

$$S(\mathcal{E}_m(\tilde{\rho})) = S\big(\mathcal{E}_m\big(\tfrac{1}{2}\rho + \tfrac{1}{2}\mathcal{S}_w\rho\mathcal{S}_w\big)\big) \\ = S\big(\tfrac{1}{2}\mathcal{E}_m(\rho) + \tfrac{1}{2}\mathcal{E}_m(\mathcal{S}_w\rho\mathcal{S}_w)\big) \\ \geqslant \tfrac{1}{2}S(\mathcal{E}_m(\rho)) + \tfrac{1}{2}S(\mathcal{E}_m(\mathcal{S}_w\rho\mathcal{S}_w)) \\ = \tfrac{1}{2}S(\mathcal{E}_m(\rho)) + \tfrac{1}{2}S(\mathcal{S}_w\mathcal{E}_m(\rho)\mathcal{S}_w) \\ = S(\mathcal{E}_m(\rho)), \tag{27}$$

where we have used arguments similar to those exploited in deriving (21), together with the covariance property (12). Relations (26) and (27) prove the upper bound provided by $\chi(\mathcal{E}_m,\{\tilde{p}_k,|\tilde{\psi}_k\rangle\})$.

3. Step III: Exploiting the structure of the output state eigenvalues

When the channel $\mathcal{E}_m$ acts on the generic state (22), it yields an output given by

$$\rho'_k = \begin{pmatrix} |a_k|^2 + (1-\eta)|d_k|^2 & a_k b_k^* & a_k c_k^* & \sqrt{\eta} a_k d_k^* \\ a_k^* b_k & |b_k|^2 & b_k c_k^* & \sqrt{\eta} b_k d_k^* \\ a_k^* c_k & b_k^* c_k & |c_k|^2 & \sqrt{\eta} c_k d_k^* \\ \sqrt{\eta} a_k^* d_k & \sqrt{\eta} b_k^* d_k & \sqrt{\eta} c_k^* d_k & \eta |d_k|^2 \end{pmatrix}. \tag{28}$$

The above density operator has at least two zero eigenvalues, due to the fact that the channel $\mathcal{E}_m$ has a noiseless subspace span$\{|01\rangle,|10\rangle\}$ which does not mix with the other subspace span$\{|00\rangle,|11\rangle\}$. The remaining two eigenvalues are given by

$$l_{k\pm} = \tfrac{1}{2}\big(1 \pm \sqrt{1 - z_k^2}\big), \\ z_k^2 = 1 - \{|a_k|^4 + 2|a_k|^2(|b_k|^2 + |c_k|^2 + |d_k|^2) \\ + [|b_k|^2 + |c_k|^2 - |d_k|^2(1-2\eta)]^2\}. \tag{29}$$

Since $l_{k\pm}$ do not depend on the phase of a_k, b_k, c_k, d_k, we can assume without loss of generality that these coefficients are real. From (29) the average output entropy is found

$$\sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)) = \sum_k p_k H_2(l_k), \tag{30}$$

where $H_2(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the Shannon binary entropy.

Starting from the mixed state (23) the map produces

$$\rho' = \mathcal{E}_m(\rho) = \begin{pmatrix} \alpha + (1-\eta)\delta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \eta\delta \end{pmatrix}, \tag{31}$$

and therefore the output entropy is given by

$$S(\mathcal{E}_m(\rho)) = -[\alpha + (1-\eta)\delta]\log_2[\alpha + (1-\eta)\delta] \\ - \beta\log_2(\beta) - \gamma\log_2(\gamma) - \eta\delta\log_2(\eta\delta). \tag{32}$$

We now modify the ensemble $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ introduced in Sec. III A2, by replacing the coefficients b_k and c_k in each state $|\tilde{\psi}_k\rangle$ with $\overline{b}_k$ and $\overline{c}_k$, where $\overline{b}_k = \pm\overline{c}_k = \sqrt{(b_k^2 + c_k^2)/2}$. We call this new ensemble $\{\overline{p}_k,|\overline{\psi}_k\rangle\}$ and the corresponding density operator $\overline{\rho}$, which is the same as $\tilde{\rho}$. Indeed,

$$\overline{\rho} = \sum_k \overline{p}_k|\overline{\psi}_k\rangle\langle\overline{\psi}_k| \\ = \begin{pmatrix} \sum_k \tilde{p}_k a_k^2 & 0 & 0 & 0 \\ 0 & \sum_k \tilde{p}_k \overline{b}_k^2 & 0 & 0 \\ 0 & 0 & \sum_k \tilde{p}_k \overline{c}_k^2 & 0 \\ 0 & 0 & 0 & \sum_k \tilde{p}_k d_k^2 \end{pmatrix} \\ = \sum_k \frac{p_k}{2}\begin{pmatrix} 2a_k^2 & 0 & 0 & 0 \\ 0 & b_k^2 + c_k^2 & 0 & 0 \\ 0 & 0 & b_k^2 + c_k^2 & 0 \\ 0 & 0 & 0 & 2d_k^2 \end{pmatrix} \\ = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\beta+\gamma) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\beta+\gamma) & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \\ = \frac{1}{2}\rho + \frac{1}{2}\rho(\beta \leftrightarrow \gamma) = \tilde{\rho}, \tag{33}$$

where we have used relations (24). The third equality comes from the fact that for any state $|\tilde{\psi}_k\rangle$ there is another one with the same occurrence probability $\tilde{p}_k = p_k/2$, which has the same a_k, d_k, but with b_k exchanged with c_k. It follows that the first term of the Holevo quantity is unchanged. This is true also for the second term. For $\{\overline{p}_k,|\overline{\psi}_k\rangle\}$ it reads

$$\sum_k \overline{p}_k S(\mathcal{E}_m(|\overline{\psi}_k\rangle\langle\overline{\psi}_k|)) = \sum_k \tilde{p}_k H_2(l'_k) \quad (34)$$

and the new eigenvalues l'_k are identical to the $l_{k\pm}$ in Eq. (29), since for real coefficients, they both depend on on the combination $b_k^2 + c_k^2$, which is unaffected by the transformation $b_k \to \overline{b}_k, c_k \to \overline{c}_k$.

Therefore the ensemble $\{\overline{p}_k,|\overline{\psi}_k\rangle\}$ produces the same Holevo quantity of $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ of Sec. III A2 [as Eqs. (33) and (34) show], but has the advantage of a simpler structure of the states in the ensemble.

4. Step IV: Optimal ensemble and the corresponding Holevo quantity

The chain of relations obtained up to here proves that the ensemble $\{\bar{p}_k,|\bar{\psi}_k\rangle\}$ makes it possible to achieve an upper bound for the Holevo quantity of a generic ensemble $\{p_\alpha,\rho_\alpha\}$. Indeed, $\{\bar{p}_k,|\bar{\psi}_k\rangle\}$ belongs to the original ensemble $\{p_\alpha,\rho_\alpha\}$, the maximization of the Holevo quantity for the former ensemble also yields the maximum over the whole set of $\{p_\alpha,\rho_\alpha\}$.

Summing up and simplifying the notation, we then have to explore ensembles $\{p_k,|\psi_k\rangle\}$ ($k \in \{1,2,\ldots,N\}$), where states have the form

$$|\psi_k\rangle = a_k|00\rangle + b_k|01\rangle \pm b_k|10\rangle + d_k|11\rangle, \quad (35)$$

with real a_k,b_k,d_k ($a_k^2 + 2b_k^2 + d_k^2 = 1$). The density matrix of such ensemble has the form

$$\rho = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \quad (36)$$

where

$$\alpha = \sum_k p_k a_k^2, \quad \beta = \sum_k p_k b_k^2, \quad \delta = \sum_k p_k d_k^2 = 1 - \alpha - 2\beta. \quad (37)$$

The output entropy is given by

$$S(\mathcal{E}_m(\rho)) = -[\alpha + (1-\eta)\delta]\log_2[\alpha + (1-\eta)\delta] - 2\beta\log_2(\beta) - \eta\delta\log_2(\eta\delta). \quad (38)$$

The average output entropy reads

$$\sum_k p_k H_2\left[\tfrac{1}{2}\left(1 + \sqrt{1 - z_k^2}\right)\right], \quad (39)$$

where

$$z_k^2 = 4d_k^2(1-\eta)\left(2b_k^2 + \eta d_k^2\right). \quad (40)$$

Finally, the Holevo quantity (16) is given by

$$\begin{aligned}\chi(\mathcal{E}_m,\{p_k,|\psi_k\rangle\}) &= -[\alpha + (1-\eta)\delta]\log_2[\alpha + (1-\eta)\delta] \\ &\quad - 2\beta\log_2(\beta) - \eta\delta\log_2(\eta\delta) \\ &\quad - \sum_k p_k H_2\left\{\frac{1}{2}\left[1 + \sqrt{1 - 4d_k^2(1-\eta)\left(2b_k^2 + \eta d_k^2\right)}\right]\right\}. \end{aligned} \quad (41)$$

In the following sections we compute the maximum of χ over two-qubit states, i.e., for single-use input states belonging to the class (35) and (36), thus deriving the classical capacity C_1 for the channel $\mathcal{E}_m$.

B. A lower bound for C_1

In order to find a lower bound for C_1, it is sufficient to compute the Holevo quantity (41) for an arbitrary ensemble. We choose ensembles of the special form

$$\{p_k,|\psi_k\rangle\} = \{p_{\varphi k},|\varphi_k\rangle\} \cup \{p_{\phi k},|\phi_k\rangle\}, \quad (42)$$

where $\sum_k(p_{\varphi k} + p_{\phi k}) = 1$, and such that $|\varphi_k\rangle \in \text{span}\{|01\rangle,|10\rangle\}$, whereas $|\phi_k\rangle \in \text{span}\{|00\rangle,|11\rangle\}$.

From (28) it is clear that the transmission of the states $|\varphi_k\rangle$ is noiseless $[S(\mathcal{E}_m(|\varphi_k\rangle\langle\varphi_k|)) = 0]$, so that

$$\sum_k p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)) = \sum_k p_{\phi k} S(\mathcal{E}_m(|\phi_k\rangle\langle\phi_k|)). \quad (43)$$

It is worth noting that, since the subspace spanned by $|01\rangle$ and $|10\rangle$ is noiseless for the channel $\mathcal{E}_m$, we can choose for the subensemble $\{p_{\varphi k},|\varphi_k\rangle\}$ any pair of mutually orthogonal states:

$$\begin{cases} p_{\varphi +} = \beta, & |\varphi_+\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle, \\ p_{\varphi -} = p_{\varphi +}, & |\varphi_-\rangle = \sin\theta|01\rangle - \cos\theta|10\rangle. \end{cases} \quad (44)$$

With this notation, the subensemble of separable states $\{(\beta,|01\rangle),(\beta,|10\rangle)\}$ and the subensemble of maximally entangled states $\{\beta,\frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle\}$ are recovered when $\theta = 0$ and $\theta = \pi/4$, respectively. All values of θ give the same contribution $-2\beta\log_2(\beta)$ to the Holevo quantity (41). Therefore, as far as we consider ensembles (42), there is no advantage in using entangled input states of span$\{|01\rangle, |10\rangle\}$.

With regard to the subensemble $\{p_{\phi k},|\phi_k\rangle\}$, it is interesting to examine two instances. First we choose a set of product states $\{(\alpha,|00\rangle),(\delta,|11\rangle)\}$, calling $\mathcal{A}$ the corresponding ensemble. In this case we obtain

$$\begin{aligned}\sum_k p_{\phi k} S(\mathcal{E}_m(|\phi_k\rangle\langle\phi_k|)) &= \alpha S(\mathcal{E}_m(|00\rangle\langle 00|)) + \delta\, S(\mathcal{E}_m(|11\rangle\langle 11|)) \\ &= \delta\, H_2(\eta), \end{aligned} \quad (45)$$

since from (28) it turns out that $\mathcal{E}_m(|00\rangle\langle 00|) = |00\rangle\langle 00|$ and $\mathcal{E}_m(|11\rangle\langle 11|) = (1-\eta)|00\rangle\langle 00| + \eta|11\rangle\langle 11|$. The Holevo quantity (41) relative to the ensemble $\mathcal{A}$ is

$$\chi(\mathcal{E}_m,\mathcal{A}) = -[\alpha + (1-\eta)\delta]\log_2[\alpha + (1-\eta)\delta] - 2\beta\log_2(\beta) - \eta\delta\log_2(\eta\delta) - \delta H_2(\eta), \quad (46)$$

so that a lower bound to C_1 is provided by

$$\chi_1^{(lb)} = \max_{\alpha,\beta,\delta} \chi(\mathcal{E}_m,\mathcal{A}), \tag{47}$$

with α,β,δ real and $\alpha + 2\beta + \delta = 1$.

Second, for the subensemble $\{p_{\phi k},|\phi_k\rangle\}$ we choose a set of entangled states,

$$p_{\phi_\pm} = \frac{\alpha+\delta}{2}, \quad |\phi_\pm\rangle = \sqrt{\frac{\alpha}{\alpha+\delta}}|00\rangle \pm \sqrt{\frac{\delta}{\alpha+\delta}}|11\rangle, \tag{48}$$

calling $\mathcal{B}$ the corresponding ensemble, for which we have

$$\sum_k p_{\phi k} S(\mathcal{E}_m(|\phi_k\rangle\langle\phi_k|)) = (\alpha+\delta) H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{\delta}{\alpha+\delta}\right)^2}\right]\right\}, \tag{49}$$

as the output states generated by $\mathcal{E}_m$ from the input states (48) have the same entropy; see Eq. (29). The Holevo quantity (41) relative to the ensemble $\mathcal{B}$ is

$$\chi(\mathcal{E}_m,\mathcal{B}) = -[\alpha+(1-\eta)\delta]\log_2[\alpha+(1-\eta)\delta] - 2\beta\log_2(\beta) - \eta\delta\log_2(\eta\delta) - (\alpha+\delta) \times H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{\delta}{\alpha+\delta}\right)^2}\right]\right\}, \tag{50}$$

yielding the lower bound for C_1 given by

$$\chi_2^{(lb)} = \max_{\alpha,\beta,\delta} \chi(\mathcal{E}_m,\mathcal{B}), \tag{51}$$

We plot the bounds (47) and (51) in Fig. 2. Ensemble $\mathcal{B}$ (thick curve in the figure) always produces a better performance than ensemble $\mathcal{A}$ (thin curve). This result is a first hint that entangled input states may be useful to improve the channel capability to convey classical information. Moreover, the classical capacity of $\mathcal{E}_m$ is at least equal to $\log_2(3)$, reflecting the fact that in the worst case ($\eta = 0$) there are three states allowing for noiseless transmission: $|00\rangle,|01\rangle,|10\rangle$. The lower bound $\log_2(3)$ is found by using them to encode three classical symbols, each one occurring with the same probability $1/3$.

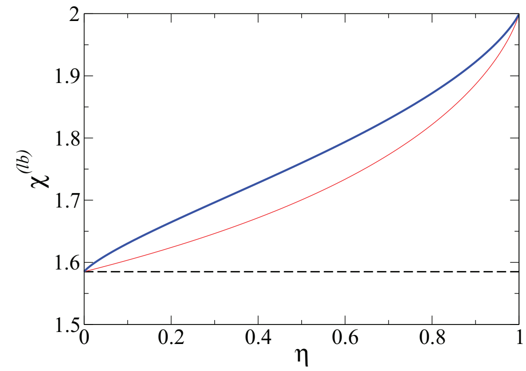

FIG. 2. (Color online) Maximum (obtained via numerical optimization) Holevo quantity relative to the ensembles $\mathcal{A}$ (thin red curve) and $\mathcal{B}$ (thick blue curve) as a function of the channel transmissivity η. In the first case, we obtain the lower bound $\chi_1^{(lb)}$ (47) to the capacity C_1, in the second the lower bound $\chi_2^{(lb)}$ (51). We also plot the trivial lower bound $\log_2 3$ (dashed line).

C. The C_1 capacity of $\mathcal{E}_m$

Now we are ready to find the optimal ensemble, whose maximum Holevo quantity gives the C_1 classical capacity of $\mathcal{E}_m$. To this end, we consider a generic ensemble $\{p_k,|\psi_k\rangle\}$ belonging to the class (35) and (36), and we replace each state $|\psi_k\rangle$ and its occurrence probability p_k in this ensemble with

$$p_k,|\psi_k\rangle \to \begin{cases} p_{\phi k} = \frac{p_k(a_k^2+d_k^2)}{2}, & |\phi_{k\pm}\rangle = \frac{a_k}{\sqrt{a_k^2+d_k^2}}|00\rangle \pm \frac{d_k}{\sqrt{a_k^2+d_k^2}}|11\rangle, \\ p_{\varphi k} = p_k b_k^2, & |\varphi_{k\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm e^{\pi i k/N}|10\rangle), \end{cases} \tag{52}$$

where the index k ranges in $\{1,N\}$. We call $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ the new ensemble. It is straightforward to prove that the density matrix of the new ensemble is equal to that of the old ensemble (35) and (36), and therefore the output entropy is unchanged: $S(\mathcal{E}_m(\sum_k \tilde{p}_k|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|)) = S(\mathcal{E}_m(\sum_k p_k|\psi_k\rangle\langle\psi_k|))$.

With regards to the second term of the Holevo quantity, we notice that the states $|\varphi_{k\pm}\rangle$ in (52) do not contribute to the average output entropy: $S(\mathcal{E}_m(|\varphi_{k\pm}\rangle\langle\varphi_{k\pm}|)) = 0$. Therefore, the average entropy for the new ensemble is

$$\sum_k \tilde{p}_k S(\mathcal{E}_m(|\tilde{\psi}_k\rangle\langle\tilde{\psi}_k|)) = \sum_k p_{\phi k}[S(\mathcal{E}_m(|\phi_{k+}\rangle\langle\phi_{k+}|)) + S(\mathcal{E}_m(|\phi_{k-}\rangle\langle\phi_{k-}|))]$$
$$= 2\sum_k p_{\phi k} S(\mathcal{E}_m(|\phi_{k+}\rangle\langle\phi_{k+}|)) = \sum_k p_k\left(a_k^2+d_k^2\right) H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{d_k^2}{a_k^2+d_k^2}\right)^2}\right]\right\}, \tag{53}$$

where we have used the fact that the states $\mathcal{E}_m(|\phi_{k\pm}\rangle\langle\phi_{k\pm}|)$ have the same entropy [see Eq. (29)]. In order to assert that the new ensemble $\{\tilde{p}_k,|\tilde{\psi}_k\rangle\}$ produces a greater Holevo quantity (16) than the one produced by $\{p_k,|\psi_k\rangle\}$ we have to prove that

$$\sum_k p_k H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4(1-\eta)d_k^2\left(2b_k^2+\eta d_k^2\right)}\right]\right\} \geqslant \sum_k p_k\left(a_k^2+d_k^2\right) H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\frac{d_k^4}{\left(a_k^2+d_k^2\right)^2}}\right]\right\}, \tag{54}$$

the left-hand side of (54) being the last term in (41). A sufficient condition for the validity of inequality (54) is that the inequality

$$H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4(1-\eta)d_k^2\left(2b_k^2+\eta d_k^2\right)}\right]\right\} \geqslant \left(a_k^2+d_k^2\right) H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\frac{d_k^4}{\left(a_k^2+d_k^2\right)^2}}\right]\right\} \tag{55}$$

holds true for any admissible value of a_k, b_k, d_k, and η. We checked it numerically and it turns out that this inequality holds; moreover, it is tight except for $\eta = 1$, or $b = 0$, or $d = 0$.

By summarizing the above results, we can state that for any ensemble $\{p_k, |\psi_k\rangle\}$, we can find a new one $\{\tilde{p}_k, |\tilde{\psi}_k\rangle\}$ of the form (52), whose Holevo quantity is at least as great. For this new ensemble the output entropy is given by (38), whereas the average output entropy is given by (53).

We can now find an upper bound to the Holevo quantity of ensemble (52) by considering its average output entropy (53), and by taking advantage of the convexity of the function $H_2[\frac{1}{2}(1+\sqrt{1-x^2})]$ [42] with respect to x:

$$\begin{aligned}
&\sum_k p_k\left(a_k^2+d_k^2\right)H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{d_k^2}{a_k^2+d_k^2}\right)^2}\right]\right\}\\
&\geqslant(\alpha+\delta)H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\sum_k p_k\frac{a_k^2+d_k^2}{\alpha+\delta}\frac{d_k^2}{a_k^2+d_k^2}\right)^2}\right]\right\}\\
&=(\alpha+\delta)H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{\delta}{\alpha+\delta}\right)^2}\right]\right\}.
\end{aligned} \tag{56}$$

The Holevo quantity of the ensemble (52) is thus upper bounded by

$$\begin{aligned}
\chi^* = \max_{\alpha,\beta,\delta}\Bigg(&-[\alpha+(1-\eta)\delta]\log_2[\alpha+(1-\eta)\delta]-2\beta\log_2(\beta)-\eta\delta\log_2(\eta\delta)\\
&-(\alpha+\delta)H_2\left\{\frac{1}{2}\left[1+\sqrt{1-4\eta(1-\eta)\left(\frac{\delta}{\alpha+\delta}\right)^2}\right]\right\}\Bigg),
\end{aligned} \tag{57}$$

This is precisely the Holevo quantity achievable by ensemble $\mathcal{B}$, Sec. III B Eqs. (50) and (51); therefore, we conclude that (57) gives the C_1 classical capacity of $\mathcal{E}_m$. In Fig. 3 we plot the values of the coefficients α,β,δ, which give the maximum of the Holevo quantity for ensemble $\mathcal{B}$, whereas the plot of C_1 as a function of η is just given by the thick curve of Fig. 2.

It is worth noting that for $\eta = 0$, the maximization problem (57) does not admit a unique solution for the coefficients α and δ; indeed, in this case, the channel deterministically transforms $|11\rangle$ into $|00\rangle$, so that any state $\sqrt{\alpha/(\alpha+\delta)}|00\rangle + \sqrt{\delta/(\alpha+\delta)}|11\rangle$ is mapped into $|00\rangle$. To obtain the maximum of the Holevo quantity (which in this case equals $\log_2 3$), we can arbitrarily choose α and δ, provided that $\alpha+\delta = 1/3$. For a noiseless channel ($\eta = 1$), Fig. 3 shows that the optimal

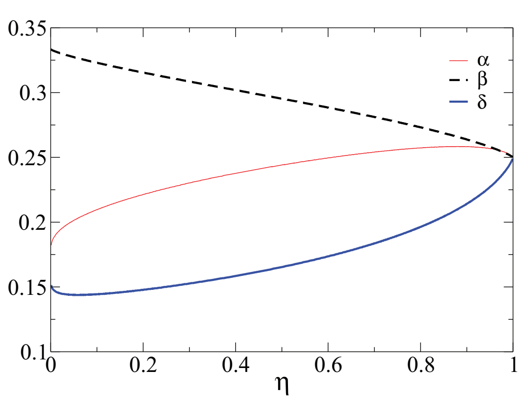

FIG. 3. (Color online) Coefficients α (thin red curve), β (dashed curve), and δ (thick blue curve) maximizing the Holevo quantity, plotted as functions of η. Such coefficients are obtained by numerically solving the optimization problem (57).

coefficients are $\alpha = \beta = \delta = \frac{1}{4}$; it means that ensemble $\mathcal{B}$ reduces to four orthogonal states, one pair inside the subspace span$\{|01\rangle,|10\rangle\}$, the other inside the subspace span$\{|00\rangle,|11\rangle\}$, each state occurring with equal probability $\frac{1}{4}$.

D. Is entanglement necessary to achieve C_1?

It is worth noting that the ensemble $\mathcal{B}$, making it possible to reach C_1, contains entangled states in the subspace $\{|00\rangle,|11\rangle\}$. This raises the following question: Is entanglement a necessary ingredient to achieve the channel capacity C_1? In Appendix A we show that the answer is positive. In particular, we show that for any $0 < \eta < 1$, only the use of entangled states makes it possible to achieve C_1 and the optimal ensemble is of the form

$$\begin{cases} p_\pm = \frac{\alpha+\delta}{2}, & |\phi_\pm\rangle = \sqrt{\frac{\alpha}{\alpha+\delta}}|00\rangle \pm \sqrt{\frac{\delta}{\alpha+\delta}}|11\rangle,\\ p_0 = \beta, & |\varphi_0\rangle = |01\rangle,\\ p_1 = \beta, & |\varphi_1\rangle = |10\rangle. \end{cases} \tag{58}$$

One can ask how much entanglement is needed in order to achieve this bound. We can answer this question for the above ensemble. It is clear that we really need entanglement only inside the subspace spanned by $\{|00\rangle, |11\rangle\}$. In Fig. 4 we plot the entropy of entanglement E_ϕ, defined as the von Neumann entropy of one of the two reduced states, obtained after tracing over one of the two qubits: $E_\phi = S(\rho_1) = S(\rho_2)$, with $\rho_{1(2)} = \mathrm{Tr}_{2(1)}(|\phi_\pm\rangle\langle\phi_\pm|)$. E_ϕ quantifies the entanglement content of the states $|\phi_\pm\rangle$ in the ensemble $\mathcal{B}$; see (48). The average entanglement required is given by $\overline{E}_\phi = (\alpha+\delta)E_\phi$, since we really need entanglement only when we use a state inside the subspace spanned by $\{|00\rangle,|11\rangle\}$, which happens with probability $\alpha+\delta$.

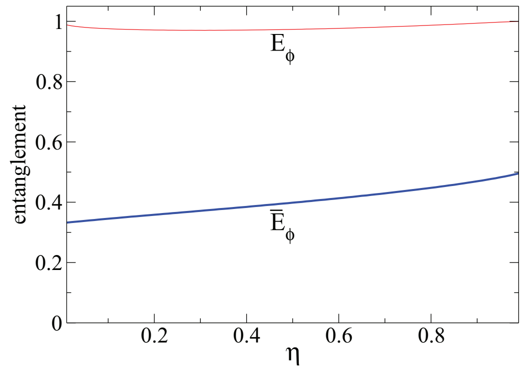

FIG. 4. (Color online) Entanglement E_ϕ (thin red curve) of the pure states $|\phi_\pm\rangle$ in the ensemble $\mathcal{B}$ and average entanglement $\overline{E}_\phi = (\alpha+\delta)E_\phi$ (thick blue curve) as a function of the transmissivity η. The values of α,β,δ are those ones solving the maximization problem (57).

E. An explicit formula for C_1

The form of the ensemble (52), which allows us to maximize the Holevo quantity of channel $\mathcal{E}_m$, tells us that we can view our memory channel as composed of two distinct and parallel channels acting on two orthogonal subspaces of the four-dimensional Hilbert space of the two-qubit system: a noiseless channel inside the subspace spanned by $\{|10\rangle,|01\rangle\}$ and a memoryless amplitude damping channel inside the subspace spanned by $\{|00\rangle,|11\rangle\}$. We denote these two channels as $\mathcal{E}_\varphi$ and $\mathcal{E}_\phi$, respectively. In other words, we have proved that, for the fully correlated amplitude damping channel $\mathcal{E}_m$, the channel capacity C_1 is obtained without involving any coherent superposition of states from these two different subspaces. This makes it possible to analytically carry out the optimization (57). Indeed, we can write

$$\begin{aligned} C_1(\mathcal{E}_m) &= \max_{\{p_k,\rho_k\}} \chi(\mathcal{E}_m,\{p_k,\rho_k\}) \\ &= \max_{\{p_{\phi k},\rho_{\phi k}\}\cup\{p_{\varphi k},\rho_{\varphi k}\}} \chi(\mathcal{E}_m,\{p_k,\rho_k\}), \end{aligned} \tag{59}$$

where $\rho_{\phi k}$ is a generic state inside the subspace $\{|00\rangle,|11\rangle\}$, whereas $\rho_{\varphi k}$ is a generic state inside the subspace $\{|01\rangle,|10\rangle\}$. Now we call $p=\sum_k p_{\phi k}$, and consequently we have that $\sum_k p_{\varphi k} = 1-p$. We can then write

$$\begin{aligned} \rho &= \sum_k p_k \rho_k \\ &= \sum_k p_{\phi k}\,\rho_{\phi k} + \sum_k p_{\varphi k}\,\rho_{\varphi k} \\ &= p\sum_k \frac{p_{\phi k}}{\sum_{k'} p_{\phi k'}}\rho_{\phi k} + (1-p)\sum_k \frac{p_{\varphi k}}{\sum_{k'} p_{\varphi k'}}\rho_{\varphi k'} \\ &= p\sum_k \tilde{p}_{\phi k}\,\rho_{\phi k} + (1-p)\sum_k \tilde{p}_{\varphi k}\,\rho_{\varphi k} \\ &= p\,\rho_\phi + (1-p)\,\rho_\varphi, \end{aligned} \tag{60}$$

where we have set

$$\begin{aligned} \tilde{p}_{\phi k} &\equiv \frac{p_{\phi k}}{\sum_{k'} p_{\phi k'}}, \quad \rho_\phi \equiv \sum_k \tilde{p}_{\phi k}\,\rho_{\phi k}, \\ \tilde{p}_{\varphi k} &\equiv \frac{p_{\varphi k}}{\sum_{k'} p_{\varphi k'}}, \quad \rho_\varphi \equiv \sum_k \tilde{p}_{\varphi k}\,\rho_{\varphi k}. \end{aligned} \tag{61}$$

Note that $\mathrm{Tr}[\rho_\phi] = \mathrm{Tr}[\rho_\varphi] = 1$. The first term of the Holevo quantity (59) is given by

$$\begin{aligned} &S\left(\mathcal{E}_m\left(\sum_k p_k\rho_k\right)\right) \\ &\quad = S(p\,\mathcal{E}_m(\rho_\phi) + (1-p)\,\mathcal{E}_m(\rho_\varphi)) \\ &\quad = H_2(p) + p\,S(\mathcal{E}_m(\rho_\phi)) + (1-p)\,S(\mathcal{E}_m(\rho_\varphi)) \\ &\quad = H_2(p) + p\,S\left(\mathcal{E}_m\left(\sum_k \tilde{p}_{\phi k}\,\rho_{\phi k}\right)\right) \\ &\qquad + (1-p)\,S\left(\mathcal{E}_m\left(\sum_k \tilde{p}_{\varphi k}\,\rho_{\varphi k}\right)\right), \end{aligned} \tag{62}$$

where the second equality is due to the fact that the two output states in the above equation are supported on orthogonal subspaces and can therefore be independently and simultaneously diagonalized. Now we turn to the second term of the Holevo quantity, namely,

$$\begin{aligned} &\sum_k p_k S(\mathcal{E}_m(\rho_k)) \\ &\quad = \sum_k p_{\phi k} S(\mathcal{E}_\phi(\rho_{\phi k})) + \sum_k p_{\varphi k} S(\mathcal{E}_\varphi(\rho_{\varphi k})) \\ &\quad = p\sum_k \tilde{p}_{\phi k} S(\mathcal{E}_\phi(\rho_{\phi k})) + (1-p)\sum_k \tilde{p}_{\varphi k} S(\mathcal{E}_\varphi(\rho_{\varphi k})), \end{aligned} \tag{63}$$

where we have used the quantities defined in (61). From (62) and (63) we obtain

$$\begin{aligned} \chi(\mathcal{E}_m,\{p_k,\rho_k\}) = H_2(p) &+ p\,\chi_\phi(\{\tilde{p}_{\phi k},\rho_{\phi k}\}) \\ &+ (1-p)\,\chi_\varphi(\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}), \end{aligned} \tag{64}$$

where we have defined

$$\begin{aligned} \chi_\phi(\{\tilde{p}_{\phi k},\rho_{\phi k}\}) &\equiv \chi(\mathcal{E}_\phi,\{\tilde{p}_{\phi k},\rho_{\phi k}\}), \\ \chi_\varphi(\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}) &\equiv \chi(\mathcal{E}_\varphi,\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}). \end{aligned} \tag{65}$$

The maximization problem (59) is therefore equivalent to

$$\begin{aligned} C_1(\mathcal{E}_m) &= \max_{\{p_k,\rho_k\}} \chi(\mathcal{E}_m,\{p_k,\rho_k\}) \\ &= \max_{\{p_{\phi k},\rho_{\phi k}\},\{p_{\varphi k},\rho_{\varphi k}\}} [H_2(p) + p\,\chi_\phi(\{\tilde{p}_{\phi k},\rho_{\phi k}\}) \\ &\quad + (1-p)\,\chi_\varphi(\{\tilde{p}_{\varphi k},\rho_{\varphi k}\})] \\ &= \max_{p\in[0,1]} \Big[H_2(p) + p \max_{\{\tilde{p}_{\phi k},\rho_{\phi k}\}} \chi_\phi(\{\tilde{p}_{\phi k},\rho_{\phi k}\}) \\ &\quad + (1-p) \max_{\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}} \chi_\varphi(\{\tilde{p}_{\varphi k},\rho_{\varphi k}\})\Big] \\ &= \max_{p\in[0,1]} \left[H_2(p) + p\,C_{\phi_1} + (1-p)\,C_{\varphi_1}\right], \end{aligned} \tag{66}$$

where C_{ϕ_1} and C_{φ_1} are, respectively, the classical product state capacity,

$$C_{\phi_1} = \max_{\{\tilde{p}_{\phi k},\rho_{\phi k}\}} \chi(\{\tilde{p}_{\phi k},\rho_{\phi k}\}), \tag{67}$$

$$C_{\varphi_1} = \max_{\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}} \chi(\mathcal{E}_m,\{\tilde{p}_{\varphi k},\rho_{\varphi k}\}). \tag{68}$$

The maximization (66) over p can then be simply achieved by studying the first derivative of $G(p) \equiv H_2(p) + p\,C_{\phi_1} +$

$(1-p)\,C_{\varphi_1}$ with respect to p:

$$\frac{\partial G(p)}{\partial p} = \log_2 \frac{1-p}{p} + C_{\phi_1} - C_{\varphi_1}. \qquad (69)$$

A maximum is found for

$$p_{\rm opt} = \frac{1}{1+2^{C_{\varphi_1}-C_{\phi_1}}} = \frac{1}{1+2^{1-C_{ad,1}}}, \qquad (70)$$

since $C_{\varphi_1}=1$ and C_{ϕ_1} is the product state capacity $C_{ad,1}$ of the memoryless amplitude damping channel (1) [42], which is given by

$$C_{ad,1} = \max_{p_1\in[0,1]} \left(H_2(\eta p_1) - H_2\{\tfrac{1}{2}[1+\sqrt{1-4\eta(1-\eta)p_1^2}]\}\right). \qquad (71)$$

It is worth noting that the optimal value of p_1 in (71) also gives the population of the single-qubit state $|1\rangle$, in the density operator describing the ensemble which maximizes the single-use (and single-qubit) Holevo quantity for the memoryless amplitude damping channel [42].

We can conclude that the C_1 capacity of the memory channel $\mathcal{E}_m$ is

$$C_1(\mathcal{E}_m) = 1 + H_2(p_{\rm opt}) - p_{\rm opt}\,(1-C_{ad,1}). \qquad (72)$$

Equation (72) provides an explicit solution to (57), once $C_{ad,1}$ is known. In Fig. 5 we show the optimal value $p_{\rm opt}$ as a function of the channel transmissivity η. Note that the value of $p_{\rm opt}$ tells us the weight of the subspace spanned by $\{|00\rangle,|11\rangle\}$ in achieving the C_1 capacity of the channel $\mathcal{E}_m$. Let us consider two limiting cases. As expected, for $\eta=0$ we have that $C_{\phi_1}=0$ and therefore by (70) we find $p_{\rm opt}=1/3$, while for $\eta=1$ we have that $C_{\phi_1}=1$ and $p_{\rm opt}=1/2$.

From the maximization procedure we depicted, it is clear that the probability $\delta/(\alpha+\delta)$, which gives the population of the state $|11\rangle$ in the density operator describing the optimal ensemble, normalized by the probability that a state picked up from this ensemble belongs to the subspace spanned by $\{|00\rangle,|11\rangle\}$, is the same of the optimal p_1 ensuring the achievement of $C_{ad,1}$ in (71) [see Eq. (67)].

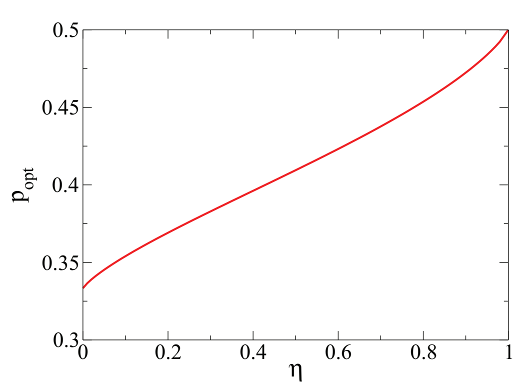

FIG. 5. (Color online) Plot of $p_{\rm opt}$ [Eq. (70)] as functions of η.

IV. QUANTUM CAPACITY

The quantum capacity Q concerns the channel ability to convey quantum information. It can be computed as [7–9]

$$Q = \lim_{n\to\infty}\frac{Q_n}{n}, \quad Q_n = \max_{\rho^{(n)}} I_c\big(\mathcal{E}_m^{\otimes n},\rho^{(n)}\big), \qquad (73)$$

where $\rho^{(n)}$ is an input state for n channel uses and

$$I_c\big(\mathcal{E}_m^{\otimes n},\rho^{(n)}\big) = S\big(\mathcal{E}_m^{\otimes n}(\rho^{(n)})\big) - S_e\big(\mathcal{E}_m^{\otimes n},\rho^{(n)}\big) \qquad (74)$$

is the *coherent information* [43]. $S_e(\mathcal{E}_m^{\otimes n},\rho^{(n)})$ is the *entropy exchange* [44], defined as

$$S_e\big(\mathcal{E}_m^{\otimes n},\rho^{(n)}\big) = S\big[\big(\mathcal{I}\otimes\mathcal{E}_m^{\otimes n}\big)(|\Psi\rangle\langle\Psi|)\big], \qquad (75)$$

where $|\Psi\rangle$ is any purification of $\rho^{(n)}$. That is, we consider the system S, described by the density operator $\rho^{(n)}$, as a part of a larger quantum system RS; $\rho = \mathrm{Tr}_{\mathsf{R}}|\Psi\rangle\langle\Psi|$ and the reference system R evolves trivially, according to the identity superoperator $\mathcal{I}$. Note that maximization (73) has to be carried out with respect to a generic density operator $\rho^{(n)}$ belonging to the Hilbert space relative to n uses of the channel $\mathcal{E}_m$ (described by the superoperator $\mathcal{E}_m^{\otimes n}$).

A. Quantum capacity for channel transmissivity $\frac{1}{2}\leqslant\eta\leqslant 1$

In order to proceed to the calculation of the quantum capacity we use the fact that the channel $\mathcal{E}_m$ is degradable [45] for $\frac{1}{2}\leqslant\eta\leqslant 1$, as shown in Appendix B. Degradability implies that regularization (73) is no longer necessary; i.e., the quantum capacity is given by the “single-letter” formula, $Q=Q_1$,

$$Q(\mathcal{E}_m) = \max_{\rho} I_c(\mathcal{E}_m,\rho), \quad \eta\in\left[\tfrac{1}{2},1\right], \qquad (76)$$

where ρ belongs to the Hilbert space relative to a single use of channel $\mathcal{E}_m$.

The coherent information is given by

$$I_c(\mathcal{E}_m,\rho) = S(\mathcal{E}_m(\rho)) - S_e(\mathcal{E}_m,\rho) = S(\rho') - S(\rho^{\mathsf{E}'}), \qquad (77)$$

where $S_e(\mathcal{E}_m,\rho)=S(\rho^{\mathsf{E}'})$ is the entropy exchange related to the channel [43]. Here ρ is a generic input state for the channel $\mathcal{E}_m$, $\rho'=\mathcal{E}_m(\rho^{\mathsf{S}})$ and $\rho^{\mathsf{E}'}$ are given by (B4) and (B5), E being a fictitious environment allowing for a unitary representation of the map $\mathcal{E}_m$ (see Appendix B).

Our goal is to find the class of input states which makes it possible to solve problem (76), i.e., to maximize the coherent information (77). To this end, we first notice that for any two-qubit density operator ρ, we can build a diagonal density operator as

$$\tilde{\rho} = \frac{1}{4}\left(\rho + \sum_{i=1}^{3}\mathcal{R}_i\,\rho\mathcal{R}_i\right) = \begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \gamma & 0\\ 0 & 0 & 0 & \delta\end{pmatrix}, \qquad (78)$$

whose coherent information is at least as large as the one related to ρ:

$$
\begin{aligned}
I_c(\mathcal{E}_m,\tilde{\rho}) &= I_c\left(\mathcal{E}_m,\frac{1}{4}\left(\rho+\sum_{i=1}^{3}\mathcal{R}_i\rho\mathcal{R}_i\right)\right)\\
&\geqslant \frac{1}{4}I_c(\mathcal{E}_m,\rho)+\frac{1}{4}\sum_{i=1}^{3}I_c(\mathcal{E}_m,\mathcal{R}_i\rho\mathcal{R}_i)\\
&= \frac{1}{4}I_c(\mathcal{E}_m,\rho)+\frac{1}{4}\sum_{i=1}^{3}S(\mathcal{E}_m(\mathcal{R}_i\rho\mathcal{R}_i))\\
&\quad -\frac{1}{4}\sum_{i=1}^{3}S_e(\mathcal{E}_m,\mathcal{R}_i\rho\mathcal{R}_i)\\
&= I_c(\mathcal{E}_m,\rho). \qquad (79)
\end{aligned}
$$

Here, the inequality derives from the fact that the coherent information of a degradable channel is a concave function [46] and we have used the covariance properties of the channel. Finally, since $\mathcal{R}_i$ can only change the sign of coherences of the input state, the von Neumann entropy of $\rho^{\mathrm{E}'}$ does not change when we replace ρ with $\mathcal{R}_i\rho\mathcal{R}_i$: $S_e(\mathcal{E}_m,\mathcal{R}_i\rho\mathcal{R}_i)=S_e(\mathcal{E}_m,\rho)$, as one can see by Eq. (B5).

Now we build a new state from $\tilde{\rho}$:

$$
\overline{\rho}=\frac{1}{2}\tilde{\rho}+\frac{1}{2}\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}}=\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \frac{\beta+\gamma}{2} & 0 & 0\\ 0 & 0 & \frac{\beta+\gamma}{2} & 0\\ 0 & 0 & 0 & \delta\end{pmatrix}. \qquad (80)
$$

This new density operator exhibits a coherent information greater than or equal to $\tilde{\rho}$, since

$$
\begin{aligned}
&I_c(\mathcal{E}_m,\overline{\rho})\\
&\quad= I_c\left(\mathcal{E}_m,\tfrac{1}{2}\tilde{\rho}+\tfrac{1}{2}\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}}\right)\\
&\quad\geqslant \tfrac{1}{2}I_c(\mathcal{E}_m,\tilde{\rho})+\tfrac{1}{2}I_c(\mathcal{E}_m,\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}})\\
&\quad= \tfrac{1}{2}I_c(\mathcal{E}_m,\tilde{\rho})+\tfrac{1}{2}[S(\mathcal{E}_m(\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}}))-S_e(\mathcal{E}_m(\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}}))]\\
&\quad= I_c(\mathcal{E}_m,\tilde{\rho}). \qquad (81)
\end{aligned}
$$

In the above equation we have again exploited the concavity of the coherent information for degradable channels in getting the inequality; then we have used the covariance property (12). For the entropy exchange we have $S_e(\mathcal{E}_m(\mathcal{S}_{\mathrm{w}}\tilde{\rho}\mathcal{S}_{\mathrm{w}}))=S_e(\mathcal{E}_m(\tilde{\rho}))$ since it does not depend on β and γ [see Eq. (B5)].

We conclude that the quantum capacity (76) can be derived by maximizing the coherent information with respect to the diagonal state,

$$
\overline{\rho}=\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \delta\end{pmatrix}, \qquad (82)
$$

since we have demonstrated that for each ρ we can construct a density operator $\bar{\rho}$ of the form (82) whose coherent information is at least as great. Therefore, for $\eta \geqslant \frac{1}{2}$ the quantum capacity is given by

$$
\begin{aligned}
Q(\mathcal{E}_m) &= \max_{\overline{\rho}^S} I_c(\mathcal{E}_m,\overline{\rho}^S)\\
&= \max_{\overline{\rho}^S}\{S(\mathcal{E}_m(\overline{\rho}^S))-S_e(\mathcal{E}_m,\overline{\rho}^S)\}\\
&= \max_{\alpha,\beta,\delta}\{-[\alpha+(1-\eta)\delta]\log_2[\alpha+(1-\eta)\delta]\\
&\quad -2\beta\log_2\beta-\eta\delta\log_2\eta\delta\\
&\quad +[1-(1-\eta)\delta]\log_2[1-(1-\eta)\delta]\\
&\quad +(1-\eta)\delta\log_2[(1-\eta)\delta]\}, \qquad (83)
\end{aligned}
$$

with the constraint $\alpha+2\beta+\delta=1$. In Fig. 6 we plot the quantum capacity Q of the channel $\mathcal{E}_m$ as a result of the maximization problem (83), and in Fig. 7 we report the relative populations of the input state (82). The results are displayed for $\eta\in[0,1]$, but we stress that (83) give us the quantum capacity only for $\eta\in[\frac{1}{2},1]$. Notice that the curve reported in Fig. 6 is higher than the one derived in Ref. [39], where only a particular class of product input states was considered.

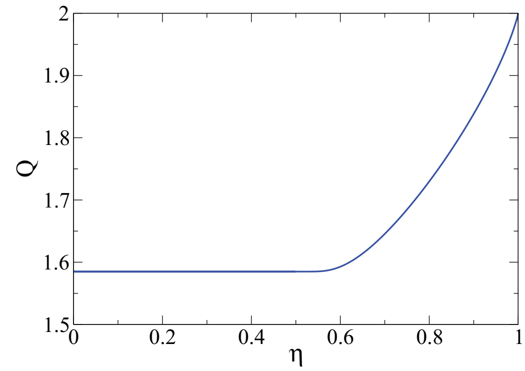

FIG. 6. (Color online) Plot of the quantum capacity Q of $\mathcal{E}_m$ as a function of η. For $\eta\geqslant 1/2$, Q is given by the numerical solution of the maximization task (83) (the searching step for α, β, δ is 10^{-4}). For $\eta<0.5$ the quantum capacity turns out to be constant and equal to $\log_2 3$.

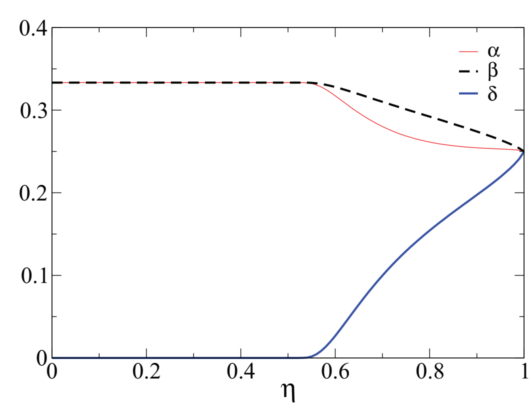

FIG. 7. (Color online) Plot of the coefficients α (thin curve), β (dashed curve), and δ (thick curve) which (numerically) solve the optimization problem (83), as functions of η.

B. Quantum capacity for channel transmissivity $0 \leqslant \eta < \frac{1}{2}$

For $\eta < 1/2$, we cannot use the subadditivity argument provided by degradability in order to find the channel quantum capacity. However, we notice that $\mathcal{E}_m$ has the property

$$\mathcal{E}_{m,\eta_2\eta_1} = \mathcal{E}_{m,\eta_2} \circ \mathcal{E}_{m,\eta_1}, \tag{84}$$

where we have used $\mathcal{E}_{m,x}$ to indicate a channel $\mathcal{E}_m$ with transmissivity x. Now we choose η_1,η_2 such that $\eta_1 = 1/2$ and $\eta_2 \in [0,1[$, then $\eta_2\eta_1 \in [0,1/2[$. By considering n channel uses and applying the quantum data processing inequality [8], we obtain

$$I_c\big(\mathcal{E}^{\otimes n}_{m,\eta_2\eta_1},\rho^{(n)}\big) \leqslant I_c\big(\mathcal{E}^{\otimes n}_{m,\frac{1}{2}},\rho^{(n)}\big), \tag{85}$$

since $\mathcal{E}^{\otimes n}_{m,\eta_2\eta_1} = \mathcal{E}^{\otimes n}_{m,\eta_2} \circ \mathcal{E}^{\otimes n}_{m,\eta_1}$. Hence, for $\eta < 1/2$, the quantum capacity is given by

$$\begin{aligned} Q(\mathcal{E}_m) &= \lim_{n\to\infty} \max_{\rho^{(n)}} I_c\big(\mathcal{E}^{\otimes n}_m,\rho^{(n)}\big) \\ &\leqslant \lim_{n\to\infty} \max_{\rho^{(n)}} I_c\big(\mathcal{E}^{\otimes n}_{m,\frac{1}{2}},\rho^{S(n)}\big) \\ &\leqslant \max_{\rho} I_c\big(\mathcal{E}_{m,\frac{1}{2}},\rho\big) = \log_2 3, \end{aligned} \tag{86}$$

where the second inequality holds since for $\eta = 1/2$ the channel is degradable, whereas the last equality is numerically provided by (83). It is easy to prove that $\log_2(3)$ is also a lower bound for the channel quantum capacity, since the three-dimensional subspace spanned by $\{|00\rangle,|01\rangle,|10\rangle\}$ is noiseless. We can therefore conclude that, for $\eta < 1/2$, $Q(\mathcal{E}_m) = \log_2 3$.

V. ENTANGLEMENT-ASSISTED CLASSICAL CAPACITY

The entanglement-assisted classical capacity C_E gives the maximum amount of classical information that can be reliably transmitted down the channel per channel use, provided the sender and the receiver share infinite prior entanglement resources. It can be computed as [11,12]

$$C_E = \max_{\rho} I(\mathcal{E}_m,\rho), \tag{87}$$

where the maximization is performed over the input state ρ for a single use of the channel $\mathcal{E}_m$ and

$$I(\mathcal{E}_m,\rho) = S(\rho) + I_c(\mathcal{E}_m,\rho) \tag{88}$$

differs from the coherent information I_c, defined in Eq. (77), by the addition of the input-state entropy $S(\rho)$. Since $S(\rho) = S(\rho^{\mathsf{R}})$ and the reference system R evolves trivially, then

$$I(\mathcal{E}_m,\rho) = S(\rho^{\mathsf{R}}) + S(\mathcal{E}_m(\rho)) - S[(\mathcal{I}\otimes\mathcal{E}_m)(|\Psi\rangle\langle\Psi|)] \tag{89}$$

is the output *quantum mutual information* [2] between the system S and the reference system R. Note that, due to the subadditivity of I [10], no regularization as in (73) is required to obtain C_E.

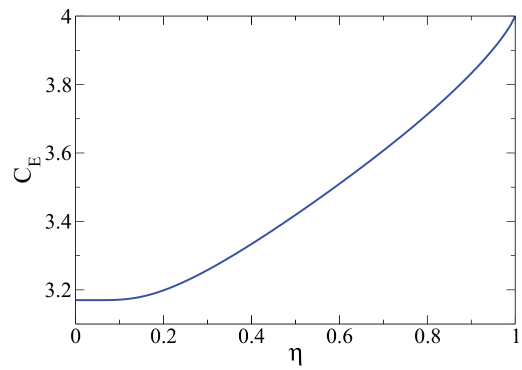

FIG. 8. (Color online) Plot of the entanglement-assisted capacity C_E of $\mathcal{E}_m$ as a function of η. C_E is obtained from the numerical solution of the maximization task (90) (the searching step for α,β,δ is 10^{-4}). For $\eta \to 0$, the entanglement-assisted capacity tends to the value $2\log_2 3$.

A. Maximization of the quantum mutual information $I(\mathcal{E}_m,\rho)$

By following an argument similar to the one exploited in deriving Eq. (83) for the quantum capacity, we obtain

$$\begin{aligned} C_E(\mathcal{E}_m) &= \max_{\overline{\rho}} I(\mathcal{E}_m,\overline{\rho}) \\ &= \max_{\overline{\rho}} \{S(\mathcal{E}_m(\overline{\rho})) + I_c(\mathcal{E}_m,\overline{\rho})\} \\ &= \max_{\alpha,\beta,\delta}\{-\alpha\log_2\alpha - \delta\log_2\delta \\ &\quad -[\alpha+(1-\eta)\delta]\log_2[\alpha+(1-\eta)\delta] \\ &\quad -4\beta\log_2\beta - \eta\delta\log_2\eta\delta \\ &\quad +[1-(1-\eta)\delta]\log_2[1-(1-\eta)\delta] \\ &\quad +(1-\eta)\delta\log_2[(1-\eta)\delta]\}, \end{aligned} \tag{90}$$

where the optimization is over a diagonal input state $\overline{\rho}$ of the form (82) (with the constraint $\alpha + 2\beta + \delta = 1$). We plot the entanglement-assisted classical capacity C_E of the channel $\mathcal{E}_m$ as a result of the maximization problem (90) in Fig. 8 and the relative populations of the optimal ensemble in Fig. 9.

Note that for $\eta = 0$ the entanglement-assisted classical capacity is $2\log_2 3$, as it turns out from the optimization problem (90); see Fig. 8. Indeed, in this case we have at our disposal a noiseless subspace, spanned by $\{|00\rangle,|01\rangle,|10\rangle\}$,

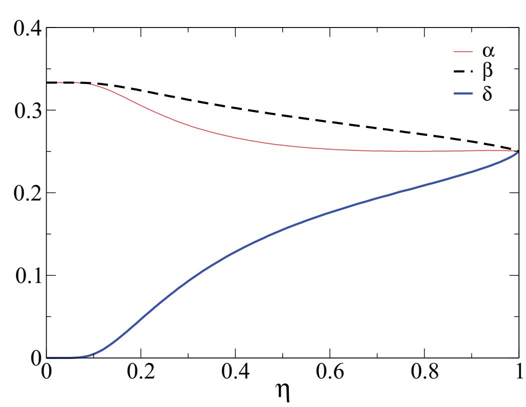

FIG. 9. (Color online) Plot of the coefficients α (thin curve), β (dashed curve), and δ (thick curve) which (numerically) solve the optimization problem (90), as functions of η.

of dimension $d = 3$. This means that we can use a quantum superdense coding protocol (see Ref. [12]) in this subspace, achieving a transmission rate of $2\log_2 d$ bits per channel use.

VI. CONCLUSIONS

In this work we have studied the behavior of a fully correlated amplitude damping channel for two qubits. We assumed that relaxation processes in the two qubits are strongly correlated, namely they only occur simultaneously for the two qubits. We have considered three types of scenarios: the transmission of classical information and of quantum information and the use of the channel in an entanglement-assisted fashion. We have derived the corresponding capacities (limiting to the single-shot capacity in the classical case), analytically studying the related maximization problems and individuating the optimal sources. In the case of classical capacity we also discussed the role of entanglement in achieving the maximum of the Holvevo quantity.

We find that the fully correlated amplitude damping channel is an interesting example of transmission of classical or quantum information through a quantum channel for which a subspace is noiseless. Since the capacity C_1 is obtained without involving any coherent superposition of states from the noiseless and the noisy subspace, it would be interesting to determine whether such result is specific for this model or more general.

A natural extension of our work would be to consider the case of amplitude damping channels with partial memory, i.e., $\mu < 1$ in Eq. (4). While the analytical solution of such model appears difficult, nontrivial bounds on the channel capacities could be computed.

ACKNOWLEDGMENTS

A.D. and G.F. acknowledge support by Grant No. PON02-00355-339123-ENERGETIC, and by EU through Grant No. PITN-GA-2009-234970. G.B. acknowledges the support by MIUR-PRIN project "Collective quantum phenomena: From strongly correlated systems to quantum simulators."

APPENDIX A: OPTIMALITY OF THE ENTANGLED ENSEMBLES FOR CLASSICAL CAPACITY

Let us consider an ensemble $\mathcal{C}_s = \{p_k, |\psi_k\rangle\}$ of separable states

$$\begin{aligned}|\psi_k\rangle &= a_k|00\rangle + b_k|01\rangle + c_k|10\rangle + d_k|11\rangle \\ &= \left(g_k|0\rangle + \sqrt{1-g_k^2}|1\rangle\right) \otimes \left(h_k|0\rangle + \sqrt{1-h_k^2}|1\rangle\right),\end{aligned} \tag{A1}$$

where we can consider $g_k, h_k \in \mathbb{R}$ [since, as shown in (29), phases do not change the eigenvalues of the output state], $g_k, h_k \in [0,1]$, and such that the average density matrix is diagonal,

$$\rho = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{A2}$$

with

$$\begin{aligned}\alpha &= \sum_k p_k a_k^2, \quad \beta = \sum_k p_k b_k^2, \\ \gamma &= \sum_k p_k c_k^2, \quad \delta = \sum_k p_k d_k^2,\end{aligned} \tag{A3}$$

and

$$\begin{aligned}a_k^2 &= g_k^2 h_k^2, \quad b_k^2 = g_k^2\left(1-h_k^2\right), \\ c_k^2 &= \left(1-g_k^2\right)h_k^2, \quad d_k^2 = \left(1-g_k^2\right)\left(1-h_k^2\right).\end{aligned} \tag{A4}$$

We want to demonstrate that for any such ensemble, we can find another ensemble $\mathcal{C}_e$, whose Holevo quantity is strictly greater than $\mathcal{C}_s$, thanks to the presence of entangled states in $\mathcal{C}_e$. We assume $\eta \in]0,1[$, since we know that for the limiting cases $\eta = 0$ and $\eta = 1$, an ensemble of separable state succeeds in achieving C_1.

We start by considering that any such ensemble must have $\alpha,\beta,\gamma,\delta \neq 0$. Indeed, since we are supposing that $\eta > 0$, we know that $C_1 > \log_2 3$ (see Fig. 2); therefore, the entropy of (A2) has to be grater than $\log_2 3$, which is impossible to achieve if even one of the parameters $\alpha,\beta,\gamma,\delta$ vanishes. Next we subdivide $\mathcal{C}_s$ in two distinct subsets, $\mathcal{C}_s = \mathcal{C}_{s_1} \cup \mathcal{C}_{s_2}$: We collect all the states state with ($g_k = 0$, $h_k = 0$) or with ($g_k = 1$, $h_k = 1$) in $\mathcal{C}_{s_2}$, all the others in $\mathcal{C}_{s_1}$.

First we turn our attention to $\mathcal{C}_{s_1}$. We operate a substitution similar to the one we applied at the beginning of Sec. III C. We replace each state $|\psi_{k_{s_1}}\rangle$ and its occurrence probability p_k in this ensemble by

$$p_k, |\psi_{k_{s_1}}\rangle \\ \rightarrow \begin{cases} p_{\phi_k} = \frac{p_k(a_k^2+d_k^2)}{2}, \; |\phi_{k\pm}\rangle = \frac{a_k}{\sqrt{a_k^2+d_k^2}}|00\rangle \pm \frac{d_k}{\sqrt{a_k^2+d_k^2}}|11\rangle, \\ p_{\varphi 0} = p_k b_k^2, \quad |\varphi_{k0}\rangle = |01\rangle, \\ p_{\varphi 1} = p_k c_k^2, \quad |\varphi_{k1}\rangle = |10\rangle. \end{cases} \tag{A5}$$

It is straightforward to see that new ensemble, which we call $\mathcal{C}_{e_1}$, has the same density matrix of $\mathcal{C}_{s_1}$, so it does not change the system output entropy. With regard to the average output entropy we note that only states $|\phi_{k\pm}\rangle$ in $\mathcal{C}_{e_1}$ contribute. The Holevo quantity for ensemble $\mathcal{C}_{e_1}$ is greater than for $\mathcal{C}_{s_1}$, since inequality (55), in which we have to replace $2b_k^2 \to b_k^2 + c_k^2$, holds and is strict. As we have numerically verified, *this is true except that for* $g_k = 1$, $h_k \neq 1$ or $g_k \neq 1$, $h_k = 1$ (by construction we have excluded cases in which $g_k = 1$, $h_k = 1$), that is, for $d_k = 0$.

Now we turn to ensemble $\mathcal{C}_{s_2}$. Its density matrix is given by

$$\rho_{s_2} = \begin{pmatrix} \alpha' & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \delta' \end{pmatrix}, \tag{A6}$$

where

$$\alpha' = \sum_{k\in s_2} p_k a_k^2, \quad \delta' = \sum_{k\in s_2} p_k d_k^2. \tag{A7}$$

We replace the ensemble $\mathcal{C}_{s_2} = \{\overline{p}_{\phi k}, |\overline{\phi}_{k\pm}\rangle\}$ with the following one, which we call $\mathcal{C}_{e_2}$:

$$\overline{p}_{\phi k} = \frac{\alpha' + \delta'}{2}, \quad |\overline{\phi}_{k\pm}\rangle = \sqrt{\frac{\alpha'}{\alpha' + \delta'}}|00\rangle \pm \sqrt{\frac{\delta'}{\alpha' + \delta'}}|11\rangle. \tag{A8}$$

The density matrix of $\mathcal{C}_{e_2}$ is equal to (A6) and therefore the system output entropy does not change. Let us turn to the average output entropy. For the ensemble $\mathcal{C}_{s_2}$ it turns out that

$$\overline{S}_{\text{out},\mathcal{C}_{s_2}} = \sum_{k\in s_2} p_k S(\mathcal{E}_m(|\psi_k\rangle\langle\psi_k|)) = \delta' H_2(\eta), \tag{A9}$$

whereas for the ensemble $\mathcal{C}_{e_2}$ we have

$$\overline{S}_{\text{out},\mathcal{C}_{e_2}} = (\alpha' + \delta')S(\mathcal{E}_m(|\overline{\phi}_{k\pm}\rangle\langle\overline{\phi}_{k\pm}|)) = (\alpha' + \delta')H_2\left\{\frac{1}{2}\left[1 + \sqrt{1 - 4\eta(1-\eta)\left(\frac{\delta'}{\alpha' + \delta'}\right)^2}\right]\right\}. \tag{A10}$$

Therefore, in order to show that by replacing $\mathcal{C}_{s_2}$ with $\mathcal{C}_{e_2}$ we increase the Holevo quantity, we must prove that

$$\delta' H_2(\eta) \geqslant (\alpha' + \delta')H_2 \times \left\{\frac{1}{2}\left[1 + \sqrt{1 - 4\eta(1-\eta)\left(\frac{\delta'}{\alpha' + \delta'}\right)^2}\right]\right\}. \tag{A11}$$

We notice that the equality holds for $\alpha'\delta' = 0$. By dividing both members of (A11) by δ' (assuming $\delta' > 0$), inequality (A11) is equivalent to

$$H_2(\eta) \geqslant x H_2\left\{\tfrac{1}{2}[1 + \sqrt{1 - 4\eta(1-\eta)x^{-2}}]\right\}, \quad \forall\ x \in [1,\infty[. \tag{A12}$$

By numerical results it turns out that this inequality is tight for any $x > 1$, that is, for any $\alpha' > 0$. That, together with the previous assumption $\delta' > 0$ and the fact the α' and δ' are populations, can be summarized as $\alpha'\delta' \neq 0$.

We can now conclude our proof that the ensemble $\mathcal{C}_e = \mathcal{C}_{e_1} \cup \mathcal{C}_{e_2}$ has a Holevo quantity strictly larger than $\mathcal{C}_s$. We observe that the two Holevo quantities can be written as

$$\chi_{\mathcal{C}_s} = S(\rho) - \overline{S}_{\text{out},\mathcal{C}_{s_1}} - \overline{S}_{\text{out},\mathcal{C}_{s_2}},$$
$$\chi_{\mathcal{C}_e} = S(\rho) - \overline{S}_{\text{out},\mathcal{C}_{e_1}} - \overline{S}_{\text{out},\mathcal{C}_{e_2}},$$

since $S_{\text{out},\mathcal{C}_s} = S_{\text{out},\mathcal{C}_e} = S(\rho)$ by construction.

As we must have $\delta \neq 0$, at least one state in $\mathcal{C}_s$ has $d_k \neq 0$; we call this state $|\xi\rangle$. Suppose first $|\xi\rangle$ belongs to the subsets $\mathcal{C}_{s_1}$: We have already proved that $\overline{S}_{\text{out},\mathcal{C}_{e_1}} < \overline{S}_{\text{out},\mathcal{C}_{s_1}}$ [inequality (55)] and therefore $\chi_{\mathcal{C}_e} > \chi_{\mathcal{C}_s}$ (since in any case $\overline{S}_{\text{out},\mathcal{C}_{e_2}} \leqslant \overline{S}_{\text{out},\mathcal{C}_{s_2}}$). We can see that in this case the ensemble $\mathcal{C}_{e_1}$ must contain at least a pair of entangled states: those states $|\phi_{k\pm}\rangle$ (A5) corresponding to $|\xi\rangle$. In fact, $|\xi\rangle$ must have $a_k \neq 0$. Actually, in the case $a_k = 0$ the inequality (55) implies that the ensemble $\mathcal{C}_{s_1}$ has a Holevo quantity smaller than the one of ensemble $\mathcal{C}_{e_1}$; in this case, $\mathcal{C}_{e_1}$ in turn exhibits a Holevo quantity of the form (46), and we know that it does not achieve C_1 (see Fig. 2), so we have to discard this case. If instead state $|\xi\rangle$ belongs to subset $\mathcal{C}_{s_2}$, we have to consider two further possibilities. (1) $\alpha' \neq 0$: Inequality (A11) is tight and therefore $\overline{S}_{\text{out},\mathcal{C}_{e_2}} < \overline{S}_{\text{out},\mathcal{C}_{s_2}}$, which implies that $\chi_{\mathcal{C}_e} > \chi_{\mathcal{C}_s}$ (since in any case $\overline{S}_{\text{out},\mathcal{C}_{e_1}} \leqslant \overline{S}_{\text{out},\mathcal{C}_{s_1}}$). We stress that in this case the states in $\mathcal{C}_{e_2}$ are entangled. (2) $\alpha' = 0$: It is simple to verify that $\mathcal{C}_s$ exhibits a Holevo quantity which is equal to the one of ensemble $\mathcal{A}$ [see Eq. (46)], and $\chi_{\mathcal{C}_s}$ is strictly less than C_1, as one can see from Fig. 2, so we can discard this case.

APPENDIX B: DEGRADABILITY OF $\mathcal{E}_m$

We consider a unitary representation of the channel $\mathcal{E}_m$,

$$\begin{aligned}
|00\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} &\longrightarrow |00\rangle^{S} \otimes |00\rangle^{\mathsf{E}},\\
|01\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} &\longrightarrow |01\rangle^{S} \otimes |00\rangle^{\mathsf{E}},\\
|10\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} &\longrightarrow |10\rangle^{S} \otimes |00\rangle^{\mathsf{E}},\\
|11\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} &\longrightarrow \sqrt{\eta}|11\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}}\\
&\qquad + \sqrt{1-\eta}|00\rangle^{\mathsf{S}} \otimes |11\rangle^{\mathsf{E}},
\end{aligned} \tag{B1}$$

where E represents a fictitious environment. When the system S is prepared in the generic pure state (22), system SE state undergoes the transformation

$$\begin{aligned}
|\psi^{\mathsf{SE}}\rangle &= |\psi_k\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} \longrightarrow\\
|\psi^{\mathsf{SE}'}\rangle &= a_k|00\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}}\\
&\quad + b_k|01\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}} + c_k|10\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}}\\
&\quad + d_k(\sqrt{\eta}|11\rangle^{\mathsf{S}} \otimes |00\rangle^{\mathsf{E}}\\
&\quad + \sqrt{1-\eta}|00\rangle^{\mathsf{S}} \otimes |11\rangle^{\mathsf{E}}).
\end{aligned} \tag{B2}$$

From (B2) we can calculate the reduced density matrix for the systems S and E; $\rho' = \text{Tr}_{\mathsf{E}}|\psi^{\mathsf{SE}'}\rangle\langle\psi^{\mathsf{SE}'}|$ is just the output state (28), whereas the reduced density matrix for the environment E is

$$\rho^{\mathsf{E}'} = \text{Tr}_{\mathsf{S}}|\psi^{\mathsf{SE}'}\rangle\langle\psi^{\mathsf{SE}'}| = \begin{pmatrix} 1 - |d_k|^2(1-\eta) & 0 & 0 & \sqrt{1-\eta}\, a_k d_k^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\eta}\, a_k^* d_k & 0 & 0 & (1-\eta)|d_k|^2 \end{pmatrix}. \tag{B3}$$

As we show in the following, it is possible to deduce $\rho_{E'}$ starting from ρ' by applying to ρ' a quantum operation and subsequently the channel $\mathcal{E}_m$ in which we have to replace the parameter η with $(1-\eta)/\eta$. This implies that the channel $\mathcal{E}_m$ is *degradable* [45] for $\eta \in [\frac{1}{2},1]$.

In order to prove this we consider a generic input state $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$; see Eq. (22). the corresponding output state is give by

$$\rho' = \begin{pmatrix} \alpha + (1-\eta)\delta & \kappa & \lambda & \sqrt{\eta}\,\varsigma \\ \kappa^* & \beta & \nu & \sqrt{\eta}\,o \\ \lambda^* & \nu^* & \gamma & \sqrt{\eta}\,\pi \\ \sqrt{\eta}\,\varsigma^* & \sqrt{\eta}\,o^* & \sqrt{\eta}\,\pi^* & \eta\delta \end{pmatrix}, \tag{B4}$$

and

$$\rho^{\mathsf{E}'} = \begin{pmatrix} 1-\delta(1-\eta) & 0 & 0 & \sqrt{1-\eta}\,\varsigma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{1-\eta}\,\varsigma^* & 0 & 0 & (1-\eta)\delta \end{pmatrix}, \tag{B5}$$

where

$$\begin{aligned} \alpha &= \sum_k |a_k|^2, \quad \beta = \sum_k |b_k|^2, \\ \gamma &= \sum_k |c_k|^2, \quad \delta = \sum_k |d_k|^2, \\ \kappa &= \sum_k p_k a_k b_k^*, \quad \lambda = \sum_k p_k a_k c_k^*, \quad \varsigma = \sum_k p_k a_k d_k^*, \\ \nu &= \sum_k p_k b_k c_k^*, \quad o = \sum_k p_k b_k d_k^*, \quad \pi = \sum_k p_k c_k d_k^*, \end{aligned} \tag{B6}$$

and moreover we set

$$\begin{aligned} &\alpha' = \alpha + (1-\eta)\delta, \quad \delta' = \eta\,\delta, \\ &\varsigma' = \sqrt{\eta}\,\varsigma, \quad o' = \sqrt{\eta}\,o, \quad \pi' = \sqrt{\eta}\,\pi. \end{aligned} \tag{B7}$$

To show that the channel $\mathcal{E}_m$ is degradable, we propose the following scheme. We add three ancillary qubits to the system S described by the state ρ' (B4); we call the ancillas A_1 and A_{23} (we collect together the second and the third ancillary qubits). Initially, the ancillas are all prepared in the state $|0\rangle$. We first apply two controlled-NOT gates, where the qubits S act as control qubits and the qubit A_1 as the target qubit. We then perform a SWAPbetween S and A_{23}, controlled by the state of the ancilla A_1. This procedure is reported below:

$$\begin{aligned} &\text{initial state} && \longrightarrow && \text{controlled-NOTs} \\ &|00^{\mathsf{S}}\rangle \otimes |0^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle && && \text{no changes} \\ &|01^{\mathsf{S}}\rangle \otimes |0^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle && && |01^{\mathsf{S}}\rangle \otimes |1^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle \\ &|10^{\mathsf{S}}\rangle \otimes |0^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle && && |10^{\mathsf{S}}\rangle \otimes |1^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle \\ &|11^{\mathsf{S}}\rangle \otimes |0^{\mathsf{A}_1}\rangle \otimes |00^{\mathsf{A}_{23}}\rangle && && \text{no changes} \\ &\longrightarrow \quad \text{controlled-SWAP} \\ &\text{no changes} \\ &|00^{\mathsf{S}}\rangle \otimes |1^{\mathsf{A}_1}\rangle \otimes |01^{\mathsf{A}_{23}}\rangle \\ &|00^{\mathsf{S}}\rangle \otimes |1^{\mathsf{A}_1}\rangle \otimes |10^{\mathsf{A}_{23}}\rangle \\ &\text{no changes.} \end{aligned} \tag{B8}$$

Exploiting the linearity of quantum operations we can transform each element of $\rho^{S'}$ as

$$\begin{aligned} \alpha'\,|00\rangle\langle 00| &\longrightarrow \alpha'\,|00\rangle\langle 00| \otimes |0^{\mathsf{A}_1}\rangle\langle 0^{\mathsf{A}_1}| \otimes |00^{\mathsf{A}_{23}}\rangle\langle 00^{\mathsf{A}_{23}}|, \\ \kappa\,|00\rangle\langle 01| &\longrightarrow \kappa\,|00\rangle\langle 00| \otimes |0^{\mathsf{A}_1}\rangle\langle 1^{\mathsf{A}_1}| \otimes |00^{\mathsf{A}_{23}}\rangle\langle 01^{\mathsf{A}_{23}}|, \\ \lambda\,|00\rangle\langle 10| &\longrightarrow \lambda\,|00\rangle\langle 00| \otimes |0^{\mathsf{A}_1}\rangle\langle 1^{\mathsf{A}_1}| \otimes |00^{\mathsf{A}_{23}}\rangle\langle 10^{\mathsf{A}_{23}}|, \\ \varsigma'\,|00\rangle\langle 11| &\longrightarrow \varsigma'\,|00\rangle\langle 11| \otimes |0^{\mathsf{A}_1}\rangle\langle 0^{\mathsf{A}_1}| \otimes |00^{\mathsf{A}_{23}}\rangle\langle 00^{\mathsf{A}_{23}}|, \\ \beta\,|01\rangle\langle 01| &\longrightarrow \beta\,|00\rangle\langle 00| \otimes |1^{\mathsf{A}_1}\rangle\langle 1^{\mathsf{A}_1}| \otimes |01^{\mathsf{A}_{23}}\rangle\langle 01^{\mathsf{A}_{23}}|, \\ \nu\,|01\rangle\langle 10| &\longrightarrow \nu\,|00\rangle\langle 00| \otimes |1^{\mathsf{A}_1}\rangle\langle 1^{\mathsf{A}_1}| \otimes |01^{\mathsf{A}_{23}}\rangle\langle 10^{\mathsf{A}_{23}}|, \\ o'\,|01\rangle\langle 11| &\longrightarrow o'\,|00\rangle\langle 11| \otimes |1^{\mathsf{A}_1}\rangle\langle 0^{\mathsf{A}_1}| \otimes |01^{\mathsf{A}_{23}}\rangle\langle 00^{\mathsf{A}_{23}}|, \\ \gamma\,|10\rangle\langle 10| &\longrightarrow \gamma\,|00\rangle\langle 00| \otimes |1^{\mathsf{A}_1}\rangle\langle 1^{\mathsf{A}_1}| \otimes |10^{\mathsf{A}_{23}}\rangle\langle 10^{\mathsf{A}_{23}}|, \\ \pi'\,|10\rangle\langle 11| &\longrightarrow \pi'\,|00\rangle\langle 11| \otimes |1^{\mathsf{A}_1}\rangle\langle 0^{\mathsf{A}_1}| \otimes |10^{\mathsf{A}_{23}}\rangle\langle 00^{\mathsf{A}_{23}}|, \\ \delta'\,|11\rangle\langle 11| &\longrightarrow \delta'\,|11\rangle\langle 11| \otimes |0^{\mathsf{A}_1}\rangle\langle 0^{\mathsf{A}_1}| \otimes |00^{\mathsf{A}_{23}}\rangle\langle 00^{\mathsf{A}_{23}}|. \end{aligned}$$

After the quantum operations (B8), tracing with respect to the ancillas we obtain

$$\begin{aligned} \rho'' &= \alpha'\,|00^{\mathsf{S}}\rangle\langle 00^{\mathsf{S}}| + \varsigma'\,|00^{\mathsf{S}}\rangle\langle 11^{\mathsf{S}}| + \varsigma'^*\,|11^{\mathsf{S}}\rangle\langle 00^{\mathsf{S}}| \\ &\quad + \beta\,|00^{\mathsf{S}}\rangle\langle 00^{\mathsf{S}}| + \gamma\,|00^{\mathsf{S}}\rangle\langle 00^{\mathsf{S}}| + \delta'\,|11^{\mathsf{S}}\rangle\langle 11^{\mathsf{S}}| \\ &= \begin{pmatrix} \alpha'+\beta+\gamma & 0 & 0 & \varsigma' \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varsigma'^* & 0 & 0 & \delta' \end{pmatrix} \\ &= \begin{pmatrix} 1-\eta\,\delta & 0 & 0 & \sqrt{\eta}\,\varsigma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\eta}\varsigma^* & 0 & 0 & \eta\,\delta \end{pmatrix}, \end{aligned} \tag{B9}$$

where we have used (B7) together with $\alpha+\beta+\gamma+\delta=1$. It is simple to see that one can obtain the state (B3) by applying the channel $\mathcal{E}_m$ to the state (B9), but replacing η with $(1-\eta)/\eta$. Indeed, we have

$$\begin{aligned} \eta\delta &\longrightarrow \eta\delta \cdot \frac{1-\eta}{\eta} = (1-\eta)\delta, \\ \sqrt{\eta}\varsigma &\longrightarrow \sqrt{\eta}\varsigma \cdot \sqrt{\frac{1-\eta}{\eta}} = \sqrt{1-\eta}\,\mu, \\ 1-\eta\delta &\longrightarrow 1-\eta\delta + \left(1-\frac{1-\eta}{\eta}\right)\cdot\eta\delta = 1-(1-\eta)\delta. \end{aligned}$$

It must, of course, happen that $0 \leqslant \frac{1-\eta}{\eta} \leqslant 1$, which means $\frac{1}{2} \leqslant \eta \leqslant 1$. We can therefore conclude that, when the transmissivity η is in the interval $[\frac{1}{2},1]$, the channel $\mathcal{E}_m$ is degradable.

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