Quantum channel detection

C. Macchiavello and M. Rossi
Dipartimento di Fisica and INFN-Sezione di Pavia, via Bassi 6, 27100 Pavia, Italy
(Received 3 September 2012; published 30 October 2013)

We present a method to detect properties of quantum channels, assuming that some *a priori* information about the form of the channel is available. The method is based on a correspondence with entanglement detection methods for multipartite density matrices based on witness operators. We first illustrate the method in the case of entanglement-breaking channels and nonseparable random unitary channels and show how it can be implemented experimentally by means of local measurements. We then study the detection of nonseparable maps and show that for pairs of systems of dimension higher than two the detection operators are not the same as in the random unitary case, highlighting a richer separability structure of quantum channels with respect to quantum states. Finally we consider the set of PPT maps, developing a technique to reveal NPT maps.

DOI: 10.1103/PhysRevA.88.042335 PACS number(s): 03.67.Hk, 03.65.Aa, 03.65.Ud, 03.67.Mn

I. INTRODUCTION

The possibility of determining properties of quantum communication channels or quantum devices is of great importance in order to be able to design and operate the channel at the best of its performances. In many realistic implementations some *a priori* information on the form of a quantum channel, or a quantum noise process, is available and it is of great interest to determine experimentally whether the channel has a certain property. The aim of this work is to propose efficient methods to detect this possibility by avoiding full quantum process tomography, which allows a complete reconstruction of the channel, but it requires a large number of measurement settings. At the same time, from the point of view of implementations, our procedure is experimentally feasible with present-day technology based on local measurements.

This work is organized as follows. In Sec. II we explain our main idea, treating as an introductory example entanglement-breaking channels. In Secs. III and IV we study the cases of separable random unitaries and separable maps, respectively. We develop a method to detect NPT channels in Sec. V and we summarize the main results in Sec. VI.

II. MAIN IDEA AND ENTANGLEMENT-BREAKING CHANNELS

In this section we show the main idea of the proposed quantum channel detection method and its link to entanglement detection methods for multipartite quantum systems. To this aim we remind the reader that quantum channels, and in general quantum noise processes, are described by completely positive (CP) and trace preserving (TP) maps $\mathscr{M}$, which can be expressed in the Kraus form [1] as

$$\mathscr{M}[\rho] = \sum_k A_k \rho A_k^\dagger, \tag{1}$$

where ρ is the density operator of the quantum system on which the channel acts and the Kraus operators $\{A_k\}$ fulfill the TP constraint $\sum_k A_k^\dagger A_k = \mathbf{1}$.

The detection method proposed is based on the use of the Choi-Jamiolkowski isomorphism [2], which gives a one-to-one correspondence between CP-TP maps acting on $\mathcal{D}(\mathcal{H})$ (the set of density operators on $\mathcal{H}$, with arbitrary finite dimension d) and bipartite density operators $C_{\mathscr{M}}$ on $\mathcal{H}\otimes\mathcal{H}$ with $\mathrm{Tr}_A[C_{\mathscr{M}}] = \mathbf{1}_B/d$. This isomorphism can be described as

$$\mathscr{M} \Longleftrightarrow C_{\mathscr{M}} = (\mathscr{M}\otimes\mathscr{I})[|\alpha\rangle\langle\alpha|], \tag{2}$$

where $\mathscr{I}$ is the identity map, and $|\alpha\rangle$ is the maximally entangled state with respect to the bipartite space $\mathcal{H}\otimes\mathcal{H}$, i.e., $|\alpha\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k\rangle|k\rangle$. This is schematically depicted in Fig. 1.

In this work, by the above isomorphism, we link some specific properties of quantum channels to properties of the corresponding Choi states $C_{\mathscr{M}}$. We consider properties that are based on a convex structure of the quantum channels.

Consider as a first simple case the class of entanglement-breaking (EB) channels [3]. A possible definition for an EB channel is based on the separability of its Choi state: A quantum channel is EB if and only if its Choi state is separable. This makes it possible to formulate a method to detect whether a quantum channel is not EB by exploiting entanglement detection methods designed for bipartite systems [4]. To this end, we remind the reader about the concept of entanglement detection via witness operators [5]: A state ρ is entangled if and only if there exists a Hermitian operator W such that $\mathrm{Tr}[W\rho] < 0$ and $\mathrm{Tr}[W\rho_{\mathrm{sep}}] \geqslant 0$ for all separable states.

As a simple example of quantum channel detection, consider the case of qubits and the single-qubit depolarizing channel, defined as

$$\Gamma_{\{p\}}[\rho] = \sum_{i=0}^{3} p_i\sigma_i\rho\sigma_i, \tag{3}$$

where σ_0 is the identity operator, $\{\sigma_i\}$ $(i = 1,2,3)$ are the three Pauli operators $\sigma_x,\sigma_y,\sigma_z$, respectively (for brevity of notation in the following, the Pauli operators are denoted by X, Y, and Z), and $p_0 = 1 - p$ (with $p \in [0,1]$), while $p_i = p/3$ for $i = 1,2,3$. Such a channel is EB for $p \geqslant 1/2$. The corresponding set of Choi bipartite density operators is given by the Werner states,

$$\rho_p = \left(1 - \frac{4}{3}p\right)|\alpha\rangle\langle\alpha| + \frac{p}{3}\mathbf{1}. \tag{4}$$

It is then possible to detect whether a depolarizing channel is not EB by exploiting an entanglement witness operator for the

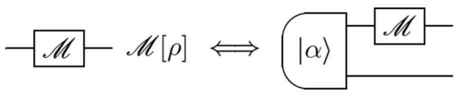

FIG. 1. Scheme showing the Choi-Jamiolkowski isomorphism. On the left is the map $\mathscr{M}$; on the right is the corresponding Choi state $C_{\mathscr{M}}$.

above set of states [4,6], which has the form

$$W_{\rm EB} = \tfrac{1}{4}(\mathbf{1}\otimes\mathbf{1} - X\otimes X + Y\otimes Y - Z\otimes Z). \qquad (5)$$

The method can then be implemented by preparing a two-qubit state in the maximally entangled state $|\alpha\rangle$, then operating with the quantum channel to be detected on one of the two qubits and measuring the operator W_{EB} acting on both qubits at the end. If the resulting average value is negative, we can then conclude that the channel under consideration is not EB.

We now prove that our method provides also a lower bound on a particular feature of EB channels recently defined in Ref. [7] as follows. Let $\mathscr{M}$ be a generic map acting on a d-dimensional system and $\mathscr{D}_\sigma$ the completely depolarizing channel defined as $\mathscr{D}_\sigma[\rho] = \sigma$, where σ is an arbitrary state. The quantity $\mu_c(\mathscr{M})$ is defined as the minimum value of the mixing probability parameter $\mu \in [0,1]$ that transforms the convex combination $(1-\mu)\mathscr{M} + \mu\mathscr{D}_\sigma$ into an EB channel, i.e., in formulas

$$\mu_c(\mathscr{M}) = \min_\sigma\{\mu|(1-\mu)\mathscr{M} + \mu\mathscr{D}_\sigma \in \text{EB}\}. \qquad (6)$$

By the Choi-Jamiolkowski isomorphism, we can rephrase the definition (6) in term of Choi states as

$$\mu_c(\mathscr{M}) = \min_\sigma\left\{\mu|(1-\mu)C_{\mathscr{M}} + \mu\sigma\otimes\frac{\mathbf{1}}{d} \in \text{Sep}\right\} \qquad (7)$$

and link this quantity to the well-known generalized robustness of entanglement. Given a state ρ, the generalized robustness of entanglement is defined [8,9] as the minimal $s > 0$ such that the state $\frac{\rho+s\sigma}{1+s}$ is separable, where σ is an arbitrary state (not necessarily separable), namely,

$$R(\rho) = \min_\sigma\left\{s|\frac{\rho+s\sigma}{1+s} \in \text{Sep}\right\}. \qquad (8)$$

This quantity can be interpreted as the minimum amount of noise necessary to wash out completely the entanglement initially present in the state ρ. Thus, by defining $p_c(\rho) = 1 - \frac{1}{1+R(\rho)}$ and interpreting ρ as the Choi state $C_{\mathscr{M}}$ corresponding to the map $\mathscr{M}$, we can bound $\mu_c(\mathscr{M})$ as

$$\mu_c(\mathscr{M}) \geqslant p_c(C_{\mathscr{M}}), \qquad (9)$$

since the minimizing set involved in the definition (7) of $\mu_c(\mathscr{M})$ is smaller than the minimizing set considered for $R(C_{\mathscr{M}})$, as can be seen in Eq. (8). By the above inequality we can derive a bound for the generalized robustness from the experimental data of an entanglement detection procedure [10] as

$$R(\rho) \geqslant |c|/w_{\max}, \qquad (10)$$

where c is measured experimentally via the expectation value of the witness, i.e., $\text{Tr}[W\rho] = c < 0$, while $w_{\max}$ is the maximal eigenvalue of the operator W. As a result, we can find that

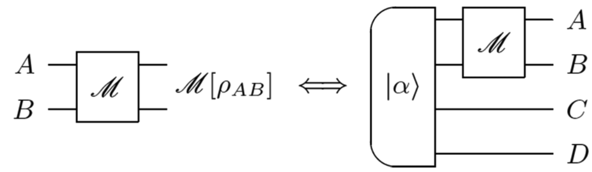

FIG. 2. Scheme of the Choi-Jamiolkowski isomorphism in the case of four-partite states. The state $|\alpha\rangle$ on the right is the maximally entangled state with respect to the bipartition AB-CD.

$$\mu_c(\mathscr{M}) \geqslant 1 - \frac{1}{1+|c|/w_{\max}}, \qquad (11)$$

which links the expectation value of the witness measured experimentally to the theoretical quantity $\mu_c(\mathscr{M})$. In the case of the depolarizing channel (3) with $p < 1/2$, by using the witness $W_{\rm EB}$ given by Eq. (5), the above bound takes the form

$$\mu_c(\Gamma_{\{p\}}) \geqslant \frac{1-2p}{2-2p}. \qquad (12)$$

In this case, however, the bound is not tight since the theoretical $\mu_c(\Gamma_{\{p\}})$ can be computed to be $\frac{2-4p}{3-4p}$ by following the method developed in [7].

III. SEPARABLE RANDOM UNITARIES

We now consider the case of random unitary (RU) channels, defined as

$$\mathscr{U}[\rho] = \sum_k p_k U_k \rho U_k^\dagger, \qquad (13)$$

where U_k are unitary operators and $p_k > 0$ with $\sum_k p_k = 1$. Notice that this kind of map includes several interesting models of quantum noisy channels, such as the already mentioned depolarizing channel or the phase-damping channel and the bit-flip channel [11]. RUs were also studied extensively and characterized in Ref. [12].

We now consider the case where the RU channel acts on a bipartite system $\rho_{\rm AB}$ as follows

$$\mathscr{V}[\rho_{\rm AB}] = \sum_k p_k(V_{k,\rm A}\otimes W_{k,\rm B})\rho_{\rm AB}(V_{k,\rm A}^\dagger \otimes W_{k,\rm B}^\dagger), \qquad (14)$$

where both $V_{k,A}$ and $W_{k,B}$ are unitary operators for all k's, acting on systems A and B respectively. Quantum channels of the above form are named separable random unitaries (SRUs) and they form a convex subset in the set of all CP-TP maps acting on bipartite systems. Interesting examples of channels of this form are given by Pauli memory channels [13].

The Choi state corresponding to quantum channels acting on bipartite systems is a four-partite state (composed of systems A, B, C, and D), as shown in Fig. 2. Notice that the state $|\alpha\rangle = \frac{1}{\sqrt{d_{\rm AB}}}\sum_{k,j=1}^{d_{\rm AB}} |k,j\rangle_{\rm AB}|k,j\rangle_{\rm CD}$ (where $d_{\rm AB} = d_{\rm A}d_{\rm B}$ is now the dimension of the Hilbert space of the bipartite system AB) can also be written as $|\alpha\rangle = |\alpha\rangle_{\rm AC}|\alpha\rangle_{\rm BD}$; namely, it is a biseparable state for the partition AC-BD of the global four-partite system. The Choi states corresponding to SRU channels therefore form a convex set, which is a subset of all biseparable states for the partition AC-BD. Since the generating set of SRUs is given by local unitaries $U_A \otimes U_B$, the generating biseparable pure states in the corresponding set of Choi states have the form

$$|U_A \otimes U_B\rangle = (U_A \otimes \mathbf{1}_C)\,|\alpha\rangle_{AC} \otimes (U_B \otimes \mathbf{1}_D)\,|\alpha\rangle_{BD}\,. \quad (15)$$

We name the set of four-partite Choi states corresponding to SRUs as $S_{\rm SRU}$. It is now possible to design detection procedures for SRU maps by employing suitable witness operators that detect the corresponding Choi state with respect to biseparable states (in AC-BD) belonging to $S_{\rm SRU}$.

We now focus on the case of a unitary transformation U acting on two d-dimensional systems. The corresponding Choi state is pure and has the form

$$|U\rangle = (U \otimes \mathbf{1})\,|\alpha\rangle\,. \quad (16)$$

Therefore, a suitable detection operator for U as a non-SRU gate can be constructed as

$$W_{{\rm SRU},U} = \alpha_{\rm SRU}^2\,\mathbf{1} - C_U, \quad (17)$$

where $C_U = |U\rangle\langle U|$, and the coefficient $\alpha_{\rm SRU}$ is the overlap between the closest biseparable state in the set $S_{\rm SRU}$ and the entangled state $|U\rangle$, namely,

$$\alpha_{\rm SRU}^2 = \max_{\mathscr{M}_{\rm SRU}} \langle U|\, C_{\mathscr{M}_{\rm SRU}}\, |U\rangle\,. \quad (18)$$

Notice that, since the maximum of a linear function over a convex set is always achieved on the extremal points, the maximum above can be always calculated by maximizing over the pure biseparable states (15) [14], i.e.,

$$\alpha_{\rm SRU} = \max_{U_A,U_B} |\langle U_A \otimes U_B|\, U\rangle| = \frac{1}{d^2}\max_{U_A,U_B} |\,{\rm Tr}[(U_A^\dagger \otimes U_B^\dagger)U]|. \quad (19)$$

As an example of the above procedure, consider the (Controlled-NOT) CNOT gate acting on a two-qubit system, defined by

$$\text{CNOT} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & X \end{pmatrix}, \quad (20)$$

with $\mathbf{1}$ representing the 2×2 identity matrix and X the usual Pauli operator. The coefficient $\alpha_{\rm SRU}$ for $U = \text{CNOT}$ can be computed as follows. The state (16) specialized for the CNOT gate is clearly not separable with respect to the split AC-BD and it can be expressed in the Schmidt decomposition regarding that split as

$$|\text{CNOT}\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AC}|\alpha\rangle_{BD} + |11\rangle_{AC}|\psi^+\rangle_{BD})\,, \quad (21)$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The above expression naturally proves that the maximum overlap with any biseparable state with respect to AC-BD cannot exceed the value of $1/\sqrt{2}$. Since the convex set $S_{\rm SRU}$ of allowed states in our optimization problem is smaller than the set of all biseparable states, this would give us only an upper bound for the maximum overlap $\alpha_{\rm SRU}$. However, two local unitary operations U_A and U_B that saturate this bound can be explicitly found, namely $U_A = S$ and $U_B = e^{-i\frac{\pi}{4}X}$, where S is the phase gate given by $S = {\rm diag}(1,i)$. This finally proves that the optimal coefficient $\alpha_{\rm SRU}$ equals $1/\sqrt{2}$ even if we restrict to the set of biseparable states $S_{\rm SRU}$. Moreover, the detection operator $W_{\rm CNOT} = \frac{1}{2}\mathbf{1} - C_{\rm CNOT}$ can be decomposed into a linear combination of local operators as

$$\begin{aligned} W_{\rm CNOT} = \frac{1}{64}(&31\,\mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1} - \mathbf{1}X\mathbf{1}X - XXX\mathbf{1} - X\mathbf{1}XX \\ &- ZZ\mathbf{1}Z + ZY\mathbf{1}Y + YYXZ + YZXY \\ &- Z\mathbf{1}Z\mathbf{1} - ZXZX + YXY\mathbf{1} + Y\mathbf{1}YX \\ &- \mathbf{1}ZZZ + \mathbf{1}YZY + XYYZ + XZYY), \end{aligned} \quad (22)$$

where for simplicity of notation the tensor product symbol has been omitted. As we can see from the above form, the CNOT can be detected by using nine different local measurements settings, namely $\{XXXX, ZZZZ, ZYZY, YXYX, YYXZ, YZXY, ZXZX, XYYZ, XZYY\}$. Actually, in the first line of the above expression the expectation values of operators $\mathbf{1}X\mathbf{1}X, XXX\mathbf{1}, X\mathbf{1}XX$ can be obtained by measuring the operator $XXXX$ and suitably processing the experimental data. Similar groupings can be done for the other terms in (22), such that the only measurement settings needed are the nine listed above. Following [4,15], it can be also easily proved that the above form is optimal in the sense that it involves the smallest number of measurement settings. From an experimental point of view, the optimal detection procedure can be implemented as follows: Prepare a four-partite qubit system in the state $|\alpha\rangle = |\alpha\rangle_{AC}|\alpha\rangle_{BD}$, apply the quantum channel to qubits A and B, and finally perform the set of nine local measurements reported above in order to measure the operator (22). If the resulting average value is negative, then the quantum channel is detected as a non-SRU map. The experimental scheme is shown in Fig. 3.

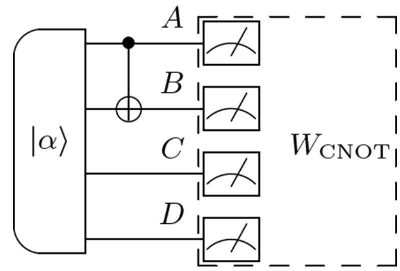

FIG. 3. Experimental scheme implementing the detection of the CNOT gate.

Notice that the number of measurements needed in this procedure is much smaller than the one required for complete quantum process tomography, since the former scales as d_{AB}^2 [6] while the latter as d_{AB}^4 [11].

The number of measurement settings in the detection scheme can be further decreased if we allow a nonoptimal detection operator, in the sense that the coefficient $\alpha_{\rm SRU}$ in $W_{\rm CNOT}$ is smaller than the maximum value. In this case, since the state $C_{\rm CNOT}$ is a stabilizer state with generators $\{XXX\mathbf{1}, \mathbf{1}X\mathbf{1}X, Z\mathbf{1}Z\mathbf{1}, ZZ\mathbf{1}Z\}$, an alternative detection operator can be derived, following the approach of Ref. [16].

The resulting suboptimal detection operator turns out to be

$$\tilde{W}_{\text{CNOT}} = 3\,\mathbf{1} - 2\left[\frac{(\mathbf{1}+XXX\,\mathbf{1})}{2}\frac{(\mathbf{1}+\mathbf{1}\,X\,\mathbf{1}\,X)}{2} + \frac{(\mathbf{1}+Z\,\mathbf{1}\,Z\,\mathbf{1})}{2}\frac{(\mathbf{1}+ZZ\,\mathbf{1}\,Z)}{2}\right], \quad (23)$$

which requires only the two local measurement settings $\{XXXX, ZZZZ\}$. The robustness of the method in the detection of the CNOT gate was analyzed in [17].

IV. SEPARABLE MAPS

We now focus on the detection of nonseparable maps. By definition, a separable map $\mathscr{M}_{\text{sep}}$ is given by

$$\mathscr{M}_{\text{sep}}[\rho_{\text{AB}}] = \sum_k (A_k \otimes B_k)\rho_{\text{AB}}(A_k^\dagger \otimes B_k^\dagger); \quad (24)$$

namely, it can be written in terms of separable Kraus operators [18]. Here we do not require the TP condition. Notice that the set of separable maps is a larger set than the set of SRUs studied above. A general map $\mathscr{M}$ acting on two qudits is not separable if and only if the corresponding Choi state $C_{\mathscr{M}}$ is entangled with respect to the splitting AC-BD [19].

Analogously to the case of SRU maps, for a unitary transformation U we can define a witness operator of the same form (17), where now the coefficient α^2_{SRU} is replaced with α_S^2 defined as

$$\alpha_S^2 = \max_{\mathscr{M}_{\text{sep}}} \langle U|C_{\mathscr{M}_{\text{sep}}}|U\rangle. \quad (25)$$

Since the set of SRUs is a subset of all separable maps, in general, $\alpha_S \geqslant \alpha_{\text{SRU}}$. The maximum in Eq. (25) is attained on pure states, which are the extremal points in the set of $C_{\mathscr{M}_{\text{sep}}}$. Since a map $\mathscr{M}$ is described by a single Kraus operator if and only if its Choi state $C_{\mathscr{M}}$ is pure [20], we can then compute the maximum on separable maps $\mathscr{M}_{\text{sep}}$ with a single Kraus operator. The calculation for α_S can then be simplified as

$$\alpha_S = \max_{A,B} |\langle A \otimes B|U\rangle| = \frac{1}{d^2}\max_{A,B} |\operatorname{Tr}[(A^\dagger \otimes B^\dagger)U]|. \quad (26)$$

Notice that now we do not require $A \otimes B$ to be TP; otherwise, both A and B would be automatically unitary. Interestingly, we now show that for a general unitary U on two-qubit systems the two coefficients α_{SRU} and α_S coincide, while for higher dimension this no longer holds.

We compute the coefficients by starting from the Schmidt decomposition of an operator O acting on two qudits, which can be written as

$$O = \sum_{i=1}^r \sigma_i A_i \otimes B_i, \quad (27)$$

where $\{A_i\}_{i=1,\dots,d^2}$ and $\{B_i\}_{i=1,\dots,d^2}$ are two orthogonal bases ($\operatorname{Tr}[A_i^\dagger A_j] = \operatorname{Tr}[B_i^\dagger B_j] = d\delta_{ij}$) for the operator space, and r is the Schmidt rank fulfilling $1 \leqslant r \leqslant d^2$. Notice that the unique Schmidt coefficients σ_i are always positive and ordered, i.e., $\sigma_1 \geqslant \cdots \geqslant \sigma_r$. As a result, if we write the unitary U in the Schmidt decomposition (27), it follows that the maximum (26) is achieved by the choice of $A \otimes B = A_1 \otimes B_1$, where A_1 and B_1 are the operators corresponding to the largest Schmidt coefficient σ_1. We then have

$$\alpha_S = \frac{1}{d^2}|\operatorname{Tr}[(A_1^\dagger \otimes B_1^\dagger)U]| = \sigma_1. \quad (28)$$

It is then interesting to establish whether the optimal separable operator $A_1 \otimes B_1$ has to be unitary as well. As mentioned above, we show that this is true for qubit systems but does no longer hold when the dimension increases. We first show that for two qubits it is always possible to find a separable unitary $U_{\text{A}} \otimes U_{\text{B}}$ such that the overlap with U achieves the maximum σ_1, namely,

$$\exists U_{\text{A}}, U_{\text{B}} \text{ such that } |\langle U_{\text{A}} \otimes U_{\text{B}}|U\rangle| = \alpha_{\text{SRU}} = \sigma_1. \quad (29)$$

This is a consequence of the Cartan decomposition [21,22] of a general unitary U acting on two qubits, given by

$$U = (V_{\text{A}} \otimes V_{\text{B}})\tilde{U}(W_{\text{A}} \otimes W_{\text{B}}), \quad (30)$$

where V_{A}, V_{B}, W_{A}, and W_{B} are single-qubit unitaries and

$$\tilde{U} = e^{i(\theta_x X\otimes X + \theta_y Y\otimes Y + \theta_z Z\otimes Z)}. \quad (31)$$

Notice that, by the definitions $c_\alpha = \cos\theta_\alpha$ and $s_\alpha = \sin\theta_\alpha$, $\tilde{U}$ takes the form

$$\tilde{U} = (c_x c_y c_z + i s_x s_y s_z)\,\mathbf{1}\otimes\mathbf{1} + (c_x s_y s_z + i s_x c_y c_z) X\otimes X + (s_x c_y s_z + i c_x s_y c_z) Y \otimes Y + (s_x s_y c_z + i c_x c_y s_z) Z \otimes Z. \quad (32)$$

According to (30), it is then straightforward to see that the above form of $\tilde{U}$ leads directly to the Schmidt decomposition of U. Actually, the magnitudes of the coefficients in front of the bipartite operators correspond to the Schmidt coefficients themselves and the phases can be reabsorbed into the Pauli operators without changing the orthogonality relations. Therefore, given a unitary U on two qubits, it is always possible to find a local unitary achieving the maximum σ_1, since there always exists a Schmidt decomposition of U involving only unitary operators as a local basis. For higher dimensional systems the above argument does not hold. Actually, already in the two-qutrit case it may happen that the maximum (28) can, in general, be attained only by local nonunitary operators. This means that the closest [under the criterion defined in (26)] separable map to a unitary U may be nonunitary.

We show an explicit example for a system of two qutrits given by the gate Z_3 defined as

$$Z_3 = \operatorname{diag}(1,1,1,1,1,1,1,1,-1), \quad (33)$$

which is unitary and not separable. We can rewrite Z_3 in the Schmidt form with Schmidt rank $r = 2$ as

$$Z_3 = \sigma_1 A_1 \otimes B_1 + \sigma_2 A_2 \otimes B_2, \quad (34)$$

where $\sigma_{1,2} = \sqrt{\frac{1}{2}(9 \pm \sqrt{17})}/3$, while the operators $A_{1,2}$ and $B_{1,2}$ are nonunitary and can be written as

$$A_{1,2} = \frac{\sqrt{3}}{\sqrt{102 \pm 22\sqrt{17}}} \times \operatorname{diag}(5 \pm \sqrt{17}, 5 \pm \sqrt{17}, 1 \pm \sqrt{17}), \quad (35)$$

$$B_{1,2} = \frac{\sqrt{3}}{\sqrt{646 \pm 150\sqrt{17}}} \times \mathrm{diag}(11 \pm 3\sqrt{17}, 11 \pm 3\sqrt{17}, 9 \pm \sqrt{17}). \quad (36)$$

From the Schmidt decomposition it immediately follows that the value of the maximum overlap is given by $\alpha_S = \sigma_1 = \sqrt{\frac{1}{2}(9+\sqrt{17})/3} \sim 0.854$. The coefficient α_{SRU} can be computed, leading to $\alpha_{\mathrm{SRU}} \sim 0.786$ [23]. Hence, this proves that the maximum attained over SRUs is always strictly smaller then the maximum achieved by separable maps, $\alpha_{\mathrm{SRU}} < \alpha_S$. We want to stress that our method is then suitable to detect the gap between separable and SRU maps, as long as $d \geqslant 3$. Actually, by the amount of violation of the expectation value of $W_{\mathrm{SRU},U}$ for detecting U, we can establish whether the detected map was separable or in addition RU too. For example, the unitary Z_3 can be detected as a non-SRU map by a witness operator of the form

$$W_{\mathrm{SRU},Z_3} = \alpha_{\mathrm{SRU}}^2 \mathbf{1} - C_{Z_3}, \quad (37)$$

where $C_{Z_3} = |Z_3\rangle\langle Z_3|$ and $\alpha_{\mathrm{SRU}} \sim 0.786$. Moreover, the expectation value of W_{SRU,Z_3} over the Choi state of the experimentally accessible map $\mathscr{M}$, i.e., $\mathrm{Tr}[W_{\mathrm{SRU},Z_3} C_{\mathscr{M}}]$, allows us to distinguish between non-SRU and nonseparable maps. Actually, $\mathscr{M}$ is detected to be non-SRU if $\mathrm{Tr}[W_{\mathrm{SRU},Z_3} C_{\mathscr{M}}] < 0$, and in addition we can say that $\mathscr{M}$ is not a separable map if $\mathrm{Tr}[W_{\mathrm{SRU},Z_3} C_{\mathscr{M}}] < \alpha_{\mathrm{SRU}}^2 - \alpha_S^2$.

V. PPT CHANNELS

In this section we consider a larger set of quantum channels, namely PPT channels. A CP map $\mathscr{M}$ acting on two qudits is positive partial transpose (PPT) if and only if the composite map $\mathscr{M}_{\mathscr{T}} = \mathscr{T}_{\mathrm{A}} \circ \mathscr{M} \circ \mathscr{T}_{\mathrm{A}}$, being $\mathscr{T}_A$ being the partial transposition map on the first system A, is CP [24,25]. Since a map $\mathscr{M}$ is CP if and only if the corresponding Choi operator $C_{\mathscr{M}}$ is positive, we can restate the above definition as follows: A CP map $\mathscr{M}$ is PPT if and only if the Choi operator $C_{\mathscr{M}_{\mathscr{T}}}$ related to the composite map $\mathscr{M}_{\mathscr{T}}$ is positive.

By the above correspondence we develop a method to detect whether a map is nonpositive partial transpose (NPT). We employ techniques already developed for the detection of entangled NPT states [26]; namely, we consider a witness operator of the form

$$W_{\mathrm{PPT}} = |\lambda_-\rangle\,\langle\lambda_-|^{\mathscr{T}_{\mathrm{A}}}, \quad (38)$$

where $|\lambda_-\rangle$ is the eigenvector of the Choi state $C_{\mathscr{M}_{\mathscr{T}}}$ corresponding to the most negative eigenvalue λ_- for an NPT map $\mathscr{M}$.

The expectation value of the above witness operator should now be measured for the Choi operator corresponding to the composite map $\mathscr{M} \circ \mathscr{T}_{\mathrm{A}}$, since the partial transposition following $\mathscr{M}$ is already taken into account in the form of the operator (38). Therefore, a crucial point of this approach is now related to the implementation of the map $\mathscr{T}_A$, which is non-CP. A possible solution is to add noise to the map $\mathscr{T}_A$ in order to make it CP, as shown in Ref. [27]. Following the approach of [27] we consider the minimal amount of depolarizing noise

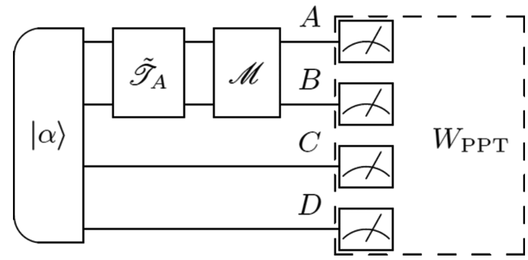

FIG. 4. Experimentally feasible scheme to implement the detection of the NPT map $\mathscr{M}$.

such that the map

$$\tilde{\mathscr{T}}_{\mathrm{A}}[\rho_{\mathrm{AB}}] = (1-p)\mathscr{T}_{\mathrm{A}}[\rho_{\mathrm{AB}}] + p\frac{\mathbf{1}_{\mathrm{AB}}}{d^2} \quad (39)$$

is CP. This is given by $p = d^3/(d^3+1)$ [27]. From an experimental point of view, we then consider the implementation of the map $\tilde{\mathscr{T}}_{\mathrm{A}}$ instead of the nonphysical map $\mathscr{T}_{\mathrm{A}}$, as shown in Fig. 4. This procedure leads to an extra contribution in the expectation value of the witness operator, related to the presence of the depolarized term in Eq. (39). The expectation value of W_{PPT} for the Choi state $C_{\mathscr{M}\circ\tilde{\mathscr{T}}_{\mathrm{A}}}$ related to the composite map $\mathscr{M} \circ \tilde{\mathscr{T}}_{\mathrm{A}}$ is given by

$$\begin{aligned}\mathrm{Tr}[W_{\mathrm{PPT}} C_{\mathscr{M}\circ\tilde{\mathscr{T}}_{\mathrm{A}}}] &= (1-p)\,\langle\lambda_-|\,C_{\mathscr{M}_{\mathscr{T}}}\,|\lambda_-\rangle + p\,\langle\lambda_-|\,\mathscr{M}_{\mathscr{T}} \\ &\quad \times \left[\frac{\mathbf{1}_{\mathrm{AB}}}{d^2}\right] \otimes \frac{\mathbf{1}_{\mathrm{CD}}}{d^2}\,|\lambda_-\rangle \\ &= (1-p)\lambda_- + p\,\langle\lambda_-|\,\mathscr{M}_{\mathscr{T}}\left[\frac{\mathbf{1}_{\mathrm{AB}}}{d^2}\right] \otimes \frac{\mathbf{1}_{\mathrm{CD}}}{d^2}\,|\lambda_-\rangle. \quad (40)\end{aligned}$$

Notice that the negative term λ_- comes from the NPT-ness of the map $\mathscr{M}_{\mathscr{T}}$, while the other term is due to the implementation of $\tilde{\mathscr{T}}_A$ in the proposed experimental procedure. The expression above clearly shows that the operator W_{PPT} can be regarded as a witness with respect to the set of PPT maps, as its expectation value is always non-negative on this set. Therefore, if the expectation value of the witness W_{PPT} is negative, the map $\mathscr{M}$ is guaranteed to be NPT.

Let us now assume that the map $\mathscr{M}$ is unital [28]. The expectation value in Eq. (40) then takes the simple form

$$\mathrm{Tr}[W_{\mathrm{PPT}} C_{\mathscr{M}\circ\tilde{\mathscr{T}}_{\mathrm{A}}}] = (1-p)\lambda_- + \frac{p}{d^4}. \quad (41)$$

In this case the addition of the depolarized term that makes the map $\mathscr{T}_{\mathrm{A}}$ physically implementable introduces only a constant shift in the expectation value of the witness. As a result, for any PPT unital map $\mathscr{M}_{\mathrm{PPT,unital}}$ we have

$$\mathrm{Tr}[W_{\mathrm{PPT}} C_{\mathscr{M}_{\mathrm{PPT,unital}}} \circ \tilde{\mathscr{T}}_{\mathrm{A}}] \geqslant \frac{p}{d^4}. \quad (42)$$

Therefore, if we know *a priori* that the map $\mathscr{M}$ to be detected is a unital map, then we are guaranteed that it is a NPT map whenever the expectation value of W_{PPT} is smaller than p/d^4.

As an illustrative example we consider again the case of the CNOT gate. Here we want to detect such a gate as a NPT map by following the experimental procedure discussed above. It is straightforward to see that the Choi state $C_{\mathrm{CNOT}_{\mathscr{T}}}$ corresponding to the map $\text{CNOT}_{\mathscr{T}} = \mathscr{T}_A \circ \text{CNOT} \circ \mathscr{T}_A$ has a single negative eigenvalue $\lambda_- = -1/2$. Since the CNOT is unital, from Eq. (41) it follows that $\text{Tr}[W_{PPT}C_{\text{CNOT}\circ\mathscr{T}_A}] = 0$, and the gap with the bound provided by Eq. (42) ($\sim$0.055 in this case) is then experimentally accessible.

VI. CONCLUSION

In conclusion, we have presented an experimentally feasible method to detect several sets of quantum channels. The proposed procedure works when some *a priori* knowledge on the quantum channel is available and is based on a link to detection methods for entanglement properties of multipartite quantum states via witness operators. The method has been first explicitly illustrated in the simple case of EB channels and then presented to detect separability properties of quantum channels. In particular, methods to reveal non-SRUs and non-separable maps have been derived, showing also the possibility to detect the gap between the sets of SRUs and separable maps. This result highlights a richer separability structure of Choi operators that has no counterpart in the separability properties of ordinary entangled/separable states. The present method can be also applied to other properties of quantum channels that rely on a convex structure and reflect on properties of the corresponding Choi states, such as, for example, completely copositive maps [29] or bientangling operations introduced in Ref. [30]. The advantage over standard quantum process tomography is that a much smaller number of measurement settings is needed in an experimental implementation. Finally, we want to point out that the proposed scheme can be implemented with current technology, for example in a quantum optical scheme [31].

ACKNOWLEDGMENT

We would like to thank Barbara Kraus for fruitful suggestions.

[1] K. Kraus, *States, Effects and Operations* (Springer, Berlin, 1983).
[2] A. Jamiolkowski, Rep. Math. Phys. **3**, 275 (1972); M.-D. Choi, Linear Algebra Appl. **10**, 285 (1975).
[3] M. Horodecki, P. W. Shor, and M. B. Ruskai, Rev. Math. Phys. **15**, 629 (2003).
[4] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Phys. Rev. A **66**, 062305 (2002).
[5] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996); B. M. Terhal, *ibid.* **271**, 319 (2000).
[6] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, J. Mod. Opt. **50**, 1079 (2003).
[7] A. De Pasquale and V. Giovannetti, Phys. Rev. A **86**, 052302 (2012).
[8] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141 (1999).
[9] M. Steiner, Phys. Rev. A **67**, 054305 (2003).
[10] F. G. S. L. Brandão, Phys. Rev. A **72**, 022310 (2005).
[11] See, for example, M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, UK, 2000).
[12] K. M. R. Audenaert and S. Scheel, New J. Phys. **10**, 023011 (2008).
[13] C. Macchiavello and G. M. Palma, Phys. Rev. A **65**, 050301(R) (2002).
[14] Recall that states given by Eq. (15) correspond to the generating points for SRU channels, which is actually a superset of the extremal points of the SRU set.
[15] O. Gühne and P. Hyllus, Int. J. Theor. Phys. **42**, 1001 (2003).
[16] G. Toth and O. Gühne, Phys. Rev. Lett. **94**, 060501 (2005).
[17] C. Macchiavello and M. Rossi, Phys. Scr., T **153**, 014044 (2013).
[18] E. M. Rains, arXiv:quant-ph/9707002.
[19] J. I. Cirac, W. Dür, B. Kraus, and M. Lewenstein, Phys. Rev. Lett. **86**, 544 (2001).
[20] This can be proven by induction over the number of Kraus operators composing the quantum map and using the Cauchy-Schwartz inequality for the Hilbert-Schmidt inner product.
[21] B. Kraus and J. I. Cirac, Phys. Rev. A **63**, 062309 (2001).
[22] M. A. Nielsen, C. M. Dawson, J. L. Dodd, A. Gilchrist, D. Mortimer, T. J. Osborne, M. J. Bremner, A. W. Harrow, and A. Hines, Phys. Rev. A **67**, 052301 (2003).
[23] In order to achieve this, two general unitaries U_A, U_B have been expressed as $U_A = \sum_{i=1}^{d^2} \alpha_i A_i$ and $U_B = \sum_{i=1}^{d^2} \beta_i B_i$, with $\{\alpha_i\}_{i=1,\dots,d^2}$, $\{\beta_i\}_{i=1,\dots,d^2}$ complex coefficients fulfilling the unitary condition, and $\{A_i\}_{i=1,\dots,d^2}$, $\{B_i\}_{i=1,\dots,d^2}$ bases coming from the Schmidt decomposition of Z_3. Numerical analysis shows that in this case it is not restrictive to consider two unitaries of the form $U_A = \alpha_1 A_1 + \alpha_2 A_2$, $U_B = \beta_1 B_1 + \beta_2 B_2$, which easily lead to a maximum overlap of $\alpha_{\text{SRU}} \sim 0.786$.
[24] E. M. Rains, Phys. Rev. A **60**, 179 (1999).
[25] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[26] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996); P. Horodecki, Phys. Lett. A **232**, 333 (1997).
[27] P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902 (2002).
[28] Notice that the expectation value given by (40) usually depends on the to-be-detected map $\mathscr{M}$. Therefore, in order to find the optimal witness, one should minimize the last term of Eq. (40) with respect to the set of all PPT maps, a hard task that drastically simplifies when the map $\mathscr{M}$ is promised to be unital.
[29] K. Zyczkowski and I. Bengtsson, Open Syst. Inf. Dyn. **11**, 3 (2004).
[30] S. Virmani, S. F. Huelga, and M. B. Plenio, Phys. Rev. A **71**, 042328 (2005).
[31] See for example: A. Chiuri, G. Vallone, N. Bruno, C. Macchiavello, D. Bruß, and P. Mataloni, Phys. Rev. Lett. **105**, 250501 (2010); A. Chiuri, V. Rosati, G. Vallone, S. Padua, H. Imai, S. Giacomini, C. Macchiavello, and P. Mataloni, *ibid.* **107**, 253602 (2011).