Quantum Zeno effect of general quantum operations

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In this paper, we show that the quantum Zeno effect can occur for generalized quantum measurements or operations. As a consequence of frequently performing nonselective measurements (or trace-preserving completely positive maps), the evolution of a certain measurement-invariant state is governed by an effective Hamiltonian defined by the measurement (or map) and the free-evolution Hamiltonian. For selective measurements, the state may change randomly with time according to measurement outcomes, but some physical quantities (operators) still evolve according to the effective Hamiltonian.

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I. INTRODUCTION

The phenomenon that the time evolution of a quantum system can be slowed down by frequent measurements, and eventually "frozen" in the large-frequency limit, is known as the quantum Zeno effect (QZE) [1,2]. As an interesting phenomenon in quantum physics, the QZE has been theoretically studied for decades and demonstrated in many experiments (see Ref. [3] for a review, and recent articles [4–6]). If instead of frequently projecting a system into its initial state, measurements project it into a multidimensional subspace that includes the initial state, the QZE allows the dynamics within the subspace, which is known as the quantum Zeno subspace effect [7]. In a recent work [6], some of us proposed a version of the QZE, which is called the *operator* QZE. In the operator QZE, the evolution of some physical quantities (operators) are frozen by frequent (noncommuting) measurements, while the quantum state may change randomly with time according to measurement outcomes.

In general, a quantum measurement corresponds to a set of measurement operators $\{M_q\}$ satisfying the completeness equation $\sum_q M_q^\dagger M_q = \mathbb{1}$ [8]. The postmeasurement state for the measurement outcome q is given by $\rho_q = p_q^{-1} M_q \rho M_q^\dagger$, where $p_q = \mathrm{Tr}(M_q \rho M_q^\dagger)$ is the probability of the outcome q, and ρ is the state of the system before the measurement. If the measurement is nonselective, which means outcomes are not recorded, the measurement transforms the state as a trace-preserving completely positive (CP) map $\mathcal{P}\rho = \sum_q M_q \rho M_q^\dagger$, where Kraus operators are measurement operators. Any trace-preserving CP map can be formalized in the operator-sum representation [8].

In this paper, we show that the QZE can occur for generalized quantum measurements or operations. By means of frequently performing nonselective measurements (or trace-preserving CP maps), if the initial state is invariant under the measurement $\mathcal{P}$, the evolution is governed by an effective Hamiltonian defined by the measurement and the free-evolution Hamiltonian. As a result of frequently performed selective measurements, the state may change randomly with time according to measurement outcomes, but some operators still evolve according to the effective dynamics. In the effective dynamics, each *measurement invariant subspace* (MIS), which is an irreducible common invariant subspace of measurement operators $\{M_q\}$, behaves like a single quantum state. Actually, each set of isomorphic MISs contains a noiseless subsystem of the map $\mathcal{P}$ [9], and the effective Hamiltonian drives the evolution of noiseless subsystems. If there is not any nontrivial invariant subspace of $\{M_q\}$ or MISs are not isomorphic with each other, the system is always totally frozen in the initial state. The quantum Zeno subspace effect corresponds to the case that MISs are one-dimensional.

The most remarkable practical application of the QZE consists in suppressing decoherence and dissipation, which is crucial for practical quantum-information processing. The QZE can protect unknown quantum states in the Zeno subspace [10]. Recently, it is shown that the decoherence can be suppressed by the Zeno subspace effect while allowing for full quantum control [11]. Some of us proposed a protocol of protecting unknown quantum states from decoherence based on the operator QZE [6], which has the advantage over previous protocols that only two-qubit measurements rather than multiqubit measurements are required. In this paper, we find that by frequently performing a quantum measurement or operation with isomorphic MISs, quantum information encoded in the noiseless subsystem associated with these isomorphic MISs can be protected while full quantum control is allowed. Compared with generating noiseless subsystems with a sequence of pulses as in the theory of dynamical decoupling [12] and other QZE-based protocols [6,10,11], the QZE of general operations significantly enlarges the set of operations that can be used to protect quantum information.

II. ZENO EFFECT OF NONSELECTIVE MEASUREMENTS

We consider a system whose free evolution is governed by the Hamiltonian H. The superoperator corresponding to the free time evolution is $\mathcal{U}(t) = e^{\mathcal{L}t}$, where the generator $\mathcal{L}\bullet = -i[H,\bullet]$ is the Liouvillian of the system. As a typical model of the QZE, we suppose the measurement is performed N times during the entire time of evolution τ at equal intervals and each measurement is performed instantly, meaning that the measurement can be implemented in a negligible amount

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of time. If the measurement is a nonselective measurement $\mathcal{P}$, the time evolution of the state reads [11]

$$\rho(\tau) = [\mathcal{P}\mathcal{U}(\tau/N)]^N \rho(0). \quad (1)$$

Here, the initial state is a *measurement-invariant operator* (MIO), i.e., $\mathcal{P}\rho(0) = \rho(0)$.

For projective measurements [1–3] or weak projective measurements [11,13], a MIO state is a state in the Zeno subspace, and the dynamics is governed by an effective Hamiltonian $H_{\pi_Z} = \pi_Z H \pi_Z$ in the limit $N \to \infty$. Here, π_Z is the projector of the Zeno subspace.

As the main result of this paper, we prove that, for any nonselective measurement $\mathcal{P}$, the state evolves driven by an effective Hamiltonian $\tilde{H}$ in the limit $N \to \infty$, i.e.,

$$\rho(\tau) = e^{\tilde{\mathcal{L}}\tau}\rho(0), \quad (2)$$

where $\tilde{\mathcal{L}}\bullet = -i[\tilde{H},\bullet]$. Here, we assume that the Hilbert space of the system is finite-dimensional, and $\|H\|_1 = J$ and $\|\tilde{H}\|_1 = \tilde{J}$ are both finite, where $\|\bullet\|_1$ denotes the trace norm of an operator. Importantly, this result also applies to any trace-preserving CP maps.

A. Measurement invariant subspaces

Before discussing MIOs, consider the following orthogonal decomposition of the Hilbert space:

$$\mathcal{H} = \left[\bigoplus_j (\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)})\right] \oplus \mathcal{H}_C. \quad (3)$$

Here, $\mathcal{H}_S^{(j)}$ and $\mathcal{H}_R^{(j)}$ are spanned by $\{|\Phi_s^{(j)}\rangle\}$ and $\{|\psi_r^{(j)}\rangle\}$, respectively, and $\mathcal{H}_C$ is spanned by $\{|\phi_l^C\rangle\}$.

The subspaces $\{\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)}\}$ are invariant subspaces of $\{M_q\}$, and each of them is composed of a set of isomorphic MISs $\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)} = \bigoplus_s \mathcal{H}_s^{(j)}$. Here, each MIS $\mathcal{H}_s^{(j)}$ is spanned by $\{|\Phi_s^{(j)}\rangle \otimes |\psi_r^{(j)}\rangle$ such that $r = 1,2,\ldots,d_R^{(j)}\}$, where $d_R^{(j)}$ is the dimension of the subsystem $\mathcal{H}_R^{(j)}$. Each set of isomorphic MISs is maximized, i.e., $\mathcal{H}_s^{(j)}$ and $\mathcal{H}_{s'}^{(j')}$ are isomorphic if and only if $j = j'$. Here, two MISs are isomorphic, meaning $\{\pi_s^{(j)} M_q \pi_s^{(j)}\}$ and $\{\pi_{s'}^{(j')} M_q \pi_{s'}^{(j')}\}$ are the same up to a unitary transformation, where $\pi_s^{(j)}$ is the projector of the subspace $\mathcal{H}_s^{(j)}$.

The complement subspace $\mathcal{H}_C$ neither is nor has a nontrivial invariant subspace of $\{M_q\}$, but it is an invariant subspace of $\{M_q^\dagger\}$. If the algebra generated by $\{M_q\}$ is a † algebra, $\mathcal{H}_C$ is always empty [14]. In general, the algebra generated by $\{M_q\}$ may not be a † algebra; thus, $\mathcal{H}_C$ could be nonempty (see the example in Sec. VI A).

If there is not any nontrivial invariant subspace of $\{M_q\}$, the Hilbert space $\mathcal{H}$ is irreducible and the decomposition reads $\mathcal{H} = (\mathcal{H}_S^{(1)} \otimes \mathcal{H}_R^{(1)}) \oplus \mathcal{H}_C$, where $\mathcal{H}_S^{(1)}$ is one-dimensional and $\mathcal{H}_C$ is empty.

With this decomposition of the Hilbert space, measurement operators read $M_q = \sum_j \pi^{(j)} M_q \pi^{(j)} + M_q \pi^C$, where $\pi^{(j)}$ (π^C) is the projector of the subspace $\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)}$ ($\mathcal{H}_C$). Up to a unitary transformation, $\pi^{(j)} M_q \pi^{(j)} = \mathbb{1}_S^{(j)} \otimes M_q^{(j)}$, where $\mathbb{1}_S^{(j)}$ ($\mathbb{1}_R^{(j)}$) is the identity operator of the subsystem $\mathcal{H}_S^{(j)}$ ($\mathcal{H}_R^{(j)}$), and $\{M_q^{(j)}\}$ are operators of the subsystem $\mathcal{H}_R^{(j)}$. Due to the completeness equation of $\{M_q\}$, the operators $\{M_q^{(j)}\}$ also obey the completeness equation $\sum_q M_q^{(j)\dagger} M_q^{(j)} = \mathbb{1}_R^{(j)}$. Because each $\mathcal{H}_s^{(j)}$ is a MIS, $\mathcal{H}_R^{(j)}$ is irreducible; i.e., $\mathcal{H}_R^{(j)}$ does not have any nontrivial invariant subspace of $\{M_q^{(j)}\}$.

A decomposition similar to that in Eq. (3) is generally used to study noiseless subsystems [9,12], and it is based on the representation theory [14] of the † algebra generated by $\{M_q, M_q^\dagger\}$, the *interaction algebra*. In this work, instead of considering the interaction algebra, we look at the algebra generated by $\{M_q\}$ for the purpose of analyzing MIOs. In fact, for unital maps, the complement subspace is always empty, and previous results of noiseless subsystems based on the interaction algebra [9,12] can be applied here. We would like to remark that whereas each S subsystem is a noiseless subsystem of the map $\mathcal{P}$ [9], not all noiseless subsystems are S subsystems that correspond to isomorphic MISs.

B. The limit of the map $\mathcal{S}_N$

To ensure the existence of MIOs, we define a map $\mathcal{S}_N = (1/N)\sum_{m=1}^N \mathcal{P}^m$, which is a trace-preserving CP map. For any operator A with a finite trace norm, $\mathcal{S}_N A$ converges to a MIO in the limit $N \to \infty$. If A is nonzero, $\mathcal{S}_\infty A = \lim_{N\to\infty} \mathcal{S}_N A$ is always a nonzero MIO.

One can prove the limit of the map $\mathcal{S}_N$ by noticing $\|\mathcal{S}_{N+1}A - \mathcal{S}_N A\|_1 = (N+1)^{-1}\|\mathcal{P}^{N+1}A - \mathcal{S}_N A\|_1 \leqslant 2(N+1)^{-1}\|A\|_1$ and $\|\mathcal{P}\mathcal{S}_N A - \mathcal{S}_N A\|_1 = N^{-1}\|\mathcal{P}^{N+1}A - A\|_1 \leqslant 2N^{-1}\|A\|_1$. Here, $\mathcal{P}^{N+1}$ and $\mathcal{S}_N$ are both trace-preserving CP maps, which do not increase the trace norm of a Hermitian operator. Notice that, whereas the operator A may not be a Hermitian operator, it can be written as a linear superposition of two Hermitian operators $A + A^\dagger$ and $-iA + iA^\dagger$.

C. Measurement-invariant operators

A MIO A is a fixed point of the map $\mathcal{P}$. If $\mathcal{P}$ is unital, i.e., $\mathcal{P}\mathbb{1} = \mathbb{1}$, A commutes with $\{M_q, M_q^\dagger\}$ [15]. We now show that for a general trace-preserving CP map $\mathcal{P}$, A can always be written as

$$A = \bigoplus_j \left(A_S^{(j)} \otimes \Lambda_R^{(j)}\right), \quad (4)$$

where $A_S^{(j)}$ is an operator of the subsystem $\mathcal{H}_S^{(j)}$, and $\Lambda_R^{(j)} = (1/d_R^{(j)})\mathcal{S}_\infty^{(j)}\mathbb{1}_R^{(j)}$ is a MIO of the subsystem $\mathcal{H}_R^{(j)}$. Here, $\mathcal{S}_\infty^{(j)} = \lim_{N\to\infty}(1/N)\sum_{m=1}^N \mathcal{P}^{(j)m}$ and $\mathcal{P}^{(j)}\bullet = \sum_q M_q^{(j)} \bullet M_q^{(j)\dagger}$ are maps of the subsystem $\mathcal{H}_R^{(j)}$. If $\mathcal{P}^{(j)}$ is unital, $\Lambda_R^{(j)} = (1/d_R^{(j)})\mathbb{1}_R^{(j)}$.

To prove Eq. (4), first, we consider Hermitian MIOs. In Appendix A, we prove that a Hermitian MIO A satisfies $\pi^C A = \pi^{(j)} A \pi^{(j')} = 0$ for $j \neq j'$, i.e., $A = \sum_j \pi^{(j)} A \pi^{(j)}$. Because each $\pi^{(j)} A \pi^{(j)}$ is an operator in the invariant subspace $\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)}$, each $\pi^{(j)} A \pi^{(j)}$ is a Hermitian MIO. In Appendix A, we also prove that $\Lambda_R^{(j)}$ is the unique Hermitian MIO of the measurement $\mathcal{P}^{(j)}$ up to a scalar factor. Therefore, $\pi^{(j)} A \pi^{(j)}$ is proportional to $\Lambda_R^{(j)}$, i.e., $\pi^{(j)} A \pi^{(j)} = A_S^{(j)} \otimes$

$\Lambda_R^{(j)}$, and the Hermitian MIO A can also be written in the form of Eq. (4).

If A is a MIO but not Hermitian, $A + A^\dagger$ and $-iA + iA^\dagger$ are two Hermitian MIOs that can be written in the form of Eq. (4). Therefore, any MIO can be written in the form of Eq. (4).

Now, we are in a condition to explain how to decompose the Hilbert space as in Eq. (3). Consider the MIO $\Lambda = \mathcal{S}_\infty \mathbb{1} = \bigoplus_j (I_S^{(j)} \otimes \Lambda_R^{(j)})$, where $I_S^{(j)} \geqslant d_R \mathbb{1}_S$ is an invertible Hermitian operator of the subsystem $\mathcal{H}_S^{(j)}$. Because $\Lambda_R^{(j)}$ is also invertible (see Appendix A), the complement subspace $\mathcal{H}_C$ is spanned by eigenstates of Λ with zero eigenvalues. Then, one can decompose the Hilbert space as Eq. (3) by applying the representation theory [14] of the † algebra generated by $\{\pi^{SR} M_q \pi^{SR}, \pi^{SR} M_q^\dagger \pi^{SR}\}$ to the subspace spanned by eigenstates of Λ with nonzero eigenvalues. Here, $\pi^{SR} = \mathbb{1} - \pi^C$ is the projector of the subspace spanned by nonzero-valued eigenstates. Actually, one can prove that the subspace spanned by zero-valued eigenstates of Λ neither is nor has a nontrivial invariant subspace of $\{M_q\}$, and each irreducible invariant subspace of $\{\pi^{SR} M_q \pi^{SR}, \pi^{SR} M_q^\dagger \pi^{SR}\}$ is an irreducible invariant subspace of $\{M_q\}$.

D. The effective Hamiltonian

The effective Hamiltonian reads

$$\tilde{H} = \bigoplus_j \left(\tilde{H}_S^{(j)} \otimes \mathbb{1}_R^{(j)}\right), \tag{5}$$

where $\tilde{H}_S^{(j)} = \mathrm{Tr}_R[\pi^{(j)} H \pi^{(j)} (\mathbb{1}_S^{(j)} \otimes \Lambda_R^{(j)})]$ is a Hermitian operator of the subsystem $\mathcal{H}_S^{(j)}$. As shown in Appendix B, the effective Hamiltonian satisfies $\mathcal{S}_\infty H \rho = \tilde{H} \rho$ and $\mathcal{S}_\infty \rho H = \rho \tilde{H}$ for any MIO state ρ. Operators that can be written in the form of Eq. (5) are called *dual* MIOs.

Driven by the effective Hamiltonian, the state initialized in a MIO state $\rho(0) = \bigoplus_j (\rho_S^{(j)}(0) \otimes \Lambda_R^{(j)})$ evolves as $\rho(t) = \bigoplus_j (\rho_S^{(j)}(t) \otimes \Lambda_R^{(j)})$, where $\rho_S^{(j)}(t) = e^{-i\tilde{H}_S^{(j)} t} \rho_S^{(j)}(0) e^{i\tilde{H}_S^{(j)} t}$. Here, $\rho(t)$ is always a MIO.

If the Hilbert space is irreducible, there is only one MIO Λ, up to a scalar factor. In this case, the state is frozen in Λ as a result of the QZE. Similarly, if S subsystems are all one-dimensional, i.e., MISs are not isomorphic with each other, the system is always frozen in the initial state.

For projective measurements, one can find that the effective Hamiltonian coincides with the one predicted by the Zeno subspace theory (see the example in Sec. VI B). For unital maps, the effective Hamiltonian $\tilde{H}_S^{(j)} = (1/d_R^{(j)}) \mathrm{Tr}_R(\pi^{(j)} H \pi^{(j)})$, which is the same as the one generated by a sequence of pulses as in the theory of dynamical decoupling [12] (see the example in Sec. VI C).

E. The effective dynamics

In order to show the emergence of effective dynamics, we suppose that even in a very short time τ/N_2, a large amount (N_1) of measurements are performed, i.e., $N = N_1 N_2$, where N_1 and N_2 are both large numbers. First, we consider the time evolution of the first time interval of τ/N_2, $\rho(\tau/N_2) = [\mathcal{P}\mathcal{U}(\tau/N)]^{N_1} \rho(0)$. After expanding the free-evolution superoperator $\mathcal{U}(\tau/N)$, we have

$$\rho(\tau/N_2) \simeq [\mathcal{P}^{N_1} + (\tau/N_2)\mathcal{T}_{N_1}]\rho(0), \tag{6}$$

where $\mathcal{T}_{N_1} = (1/N_1)\sum_{m=1}^{N_1} \mathcal{P}^m \mathcal{L} \mathcal{P}^{(N_1-m)}$. Because the initial state $\rho(0)$ is a MIO,

$$\rho(\tau/N_2) \simeq [1 + (\tau/N_2)\mathcal{S}_{N_1}\mathcal{L}]\rho(0). \tag{7}$$

If N_1 is large enough, $\mathcal{S}_{N_1}\mathcal{L}\rho(0) \simeq \mathcal{S}_\infty \mathcal{L}\rho(0) = \tilde{\mathcal{L}}\rho(0)$ (see Sec. II D), and

$$\rho(\tau/N_2) \simeq [1 + (\tau/N_2)\tilde{\mathcal{L}}]\rho(0) \simeq e^{\tilde{\mathcal{L}}\tau/N_2}\rho(0), \tag{8}$$

where the right-hand side is a MIO. For subsequent time intervals, the same reasoning applies. Therefore, $\rho(\tau) \simeq e^{\tilde{\mathcal{L}}\tau}\rho(0)$.

A rigorous analysis in Appendix C shows that $\rho(\tau) = e^{\tilde{\mathcal{L}}\tau}\rho(0) + \Delta$, where

$$\|\Delta\|_1 \leqslant (\delta_H + \delta_{\tilde{H}} + \delta). \tag{9}$$

Here, we have $\delta_H = N_2[e^{2J\tau/N_2} - (1 + 2J\tau/N_2)]$, similarly $\delta_{\tilde{H}} = N_2[e^{2\tilde{J}\tau/N_2} - (1 + 2\tilde{J}\tau/N_2)]$, and $\delta = (\tau/N_2)\sum_{n=1}^{N_2} \|(\mathcal{S}_{N_1}\mathcal{L} - \tilde{\mathcal{L}})\rho_n\|_1$, where $\rho_n = e^{\tilde{\mathcal{L}}(n-1)\tau/N_2}\rho(0)$ is a MIO. Without loss of generality, we set $N_1, N_2 = \sqrt{N}$. Then, in the limit $N \to \infty$, all of δ_H, $\delta_{\tilde{H}}$, and δ vanish.

III. ZENO EFFECT OF SELECTIVE MEASUREMENTS

If measurement outcomes are recorded, the final state $\rho(\tau;\{q\})$ depends on all measurement outcomes $\{q\}$ during the entire evolution. The final state may not be a MIO. And even if the driven Hamiltonian H is absent, the state may change according to outcomes during the evolution. In the limit $N \to \infty$, the evolution of the state with selective measurements reads $\rho(\tau;\{q\}) = \bigoplus_j [\rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})]$, where $\rho_S^{(j)}(\tau)$ is the state of the S subsystem that evolves driven by the effective Hamiltonian, and $\rho_R^{(j)}(\{q\})$ is the state of the subsystem $\mathcal{H}_R^{(j)}$ whose evolution depends on the measurement outcomes (see Appendix D). We would like to remark that, due to the probabilistic nature of $\mathrm{Tr}\rho_R^{(j)}(\{q\})$, the probability of finding the state in the subspace $\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)}$, $\mathrm{Tr}[\rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})]$, depends on measurement outcomes.

IV. OPERATOR QUANTUM ZENO DYNAMICS

If the initial state is a product state of two subsystems, $\rho(0) = \rho_S^{(j)}(0) \otimes \Lambda_R^{(j)}$, the state is always confined in the subspace $\mathcal{H}_S^{(j)} \otimes \mathcal{H}_R^{(j)}$; i.e., $\rho(\tau) = \rho_S^{(j)}(\tau) \otimes \Lambda_R^{(j)}$ for the case of nonselective measurements. For selective measurements, $\rho(\tau;\{q\}) = \rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})$. In this case, for a dual MIO $B = \bigoplus_j (B_S^{(j)} \otimes \mathbb{1}_R^{(j)})$, we have that $\mathrm{Tr}[\rho(\tau)B] = \mathrm{Tr}[\rho(\tau;\{q\})B]$. Therefore, for product-state initial states, we can define the effective evolution of operators $B(\tau) = e^{-\tilde{\mathcal{L}}\tau}B$, so that for both selective and nonselective measurements $\mathrm{Tr}[\rho(\tau)B] = \mathrm{Tr}[\rho(\tau;\{q\})B] = \mathrm{Tr}[\rho(0)B(\tau)]$; i.e., some operators always evolve according to the effective dynamics for both selective and nonselective measurements.

V. ZENO QUANTUM MEMORY WITH GENERAL MEASUREMENTS

An important application of the QZE is protecting quantum states from decoherence [6,10,11]. In general, the free evolution of a quantum memory is governed by a Hamiltonian $H = H_0 + H_{\rm noise}$, where the control Hamiltonian H_0 drives the evolution of the stored quantum state, and the noise Hamiltonian $H_{\rm noise}$ induces decoherence due to the coupling with the environment.

If a measurement $\mathcal{P}$ has a multidimensional S subsystem, e.g., $\mathcal{H}_S^{(1)}$, the quantum state stored in the subsystem $\mathcal{H}_S^{(1)}$ can be protected from decoherence by frequently performing the measurement $\mathcal{P}$ when the corresponding effective noise Hamiltonian $\tilde{H}^{(1)}_{S,\rm noise} \propto \mathbb{1}_S^{(1)}$. Here, $\tilde{H}^{(1)}_{S,\rm noise} = {\rm Tr}_R[\pi^{(1)} H_{\rm noise}\pi^{(1)}(\mathbb{1}_S^{(1)} \otimes \Lambda_R^{(1)})]$. With a control Hamiltonian satisfying $\pi^{(1)}H_0\pi^{(1)} = \tilde{H}^{(1)}_{S,0} \otimes \mathbb{1}_R^{(1)}$, the evolution of the stored quantum state is governed by $\tilde{H}^{(1)}_{S,0}$. Therefore, the stored quantum state can be fully controlled.

VI. EXAMPLES

A. Example 1: Decay channel

We consider a system with three states $|g_1\rangle$, $|g_2\rangle$, and $|e\rangle$. Measurement operators are $M_1 = |g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|$, $M_2 = (1/\sqrt{2})|g_1\rangle\langle e|$, and $M_3 = (1/\sqrt{2})|g_2\rangle\langle e|$. In this example, $\mathcal{H}^C$ is nonempty and only includes the state $|e\rangle$, and $|g_1\rangle$ and $|g_2\rangle$ form two isomorphic one-dimensional MISs, respectively. Any state initialized in the subspace spanned by $|g_1\rangle$ and $|g_2\rangle$ is a MIO. As a result of the QZE, the evolution of such an initial state is frozen in the subspace.

B. Example 2: Zeno subspace

For a projective measurement $\mathcal{P}\bullet = \sum_j \pi^{(j)} \bullet \pi^{(j)}$, where the projectors $\{\pi^{(j)}\}$ commute pairwise, each common eigenstate of $\{\pi^{(j)}\}$ forms a one-dimensional MIS, and states in the same subspace $\pi^{(j)}$ are isomorphic. In this example, $\tilde{H} = \sum_j \pi^{(j)} H \pi^{(j)}$. If the state is initialized in the subspace $\pi^{(j)}$, the evolution is driven by the effective Hamiltonian $\pi^{(j)}H\pi^{(j)}$, which coincides with the Zeno subspace theory [7].

C. Example 3: Symmetrizing operation

A symmetrizing operation [12] reads $\mathcal{P}\bullet = (1/|G|)\sum_{g\in G} g \bullet g^\dagger$, where G is a group and $|G|$ is the number of group elements. In the theory of dynamical decoupling, the symmetrizing operation describes the effect of a sequence of pulses used for generating noiseless subsystems. Here, the symmetrizing operation is supposed to be implemented as a general measurement (or trace-preserving CP map). In this example, MISs could be multidimensional if the group has multidimensional irreducible representations (is non-Abelian), and the effective Hamiltonian $\tilde{H} = \mathcal{P}H$.

D. Example 4: Bacon-Shor code

To illustrate the quantum control and the protection on a logical qubit encoded in an S subsystem, we consider the 3×3 Bacon-Shor code [16,17] (see the inset of Fig. 1) as an example. For the 3×3 Bacon-Shor code, only one logical qubit is encoded in nine physical qubits and the Hilbert space can be decomposed as $\mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_G$, where $\mathcal{H}_L$ is the Hilbert space of the logical qubit, and $\mathcal{H}_G$ is the Hilbert space of eight gauge qubits. Logical Pauli operators are $Z_L = \sigma_2^z\sigma_5^z\sigma_8^z$ and $X_L = \sigma_4^x\sigma_5^x\sigma_6^x$, where σ_i^z and σ_i^x are Pauli operators of the ith physical qubit. In the figure inset, each blue (red) bond represents a gauge operator $\sigma_i^z\sigma_j^z$ ($\sigma_i^x\sigma_j^x$).

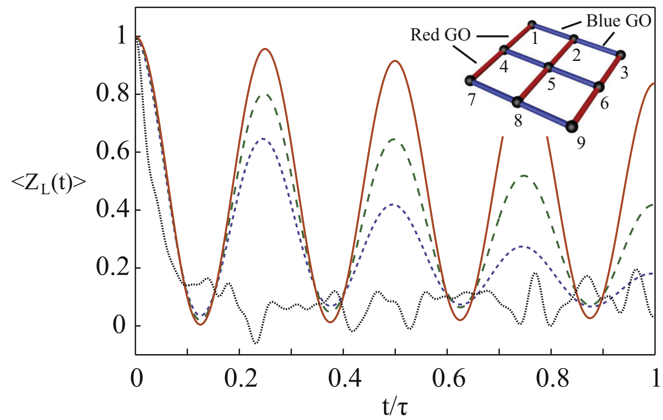

FIG. 1. (Color online) The expected value of the logical operator Z_L of the 3×3 Bacon-Shor code. In the inset, each black sphere represents a physical qubit, and each (blue or red) bond represents a gauge operator (GO). The overall measurement is constructed by projectively measuring blue GOs first and then red GOs, and it is performed N times during the entire time of evolution τ at equal interval. Here, the black dotted, blue short-dashed, green long-dashed, and red solid lines correspond $N = 0$, 500, 1000, and 5000, respectively. In this simulation, a total of eight Hadamard gates are performed.

The idea of using the QZE to protect logical qubits of the Bacon-Shor code is first mentioned in Ref. [11]. By frequently measuring gauge operators, decoherence induced by one-local and two-local noises can be suppressed [6]. Hence, we employ the measurement $\mathcal{P} = \cdots \mathcal{P}_{c_2}\mathcal{P}_{c_1}$ to protect the logical qubit, where $c_1, c_2, \ldots$ are gauge operators. The measurement of the gauge operator c reads $\mathcal{P}_c(\zeta)\bullet = [(1+\zeta)/2]\bullet + [(1-\zeta)/2]c \bullet c$, where $0 \leqslant \zeta < 1$. These two-qubit measurements can be implemented with two-qubit noisy interactions [6]. When $\zeta = 0$ the measurement $\mathcal{P}_c(\zeta)$ is a projective measurement, and when $\zeta > 0$ the measurement $\mathcal{P}_c(\zeta)$ corresponds to a weak measurement [18–20]. Weak measurements can protect quantum states, which has been proved in protocols based on the Zeno subspace [11], while evidence has been found numerically for the protocol based on the operator QZE [6].

For the measurement $\mathcal{P}$, the subsystem $\mathcal{H}_L$ and the subsystem $\mathcal{H}_G$ correspond to an S subsystem and an R subsystem, respectively. Because $\mathcal{P}$ is unital, any MIO can be written as $A = A_L \otimes \mathbb{1}_G/32$, and the effective Hamiltonian reads $\tilde{H} = ({\rm Tr}_G H) \otimes \mathbb{1}_G/32$. For logical operators, $({\rm Tr}_G Z_L) \otimes \mathbb{1}_G/32 = Z_L$ and $({\rm Tr}_G X_L) \otimes \mathbb{1}_G/32 = X_L$. For any one-local and two-local Pauli operators, ${\rm Tr}_G\sigma_i^\alpha = {\rm Tr}_G(\sigma_i^\alpha\sigma_j^\beta) = 0$.

As an example, we consider performing Hadamard gates via the control Hamiltonian $H_0 = (\omega/\sqrt{2})(Z_L + X_L)$, and the decoherence is induced by the noise Hamiltonian

$$H_{\text{noise}} = \omega\left(\sum_i \sum_{\alpha=x,y,z} \sigma_i^\alpha + \sum_{(i,j)} \sum_{\alpha,\beta=x,y,z} \sigma_i^\alpha \sigma_j^\beta\right), \quad (10)$$

where the first (second) term corresponds to one-local (two-local) noises, and (i,j) are two neighboring qubits. By frequently measuring gauge operators, decoherence of the logical qubit can be suppressed while logical operations (Hadamard gates) are performed, as shown in Fig. 1.

VII. DISCUSSIONS

We have shown that the QZE can occur for generalized quantum measurements or operations. While in this paper measurements are supposed to be performed instantly, this time scale needs to be considered in future works.

We used the trace norm rather than the operator norm to describe the Hamiltonian strength. Although for the finite-dimensional Hilbert space, a finite trace norm implies a finite operator norm for Hermitian operators, using the operator norm may be helpful in improving the bound in Eq. (9).

Besides suppressing decoherence, there are many other potential applications of the QZE [21–30].

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APPENDIX A: MEASUREMENT-INVARIANT OPERATORS

First, we prove a lemma that is very useful for our discussions about MIOs. We consider a Hermitian MIO A in a Hilbert space that can be decomposed as $\mathcal{H} = \mathcal{H}^X \oplus \mathcal{H}^I$, where $\mathcal{H}^I$ is an invariant subspace of $\{M_q\}$, i.e., $\pi^X M_q \pi^I = 0$. Here, π^X (π^I) is the projector of the subspace $\mathcal{H}^X$ $(\mathcal{H}^I)$. Because A is a Hermitian operator, $\pi^X A \pi^X$ can be diagonalized. According to eigenstates of $\pi^X A \pi^X$, we can further decompose the Hilbert space as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_- \oplus \mathcal{H}^I$, where $\mathcal{H}_\eta$ $(\mathcal{H}_I)$ is spanned by $\{|\varphi_l^{(\eta)}\rangle\}$ $(\{|\varphi_l^I\rangle\})$; $\eta = +$, 0, and $-$ correspond to positive, zero, and negative eigenvalues of $\pi^X A \pi^X$, respectively. Then, A can be written as $A = A_+ - A_- + A_I$, where

$$A_\pm = \sum_l \lambda_l^{(\pm)} |\varphi_l^{(\pm)}\rangle\langle\varphi_l^{(\pm)}| \quad (A1)$$

and

$$A_I = \pi^I A \pi^I + (\pi^I A \pi^{(+)} + \pi^I A \pi^{(0)} + \pi^I A \pi^{(-)} + \text{H.c.}). \quad (A2)$$

Here, $\{\lambda_l^{(\pm)}\}$ are all positive, and $\pi^{(\eta)} = \sum_l |\varphi_l^{(\eta)}\rangle\langle\varphi_l^{(\eta)}|$ is the projector of the subspace $\mathcal{H}_\eta$.

Lemma 1. $\mathcal{H}_\pm$ are two invariant subspaces of $\{M_q\}$, and $A_\pm$ are both MIOs.

Proof. Because $\mathcal{P}$ is a trace-preserving CP map, $\text{Tr}(\mathcal{P}A_+) = \text{Tr}A_+$, where $\text{Tr}A_+ = \sum_l \lambda_l^{(+)}$ and

$$\text{Tr}(\mathcal{P}A_+) = \text{Tr}(\pi^{(+)}\mathcal{P}A_+) + \text{Tr}(\pi^{(0)}\mathcal{P}A_+) + \text{Tr}(\pi^{(-)}\mathcal{P}A_+) + \text{Tr}(\pi^I\mathcal{P}A_+). \quad (A3)$$

Because A is a MIO, $\mathcal{P}A = A$ and $\text{Tr}(\pi^{(+)}\mathcal{P}A) = \text{Tr}(\pi^{(+)}A)$, where $\text{Tr}(\pi^{(+)}A) = \sum_l \lambda_l^{(+)}$. By noticing $\mathcal{H}_I$ is an invariant subspace of $\{M_q\}$ $(\pi^{(+)}M_q\pi^I = 0)$, we have $\text{Tr}(\pi^{(+)}\mathcal{P}A_I) = 0$ and

$$\text{Tr}(\pi^{(+)}\mathcal{P}A) = \text{Tr}(\pi^{(+)}\mathcal{P}A_+) - \text{Tr}(\pi^{(+)}\mathcal{P}A_-). \quad (A4)$$

Combining Eqs. (A3) and (A4), we have

$$\text{Tr}(\pi^{(0)}\mathcal{P}A_+) + \text{Tr}(\pi^{(-)}\mathcal{P}A_+) + \text{Tr}(\pi^I\mathcal{P}A_+) = -\text{Tr}(\pi^{(+)}\mathcal{P}A_-), \quad (A5)$$

where each term on the left-hand side is non-negative while the term on the right-hand side is nonpositive (A_+ and A_- are both positive and $\mathcal{P}$ is a positive map), which implies all terms are zero. Because

$$\text{Tr}(\pi^{(0)}\mathcal{P}A_+) = \sum_{q,l,l'} \lambda_l^{(+)} |\langle\varphi_{l'}^{(0)}|M_q|\varphi_l^{(+)}\rangle|^2 = 0, \quad (A6)$$

$$\text{Tr}(\pi^{(-)}\mathcal{P}A_+) = \sum_{q,l,l'} \lambda_l^{(+)} |\langle\varphi_{l'}^{(-)}|M_q|\varphi_l^{(+)}\rangle|^2 = 0, \quad (A7)$$

$$\text{Tr}(\pi^I\mathcal{P}A_+) = \sum_{q,l,l'} \lambda_l^{(+)} |\langle\varphi_{l'}^I|M_q|\varphi_l^{(+)}\rangle|^2 = 0, \quad (A8)$$

we have

$$\langle\varphi_{l'}^{(0)}|M_q|\varphi_l^{(+)}\rangle = \langle\varphi_{l'}^{(-)}|M_q|\varphi_l^{(+)}\rangle = \langle\varphi_{l'}^I|M_q|\varphi_l^{(+)}\rangle = 0. \quad (A9)$$

Similarly,

$$\langle\varphi_{l'}^{(+)}|M_q|\varphi_l^{(-)}\rangle = \langle\varphi_{l'}^{(0)}|M_q|\varphi_l^{(-)}\rangle = \langle\varphi_{l'}^I|M_q|\varphi_l^{(-)}\rangle = 0. \quad (A10)$$

Therefore, $\mathcal{H}_\pm$ are two invariant subspaces of $\{M_q\}$.

Because $\mathcal{H}_I$ and $\mathcal{H}_-$ are invariant subspaces, $\pi^{(+)}M_q\pi^I = \pi^{(+)}M_q\pi^{(-)} = 0$. Thus, $\pi^{(+)}(\mathcal{P}A_I)\pi^{(+)} = \pi^{(+)}(\mathcal{P}A_-)\pi^{(+)} = 0$. Then, we have $A_+ = \pi^{(+)}A\pi^{(+)} = \pi^{(+)}(\mathcal{P}A)\pi^{(+)} = \pi^{(+)}(\mathcal{P}A_+)\pi^{(+)}$. Because $\mathcal{H}_+$ is an invariant subspace, $\pi^{(+)}(\mathcal{P}A_+)\pi^{(+)} = \mathcal{P}A_+$. Therefore, $\mathcal{P}A_+ = A_+$, and A_+ is a MIO. Similarly, $\mathcal{P}A_- = A_-$, and A_- is a MIO. ■

Now, we apply Lemma 1 to the case that $\mathcal{H} = \mathcal{H}_X$ and $\mathcal{H}_I$ is empty. For any Hermitian MIO, positive eigenvalues and negative eigenvalues correspond to two invariant subspaces of $\{M_q\}$, respectively. And, any Hermitian MIO can be written as a linear superposition of two positive Hermitian MIOs.

1. The unique MIO of the map $\mathcal{P}^{(j)}$

If there exists a Hermitian MIO $\Lambda_R^{(j)\prime}$ which is linearly independent with $\Lambda_R^{(j)}$, one can compose a third nonzero Hermitian MIO $\Lambda_R^{(j)\prime\prime}$ whose trace vanishes, as a linear superposition of $\Lambda_R^{(j)}$ and $\Lambda_R^{(j)\prime}$. MIO $\Lambda_R^{(j)\prime\prime}$ must have positive and negative eigenvalues. The map $\mathcal{P}^{(j)}$ is a map in the subsystem $\mathcal{H}_R^{(j)}$. Now by applying Lemma 1 to the map $\mathcal{P}^{(j)}$, we can find that positive-valued and negative-valued eigenstates of $\Lambda_R^{(j)\prime\prime}$ form two invariant subspaces of $\{M_q^{(j)}\}$. However, $\mathcal{H}_R^{(j)}$

is irreducible. Therefore, $\Lambda_R^{(j)}$ is the unique Hermitian MIO up to a scalar factor.

2. The complement subspace

As shown in the main text, the Hilbert space can be decomposed as $\mathcal{H}=\mathcal{H}^C\oplus\mathcal{H}^{SR}$, where $\mathcal{H}^C$ is the complement subspace and $\mathcal{H}^{SR}=\bigoplus_j(\mathcal{H}_S^{(j)}\otimes\mathcal{H}_R^{(j)})$ is an invariant subspace of $\{M_q\}$. Then, we can apply Lemma 1 to the case that $\mathcal{H}^X=\mathcal{H}^C$ and $\mathcal{H}^I=\mathcal{H}^{SR}$. Without loss of generality, we consider a positive Hermitian MIO. For a positive Hermitian MIO A, eigenvalues of $\pi^C A\pi^C$ must be all zero; otherwise, the complement subspace includes one invariant subspace of $\{M_q\}$ (there are not any negative eigenvalues). In other words, $\pi^C A\pi^C=0$. Here, π^C (π^{SR}) is the projector of the subspace $\mathcal{H}^C$ ($\mathcal{H}^{SR}$). Because A is positive, all off-diagonal elements between two subspaces $\mathcal{H}^C$ and $\mathcal{H}^{SR}$ are also zero, i.e., $\pi^{SR}A\pi^C=\pi^C A\pi^{SR}=0$. Therefore, for any Hermitian MIO A, we have $A=\pi^{SR}A\pi^{SR}$ and $\pi^C A=A\pi^C=0$ (any Hermitian MIO can be written as a linear superposition of two positive MIOs).

3. Off-diagonal elements between two MISs

In general, we can rewrite the decomposition as $\mathcal{H}=\mathcal{H}^C\oplus\mathcal{H}_1\oplus\mathcal{H}_2\oplus\mathcal{H}_3\oplus\cdots$, where $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\ldots$ are MISs. Because the complement subspace is irrelevant for a Hermitian MIO A ($\pi^C A=A\pi^C=0$), the Hermitian MIO can be written as $A=\sum_{i,i'}\pi_i A\pi_{i'}$, where π_i is the projector of the MIS $\mathcal{H}_i$ and $i=1,2,3,\ldots$. Because $\{\mathcal{H}_i\}$ are MISs, $\pi^C M_q\pi_i=0$ and $\pi_i M_q\pi_{i'}=0$ if $i\neq i'$. Thus, $\pi_i A\pi_{i'}=\pi_i\mathcal{P}(A)\pi_{i'}=\pi_i[\mathcal{P}(\pi_i A\pi_{i'})]\pi_{i'}=\mathcal{P}(\pi_i A\pi_{i'})$, and $\{\pi_i A\pi_{i'}\}$ are MIOs. Without loss of generality, we consider two MISs $\mathcal{H}_1$ and $\mathcal{H}_2$. In the following, we prove that, if the Hermitian MIO $A_{12}=\pi_1 A\pi_2+\pi_2 A\pi_1$ is nonzero, $\mathcal{H}_1$ and $\mathcal{H}_2$ must be isomorphic. Hence, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are not isomorphic, $\pi_1 A\pi_2=\pi_2 A\pi_1=0$. Therefore, $\pi^{(j)}A\pi^{(j')}=0$ if $j\neq j'$.

If the Hermitian MIO A_{12} is nonzero, there must exist two nonempty invariant subspaces $\mathcal{H}_+$ and $\mathcal{H}_-$ corresponding to positive and negative eigenvalues of A_{12}, respectively, as a consequence of Lemma 1 ($\mathcal{H}^X=\mathcal{H}_1\oplus\mathcal{H}_2$ is an invariant subspace of $\{M_q\}$ and $\mathcal{H}^I$ is empty). Here, we would like to remark that $\mathrm{Tr}A_{12}=0$. For convenience, we denote eigenstates of A_{12} with positive eigenvalues as vectors $\{\binom{\mathbf{u}_l}{\mathbf{v}_l}\}$, where the vector $\mathbf{u}_l$ ($\mathbf{v}_l$) corresponds to a state in the subspace $\mathcal{H}_1$ ($\mathcal{H}_2$). In the subspace $\mathcal{H}_1\oplus\mathcal{H}_2$, measurement operators can be represented as $M_q^{(12)}=\left(\begin{smallmatrix}M_q^{(1)}&0\\0&M_q^{(2)}\end{smallmatrix}\right)$, where $M_q^{(1)}=\pi_1 M_q\pi_1$ and $M_q^{(2)}=\pi_1 M_q\pi_1$ are matrices which are the same as measurement operators of corresponding R systems, respectively.

Because $\mathcal{H}_+$ is an invariant subspace of $\{M_q\}$, we have

$$\begin{pmatrix}M_q^{(1)}&0\\0&M_q^{(2)}\end{pmatrix}\begin{pmatrix}\mathbf{u}_l\\\mathbf{v}_l\end{pmatrix}=\sum_{l'}\alpha_{l,l'}\begin{pmatrix}\mathbf{u}_{l'}\\\mathbf{v}_{l'}\end{pmatrix},\qquad\text{(A11)}$$

which indicates that $M_q^{(1)}\mathbf{u}_l=\sum_{l'}\alpha_{l,l'}\mathbf{u}_{l'}$ and $M_q^{(2)}\mathbf{v}_l=\sum_{l'}\alpha_{l,l'}\mathbf{v}_{l'}$. We would like to remark that $\{\mathbf{u}_l\}$ and $\{\mathbf{v}_l\}$ are decoupled under $M_q^{(12)}$. Hence, $\{\mathbf{u}_l\}$ and $\{\mathbf{v}_l\}$ are invariant subspaces of $\{M_q^{(1)}\}$ and $\{M_q^{(2)}\}$, respectively. The rank of $\{\mathbf{u}_l\}$ ($\{\mathbf{v}_l\}$) must be the same as the dimension of $\mathcal{H}_1$ ($\mathcal{H}_2$); otherwise, $\mathcal{H}_1$ ($\mathcal{H}_2$) is reducible. It is similar for the subspace corresponding to negative eigenvalues. Therefore, the dimensions of $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_+$, and $\mathcal{H}_-$, and the ranks of $\{\mathbf{u}_l\}$ and $\{\mathbf{v}_l\}$, must be the same. And $\{\mathbf{u}_l\}$ ($\{\mathbf{v}_l\}$) is a set of linearly independent vectors.

Because the ranks of $\{\mathbf{u}_l\}$ and $\{\mathbf{v}_l\}$ are the same and each of them is a set of linearly independent vectors, we can define an invertible transformation T satisfying $T\mathbf{u}_l=\mathbf{v}_l$, so that $M_q^{(1)}=T^{-1}M_q^{(2)}T$ and $M_q^{(2)}=TM_q^{(1)}T^{-1}$. Because $\{M_q^{(2)}\}$ satisfy the completeness equation, we have

$$\sum_q M_q^{(2)\dagger}M_q^{(2)}=\sum_q T^{\dagger-1}M_q^{(1)\dagger}T^\dagger TM_q^{(1)}T^{-1}=\mathbb{1}_v,\qquad\text{(A12)}$$

which means $\sum_q M_q^{(1)\dagger}T^\dagger TM_q^{(1)}=T^\dagger T$, i.e., $T^\dagger T$ is a Hermitian invariant operator of the dual map. Here, $\mathbb{1}_v$ is the identical operator of the vector space spanned by $\{\mathbf{u}_l\}$ (or $\{\mathbf{v}_l\}$). In the next section, we show $T^\dagger T$ is proportional to $\mathbb{1}_v$. Therefore, T is proportional to a unitary transformation and two subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic.

4. Dual measurement-invariant operators

A dual map in the MIS $\mathcal{H}_1$ reads $\mathcal{P}^{(1)\dagger}\bullet=\sum_q M_q^{(1)\dagger}\bullet M_q^{(1)}$. Because $\{M_q^{(1)}\}$ satisfy the completeness equation, $\mathbb{1}_v$ is a dual MIO, i.e., $\sum_q M_q^{(1)\dagger}\mathbb{1}_v M_q^{(1)}=\mathbb{1}_v$. If there exists a Hermitian dual MIO that is linearly independent with $\mathbb{1}_v$, we can show that $\mathcal{H}_1$ is reducible. Therefore, all dual MIOs of the MIS $\mathcal{H}_1$ are proportional to $\mathbb{1}_v$.

We suppose $\bar{D}$ is a Hermitian dual MIO that is linearly independent with $\mathbb{1}_v$. Then, we always have another nonzero dual MIO $D=\bar{D}-\bar{\lambda}_{\min}\mathbb{1}_v$, where $\bar{\lambda}_{\min}$ is the minimal eigenvalue of $\bar{D}$. The dual MIO D can be written as $D=\sum_l\lambda_l^{(+)}\mathbf{w}_l^{(+)}\mathbf{w}_l^{(+)\dagger}$, where $\{\lambda_l^{(+)}\}$ are all positive, and $\{\mathbf{w}_l^{(+)}\}$ ($\{\mathbf{w}_l^{(0)}\}$) are eigenstates of D with positive (zero) eigenvalues. We would like to remark that $\{\mathbf{w}_l^{(+)}\}$ and $\{\mathbf{w}_l^{(0)}\}$ are both nonempty. Because $\sum_q M_q^{(1)\dagger}DM_q^{(1)}=D$,

$$\sum_l\mathbf{w}_l^{(0)\dagger}\sum_q M_q^{(1)\dagger}DM_q^{(1)}\mathbf{w}_l^{(0)}=\sum_{q,l,l'}\lambda_{l'}^{(+)}|\mathbf{w}_{l'}^{(+)\dagger}M_q^{(1)}\mathbf{w}_l^{(0)}|^2=0.\qquad\text{(A13)}$$

Therefore, $\mathbf{w}_{l'}^{(+)\dagger}M_q^{(1)}\mathbf{w}_l^{(0)}=0$ and $\{\mathbf{w}_l^{(0)}\}$ is an invariant subspace of $\{M_q^{(1)}\}$.

APPENDIX B: THE EFFECTIVE HAMILTONIAN

Because $\{\mathcal{H}_S^{(j)}\otimes\mathcal{H}_R^{(j)}\}$ are invariant subspaces of $\{M_q\}$,

$$M_q=M_q\pi^C+\sum_j\pi^{(j)}M_q\pi^{(j)}.\qquad\text{(B1)}$$

Then,

$$M_q^\dagger M_q=\pi^C M_q^\dagger M_q\pi^C+\sum_j\pi^C M_q^\dagger\pi^{(j)}M_q\pi^{(j)}+\sum_j\pi^{(j)}M_q^\dagger\pi^{(j)}M_q\pi^C+\sum_j\pi^{(j)}M_q^\dagger\pi^{(j)}M_q\pi^{(j)}.\qquad\text{(B2)}$$

Due to the completeness equation, we have

$$\sum_q \pi^C M_q^\dagger M_q \pi^C = \pi^C, \quad \text{(B3)}$$

$$\sum_q \pi^{(j)} M_q^\dagger \pi^{(j)} M_q \pi^{(j)} = \pi^{(j)}, \quad \text{(B4)}$$

and

$$\sum_q \pi^C M_q^\dagger \pi^{(j)} M_q \pi^{(j)} = \sum_q \pi^{(j)} M_q^\dagger \pi^{(j)} M_q \pi^C = 0. \quad \text{(B5)}$$

Lemma 2. For any operator A, if $\pi^C A \pi^C = 0$ and $\mathrm{Tr}_R(\pi^{(j)} A \pi^{(j)}) = \tilde{A}^{(j)}$, $\pi^C \mathcal{P}(A) \pi^C = 0$ and $\mathrm{Tr}_R[\pi^{(j)} \mathcal{P}(A) \pi^{(j)}] = \tilde{A}^{(j)}$.

Proof. Using Eq. (B1), we have

$$\begin{aligned}\pi^C \mathcal{P}(A) \pi^C &= \sum_q \pi^C M_q A M_q^\dagger \pi^C \\ &= \sum_q \pi^C M_q \pi^C A \pi^C M_q^\dagger \pi^C = 0, \quad \text{(B6)}\end{aligned}$$

and

$$\begin{aligned}\pi^{(j)} \mathcal{P}(A) \pi^{(j)} &= \sum_q \pi^{(j)} M_q A M_q^\dagger \pi^{(j)} \\ &= \sum_q \pi^{(j)} M_q \pi^{(j)} A \pi^{(j)} M_q^\dagger \pi^{(j)} \\ &\quad + \pi^{(j)} M_q \pi^C A \pi^{(j)} M_q^\dagger \pi^{(j)} \\ &\quad + \pi^{(j)} M_q \pi^{(j)} A \pi^C M_q^\dagger \pi^{(j)}. \quad \text{(B7)}\end{aligned}$$

By noticing $\pi^{(j)} M_q \pi^{(j)} = \mathbb{1}_S^{(j)} \otimes M_q^{(j)}$ and using Eqs. (B4) and (B5), we have

$$\begin{aligned}&\mathrm{Tr}_R\left(\sum_q \pi^{(j)} M_q \pi^{(j)} A \pi^{(j)} M_q^\dagger \pi^{(j)}\right) \\ &= \mathrm{Tr}_R\left(\sum_q \pi^{(j)} M_q^\dagger \pi^{(j)} M_q \pi^{(j)} A \pi^{(j)}\right) \\ &= \mathrm{Tr}_R(\pi^{(j)} A \pi^{(j)}) = \tilde{A}^{(j)} \quad \text{(B8)}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Tr}_R\left(\sum_q \pi^{(j)} M_q \pi^C A \pi^{(j)} M_q^\dagger \pi^{(j)}\right) \\ &= \mathrm{Tr}_R\left(\sum_q \pi^{(j)} M_q^\dagger \pi^{(j)} M_q \pi^C A \pi^{(j)}\right) = 0, \quad \text{(B9)} \\ &\mathrm{Tr}_R\left(\sum_q \pi^{(j)} M_q \pi^{(j)} A \pi^C M_q^\dagger \pi^{(j)}\right) \\ &= \mathrm{Tr}_R\left(\sum_q \pi^{(j)} A \pi^C M_q^\dagger \pi^{(j)} M_q \pi^{(j)}\right) = 0. \quad \text{(B10)}\end{aligned}$$

Here, we have used that for any operator X and operator Y_R in the subsystem R, $\mathrm{Tr}_R[X(\mathbb{1}_S \otimes Y_R)] = \mathrm{Tr}_R[(\mathbb{1}_S \otimes Y_R)X]$. Therefore, $\mathrm{Tr}_R[\pi^{(j)} \mathcal{P}(A) \pi^{(j)}] = \tilde{A}^{(j)}$. ■

Lemma 3. For an operator A, if $\pi^C A \pi^C = 0$ and $\mathrm{Tr}_R(\pi^{(j)} A \pi^{(j)}) = \tilde{A}^{(j)}$, $\mathrm{Tr}_R[\pi^{(j)} \mathcal{S}_\infty(A) \pi^{(j)}] = \tilde{A}^{(j)}$ and $\pi^{(j)} \mathcal{S}_\infty(A) \pi^{(j)} = \tilde{A}^{(j)} \otimes \Lambda_R^{(j)}$.

Proof. Using Lemma 2, we have $\pi^C \mathcal{P}^m(A) \pi^C = 0$ and $\mathrm{Tr}_R[\pi^{(j)} \mathcal{P}^m(A) \pi^{(j)}] = \tilde{A}^{(j)}$ for any m. Hence, $\mathrm{Tr}_R[\pi^{(j)} \mathcal{S}_\infty(A) \pi^{(j)}] = \tilde{A}^{(j)}$. Because $\mathcal{S}_\infty(A)$ is a MIO and $\mathrm{Tr}\Lambda_R^{(j)} = 1$, $\pi^{(j)} \mathcal{S}_\infty(A) \pi^{(j)} = \tilde{A}^{(j)} \otimes \Lambda_R^{(j)}$. ■

Effective Hamiltonian. If the state ρ is a MIO, $\rho\pi^C = 0$. Hence, $\pi^C H \rho \pi^C = 0$. Using Lemma 3, we have $\pi^{(j)} \mathcal{S}_\infty(H\rho) \pi^{(j)} = \widetilde{H\rho}^{(j)} \otimes \Lambda_R^{(j)}$, where $\widetilde{H\rho}^{(j)} = \mathrm{Tr}_R(\pi^{(j)} H \rho \pi^{(j)})$.

We suppose the MIO $\rho = \bigoplus_j (\rho_S^{(j)} \otimes \Lambda_R^{(j)})$, and $\pi^{(j)} H \pi^{(j)} = \sum_{s,s'} |\Phi_s^{(j)}\rangle\langle\Phi_{s'}^{(j)}| \otimes H_{s,s'}^{(j)}$. Then, $\rho\pi^{(j)} = \pi^{(j)} \rho \pi^{(j)} = (\rho_S^{(j)} \otimes \Lambda_R^{(j)})$, and

$$\begin{aligned}\mathrm{Tr}_R(\pi^{(j)} H \rho \pi^{(j)}) &= \mathrm{Tr}_R\left[\pi^{(j)} H \pi^{(j)} \left(\rho_S^{(j)} \otimes \Lambda_R^{(j)}\right)\right] \\ &= \sum_{s,s'} \mathrm{Tr}_R\left(H_{s,s'}^{(j)} \Lambda_R^{(j)}\right) \left|\Phi_s^{(j)}\right\rangle\!\left\langle\Phi_{s'}^{(j)}\right| \rho_S^{(j)}.\end{aligned} \quad \text{(B11)}$$

Therefore, $\mathrm{Tr}_R[\pi^{(j)} \mathcal{S}_\infty(H\rho) \pi^{(j)}] = \tilde{H}_S^{(j)} \rho_S^{(j)}$, where

$$\begin{aligned}\tilde{H}_S^{(j)} &= \sum_{s,s'} \mathrm{Tr}_R\left(H_{s,s'}^{(j)} \Lambda_R^{(j)}\right) \left|\Phi_s^{(j)}\right\rangle\!\left\langle\Phi_{s'}^{(j)}\right| \\ &= \mathrm{Tr}_R\left[\pi^{(j)} H \pi^{(j)} \left(\mathbb{1}_S^{(j)} \otimes \Lambda_R^{(j)}\right)\right]. \quad \text{(B12)}\end{aligned}$$

Because $\mathcal{S}_\infty(H\rho)$ is a MIO, $\mathcal{S}_\infty H\rho = \tilde{H}\rho$, where $\tilde{H} = \bigoplus_j (\tilde{H}_S^{(j)} \otimes \mathbb{1}_R^{(j)})$. Similarly, $\mathcal{S}_\infty \rho H = \rho \tilde{H}$.

APPENDIX C: THE PROOF OF THE ZENO EFFECT WITH NONSELECTIVE MEASUREMENTS

As we show in the following, Δ includes three parts for each time interval of τ/N_2, and

$$\Delta = \sum_{n=1}^{N_2} [\Delta_{\mathrm{I}}(n) + \Delta_{\mathrm{II}}(n) + \Delta_{\mathrm{III}}(n)]. \quad \text{(C1)}$$

By using the notation $\mathcal{V}_{N_1} = [\mathcal{P}\mathcal{U}(\tau/N)]^{N_1}$, we have

$$\begin{aligned}\rho(\tau) &= \mathcal{V}_{N_1}^{N_2} \rho(0) \\ &= \mathcal{V}_{N_1}^{N_2-1} e^{\tilde{\mathcal{L}}\tau/N_2} \rho(0) + \Delta_{\mathrm{I}}(1) + \Delta_{\mathrm{II}}(1) + \Delta_{\mathrm{III}}(1) \\ &= \mathcal{V}_{N_1}^{N_2-2} e^{\tilde{\mathcal{L}}2\tau/N_2} \rho(0) + \Delta_{\mathrm{I}}(1) + \Delta_{\mathrm{II}}(1) + \Delta_{\mathrm{III}}(1) \\ &\quad + \Delta_{\mathrm{I}}(2) + \Delta_{\mathrm{II}}(2) + \Delta_{\mathrm{III}}(2) \\ &\quad \ldots \\ &= e^{\tilde{\mathcal{L}}\tau} \rho(0) + \sum_{n=1}^{N_2} [\Delta_{\mathrm{I}}(n) + \Delta_{\mathrm{II}}(n) + \Delta_{\mathrm{III}}(n)]. \quad \text{(C2)}\end{aligned}$$

For each time interval of τ/N_2,

$$\begin{aligned}&\mathcal{V}_{N_1}^{N_2-n+1} e^{\tilde{\mathcal{L}}(n-1)\tau/N_2} \rho(0) \\ &= \mathcal{V}_{N_1}^{N_2-n} [1 + (\tau/N_2)\mathcal{S}_{N_1}\mathcal{L}] e^{\tilde{\mathcal{L}}(n-1)\tau/N_2} \rho(0) + \Delta_{\mathrm{I}}(n) \\ &= \mathcal{V}_{N_1}^{N_2-n} [1 + (\tau/N_2)\tilde{\mathcal{L}}] e^{\tilde{\mathcal{L}}(n-1)\tau/N_2} \rho(0) + \Delta_{\mathrm{I}}(n) + \Delta_{\mathrm{II}}(n) \\ &= \mathcal{V}_{N_1}^{N_2-n} e^{\tilde{\mathcal{L}}n\tau/N_2} \rho(0) + \Delta_{\mathrm{I}}(n) + \Delta_{\mathrm{II}}(n) + \Delta_{\mathrm{III}}(n). \quad \text{(C3)}\end{aligned}$$

Here,

$$\Delta_{\mathrm{I}}(n) = \mathcal{V}_{N_1}^{N_2-n} \{\mathcal{V}_{N_1} - [1 + (\tau/N_2)\mathcal{S}_{N_1}\mathcal{L}]\} e^{\tilde{\mathcal{L}}(n-1)\tau/N_2} \rho(0), \quad \text{(C4)}$$

$$\Delta_{\rm II}(n) = \mathcal{V}_{N_1}^{N_2-n}\{[1+(\tau/N_2)\mathcal{S}_{N_1}\mathcal{L}] - [1+(\tau/N_2)\tilde{\mathcal{L}}]\} \times e^{\tilde{\mathcal{L}}(n-1)\tau/N_2}\rho(0), \quad \text{(C5)}$$

and

$$\Delta_{\rm III}(n) = \mathcal{V}_{N_1}^{N_2-n}\{[1+(\tau/N_2)\tilde{\mathcal{L}}] - e^{\tilde{\mathcal{L}}\tau/N_2}\}e^{\tilde{\mathcal{L}}(n-1)\tau/N_2}\rho(0). \quad \text{(C6)}$$

1. The norm of $\Delta_{\rm I}(n)$

As shown in the main text, $\rho_n = e^{\tilde{\mathcal{L}}(n-1)\tau/N_2}\rho(0)$ is a MIO. Thus,

$$\Delta_{\rm I}(n) = \mathcal{V}_{N_1}^{N_2-n}\{\mathcal{V}_{N_1} - [\mathcal{P}^{N_1} + (\tau/N)\sum_{m=1}^{N_1}\mathcal{P}^m\mathcal{L}\mathcal{P}^{(N_1-m)}]\}\rho_n. \quad \text{(C7)}$$

Because unitary operations ($\mathcal{U}$) and trace-preserving CP maps ($\mathcal{P}$) do not increase the trace norm of a Hermitian operator (see the last paragraph of this section for an explanation), $\mathcal{V}_{N_1}$ do not increase the trace norm of a Hermitian operator, and we have

$$\|\Delta_{\rm I}(n)\|_1 \leqslant \left\| \left\{ [\mathcal{P}\mathcal{U}(\tau/N)]^{N_1} - \left[\mathcal{P}^{N_1} + (\tau/N)\sum_{m=1}^{N_1}\mathcal{P}^m\mathcal{L}\mathcal{P}^{(N_1-m)}\right]\right\}\rho_n \right\|_1. \quad \text{(C8)}$$

After expanding evolution operators, we have

$$\|\Delta_{\rm I}(n)\|_1 \leqslant \left\| \left\{ \left[\mathcal{P}\sum_{l=0}^{\infty}\frac{(\tau/N)^l}{l!}\mathcal{L}^l\right]^{N_1} - \left[\mathcal{P}^{N_1} + (\tau/N)\sum_{m=1}^{N_1}\mathcal{P}^m\mathcal{L}\mathcal{P}^{(N_1-m)}\right]\right\}\rho_n \right\|_1, \quad \text{(C9)}$$

where terms of the second part are all included in the expansion of the first part (corresponding to the term without $\mathcal{L}$ and terms with only one $\mathcal{L}$ of the first part). After further expanding,

$$\|\Delta_{\rm I}(n)\|_1 \leqslant \sum_{\{n_i\}}\sum_{\{m_i\}}\alpha_{\{n_i\}\{m_i\}}\|\mathcal{P}^{m_{N_1}}\mathcal{L}^{n_{N_1}}\cdots\mathcal{P}^{m_2}\mathcal{L}^{n_2}\mathcal{P}^{m_1}\mathcal{L}^{n_1}\mathcal{P}^{m_0}\rho_n\|_1, \quad \text{(C10)}$$

where $\{n_i\}$ and $\{m_i\}$ are some strings of non-negative integers ($\sum_i n_i \geqslant 2$ and $\sum_i m_i = N_1$) and $\{\alpha_{\{n_i\}\{m_i\}}\}$ are all positive real coefficients. Again, because trace-preserving CP maps do not increase the trace norm of a Hermitian operator, we have

$$\|\Delta_{\rm I}(n)\|_1 \leqslant \sum_{\{n_i\}}\sum_{\{m_i\}}\alpha_{\{n_i\}\{m_i\}}(2J)^{\sum_i n_i}\|\rho_n\|_1, \quad \text{(C11)}$$

where the right-hand side can be obtained by replacing $\mathcal{P}$ with 1, $\mathcal{L}$ with $2J$, and ρ_n with $\|\rho_n\|_1$ in the right-hand side of Eq. (C9), i.e.,

$$[e^{2J\tau/N_2} - (1+2J\tau/N_2)]\|\rho_n\|_1 = \sum_{\{n_i\}}\sum_{\{m_i\}}\alpha_{\{n_i\}\{m_i\}}(2J)^{\sum_i n_i}\|\rho_n\|_1. \quad \text{(C12)}$$

Because $\|\rho_n\|_1 = 1$, we have

$$\|\Delta_{\rm I}(n)\|_1 \leqslant [e^{2J\tau/N_2} - (1+2J\tau/N_2)]. \quad \text{(C13)}$$

Trace norm and the trace-preserving CP map. For a Hermitian operator, the trace norm is the sum of the absolute values of eigenvalues. A Hermitian operator A can be decomposed as $A = A_+ - A_-$, where A_+ and A_- are two positive Hermitian operators corresponding to positive eigenvalues and negative eigenvalues of A, respectively. Then, $\|A_\pm\|_1 = {\rm Tr}A_\pm$ and $\|A\|_1 = {\rm Tr}(A_+ + A_-)$. Because $\mathcal{P}A_\pm$ are also positive Hermitian operators, $\|\mathcal{P}A\|_1 \leqslant \|\mathcal{P}A_+\|_1 + \|\mathcal{P}A_-\|_1 = {\rm Tr}[\mathcal{P}(A_+ + A_-)] = \|A\|_1$.

2. The norm of $\Delta_{\rm II}(n)$

It is straightforward that

$$\|\Delta_{\rm II}(n)\|_1 \leqslant (\tau/N_2)\|(\mathcal{S}_{N_1}\mathcal{L} - \tilde{\mathcal{L}})\rho_n\|_1. \quad \text{(C14)}$$

3. The norm of $\Delta_{\rm III}(n)$

Similar to $\Delta_{\rm I}(n)$, after expanding, one can find that

$$\|\Delta_{\rm III}(n)\|_1 \leqslant [e^{2\tilde{J}\tau/N_2} - (1+2\tilde{J}\tau/N_2)]. \quad \text{(C15)}$$

4. The norm of Δ

In summary,

$$\begin{aligned}\|\Delta\|_1 &\leqslant \sum_{n=1}^{N_2}\|\Delta_{\rm I}(n)\|_1 + \|\Delta_{\rm II}(n)\|_1 + \|\Delta_{\rm III}(n)\|_1 \\ &\leqslant N_2\{[e^{2J\tau/N_2} - (1+2J\tau/N_2)] \\ &\quad + [e^{2\tilde{J}\tau/N_2} - (1+2\tilde{J}\tau/N_2)]\} \\ &\quad + (\tau/N_2)\sum_{n=1}^{N_2}\|(\mathcal{S}_{N_1}\mathcal{L} - \tilde{\mathcal{L}})\rho_n\|_1. \end{aligned} \quad \text{(C16)}$$

APPENDIX D: THE PROOF OF THE ZENO EFFECT WITH SELECTIVE MEASUREMENTS

First, we consider an initial state that is a product state of two subsystems, e.g., $\rho(0) = \rho_S^{(j)}(0)\otimes\Lambda_R^{(j)}$. Without loss of generality, we suppose $\rho_S^{(j)}(0) = \sum_s w_s|\Phi_s^{(j)}\rangle\langle\Phi_s^{(j)}|$. By introducing a virtual system $\mathcal{H}_{\bar{S}}^{(j)}$ spanned by $\{|\bar{\Phi}_s^{(j)}\rangle\}$, the state $\rho_S^{(j)}(0)$ can be represented as the reduced state of a pure state $|\Psi(0)\rangle = \sum_s\sqrt{w_s}|\bar{\Phi}_s^{(j)}\rangle\otimes|\Phi_s^{(j)}\rangle$ in the space $\mathcal{H}_{\bar{S}}^{(j)}\otimes\mathcal{H}_S^{(j)}$, i.e., $\rho_S^{(j)}(0) = {\rm Tr}_{\bar{S}}|\Psi(0)\rangle\langle\Psi(0)|$. Then, the initial state in the extended Hilbert space $\mathcal{H}_{\bar{S}}^{(j)}\otimes\mathcal{H}_S^{(j)}\otimes\mathcal{H}_R^{(j)}$ is $\rho_{\rm ext}(0) = |\Psi(0)\rangle\langle\Psi(0)|\otimes\Lambda_R^{(j)}$.

For nonselective measurements, the final state in the extended Hilbert space is $\rho_{\rm ext}(\tau) = |\Psi(\tau)\rangle\langle\Psi(\tau)|\otimes\Lambda_R^{(j)}$, where $|\Psi(\tau)\rangle = e^{-i\mathbb{1}_{\bar{S}}\otimes\bar{H}_S^{(j)}\tau}|\Psi(0)\rangle$ and $\mathbb{1}_{\bar{S}}$ is the identity operator of the virtual subsystem. And the state $|\Psi(\tau)\rangle$ satisfies $\mathrm{Tr}_{\bar{S}}|\Psi(\tau)\rangle\langle\Psi(\tau)| = \rho_S^{(j)}(\tau) = e^{-i\tilde{H}_S^{(j)}\tau}\rho_S^{(j)}(0)e^{i\tilde{H}_S^{(j)}\tau}$.

For selective measurements, we suppose the final state in the extended Hilbert space is $\rho_{\mathrm{ext}}(\tau;\{q\})$. The final states for nonselective measurements and selective measurements satisfy $\rho_{\mathrm{ext}}(\tau) = \sum_{\{q\}} \rho_{\mathrm{ext}}(\tau;\{q\})$. Here, states $\rho_{\mathrm{ext}}(\tau;\{q\})$ are not normalized. Hence, $|\Psi(\tau)\rangle\langle\Psi(\tau)| = \sum_{\{q\}} \mathrm{Tr}_R\,\rho_{\mathrm{ext}}(\tau;\{q\})$. Because $|\Psi(\tau)\rangle\langle\Psi(\tau)|$ is a pure state, $\mathrm{Tr}_R\,\rho_{\mathrm{ext}}(\tau;\{q\}) \propto |\Psi(\tau)\rangle\langle\Psi(\tau)|$ for any outcomes, i.e., $\rho_{\mathrm{ext}}(\tau;\{q\}) = |\Psi(\tau)\rangle\langle\Psi(\tau)| \otimes \rho_R^{(j)}(\{q\})$. Using $\rho(\tau;\{q\}) = \mathrm{Tr}_{\bar{S}}\,\rho_{\mathrm{ext}}(\tau;\{q\})$, one can find that for the product-state initial state, $\rho(\tau;\{q\}) = \rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})$.

In general, a MIO initial state is a linear superposition of product-state initial states, i.e., $\rho(0) = \bigoplus_j (\rho_S^{(j)}(0) \otimes \Lambda_R^{(j)})$. Then, the final state for selective measurements is also a linear superposition of $\rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})$, i.e., $\rho(\tau;\{q\}) = \bigoplus_j [\rho_S^{(j)}(\tau) \otimes \rho_R^{(j)}(\{q\})]$.

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