Epistemic restrictions in Hilbert space quantum mechanics

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(Received 28 August 2013; published 31 October 2013)

A resolution of the quantum measurement problem(s) using the consistent histories interpretation yields in a rather natural way a restriction on what an observer can know about a quantum system, one that is also consistent with some results in quantum information theory. This analysis provides a quantum mechanical understanding of some recent work that shows that certain kinds of quantum behavior are exhibited by a fully classical model if by hypothesis an observer's knowledge of its state is appropriately limited.

DOI: 10.1103/PhysRevA.88.042122 PACS number(s): 03.65.Ta, 03.65.Ca, 03.67.−a

I. INTRODUCTION

The problem of understanding the quantum world continues to give rise to numerous debates. While the tools of textbook quantum theory allow us to calculate probabilities of measurement outcomes in agreement with experiment, the problem of understanding these in terms of microscopic quantum phenomena continues to perplex beginning students as well as their teachers. There are (at least) two distinct strategies for exploring these questions. One starts with classical physics, which is reasonably well understood, both its mathematical structure and its physical or intuitive interpretation, and tries to see how far classical ideas can be pushed into the quantum domain before they fail. This helps locate the classical-quantum boundary, and identify which classical concepts remain useful once it has been crossed, and which must be abandoned or radically modified. A second strategy starts from a consistent formulation of microscopic quantum theory, and seeks to apply it to larger systems to see how classical physics emerges as a suitable, and sometimes extremely good, approximation to quantum theory at the macroscopic level.

Much current research on hidden variable models represents the first strategy. In a version pioneered by Bell, one starts with hypotheses which seem plausible in classical physics and uses them to deduce consequences, typically inequalities, whose violation by quantum theory and experiment shows that one or more of the assumptions made in the derivation do not apply to the real quantum world. While some of the resulting claims, such as that the quantum world is nonlocal or contextual, do not stand up under scrutiny [1–3], this research should nonetheless help us better understand quantum mysteries provided the classical ideas and assumptions underlying the hidden variables approach are clearly and properly identified.

Classical ideas are made quite explicit in Spekkens "toy theory" approach [4], where by hypothesis an observer can have only a limited knowledge of the actual (ontic) state represented by some collection of classical variables. This idea has recently been extended in a very careful study [5] of coupled classical harmonic oscillators, by assuming that an observer's knowledge, in the form of a probability distribution, is limited by an *epistemic restriction* that resembles a quantum uncertainty principle. This restriction allows the authors to reproduce in an explicitly classical model a number of "weird" effects previously thought to lie wholly in the domain of quantum physics. To be sure, this approach does not reproduce the entire gamut of quantum phenomena, but the results encourage the authors to believe, as stated in their Introduction, that there might be an axiomatization of quantum theory in which the first axiom states a fundamental restriction on how much observers can know about a system, and the second embodies some novel principle about quantum reality (rather than knowledge thereof). They then add, "Ultimately, the first axiom ought to be derivable from the second because what one physical system can know about another ought to be a consequence of the nature of the dynamical laws."

We shall show that this is not a vain hope. The "novel principle" has already appeared in the physics literature as part of an approach embodying the second strategy mentioned above, the effort to understand how classical physics is an approximation to a more exact underlying quantum theory when the latter is properly understood and interpreted. What is known as the "consistent" or "decoherent" histories—hereafter referred to simply as "histories"—program, introduced in [6–8], provides, on the one hand, a fully consistent and paradox free (so far as is known at present) approach to microscopic quantum phenomena, and on the other a means for showing that the laws of classical mechanics are in appropriate circumstances a good approximation to the underlying and more exact quantum physics. In particular the *single framework rule* of Hilbert space quantum mechanics is a novel principle (relative to classical physics) that leads in a rather natural way to an epistemic restriction of a quite fundamental sort: what an observer can know is limited by the nature of quantum reality, since that which does not exist also cannot be known.

The remainder of this paper is organized as follows. The literature pertaining to the histories approach is not discussed in [4,5]; therefore, Sec. II contains a brief summary of the relevant principles of the histories approach, including the single framework rule. For additional details we refer the reader to other summaries as well as more extensive treatments of the basic ideas; the following are listed in order of increasing length: [9–13]. In Sec. III we show how these principles resolve the measurement problem(s) of quantum foundations, leading in a rather natural way to restrictions on what can be learned using measurements. Section IV argues that the results of Sec. III are consistent with quantum information theory. The results are summarized in the concluding Sec. V.

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II. HILBERT SPACE QUANTUM MECHANICS

A. Quantum properties

A key idea that goes back to von Neumann, Ch. III of [14], is that a *physical property*—something which can be true or false, such as "the energy is between 2 and 3 J"—is represented in quantum mechanics by a (closed) *subspace* of the quantum Hilbert space or, equivalently, by the *projector* (orthogonal projection operator) onto this subspace. (Here and later we assume a finite-dimensional Hilbert space; infinite dimensions complicates the mathematics without resolving any of the quantum conceptual difficulties.) A *physical variable* or *observable*, such as energy or angular momentum, is represented by a Hermitian operator which can be written in the form

$$A = \sum a_j P_j, \quad P_j = P_j^\dagger = P_j^2, \quad \sum_j P_j = I. \qquad (1)$$

Here the $\{P_j\}$ are a collection of projectors that form a (projective) *decomposition of the identity* operator I, and the $\{a_j\}$, each of which occurs but once in the sum, so $j \neq k$ means $a_j \neq a_k$, are the eigenvalues of A. The property that the physical variable A takes on or possesses the value a_j, thus $A = a_j$, corresponds to the projector P_j or, equivalently, the subspace $\mathcal{P}_j$ onto which P_j projects.

In classical mechanics a property corresponds to a set of points $\mathcal{P}$ in a classical phase space Γ, as in Fig. 1(a), and the classical counterpart of a projector is an *indicator function* $P(\gamma)$ on the phase space, which takes the value 1 if γ lies inside $\mathcal{P}$ and 0 if γ lies in the *complement* $\mathcal{P}^c$ of $\mathcal{P}$, the points in Γ that are not in $\mathcal{P}$. Obviously the two descriptions, using the set $\mathcal{P}$ or the indicator P, are equivalent, and set-theoretic operations on sets correspond to arithmetic operations on indicators. Thus the indicator of the intersection $\mathcal{P} \cap \mathcal{Q}$, the property "$P$ AND Q" or $P \wedge Q$, is the product PQ, and the indicator for $\mathcal{P}^c$, the property "NOT P" or $\neg P$, is given by $I - P$, where I is the function on Γ everywhere equal to 1.

In quantum mechanics, again following von Neumann, the negation $\neg P$ of a property P is given not by the set-theoretic complement $\mathcal{P}^c$ of the subspace $\mathcal{P}$, but instead by its *orthogonal complement* $\mathcal{P}^\perp$, the collection of all kets (vectors) which are orthogonal to those in $\mathcal{P}$. Indeed, $\mathcal{P}^c$ is not a subspace, whereas $\mathcal{P}^\perp$ is a subspace with projector $I - P$.

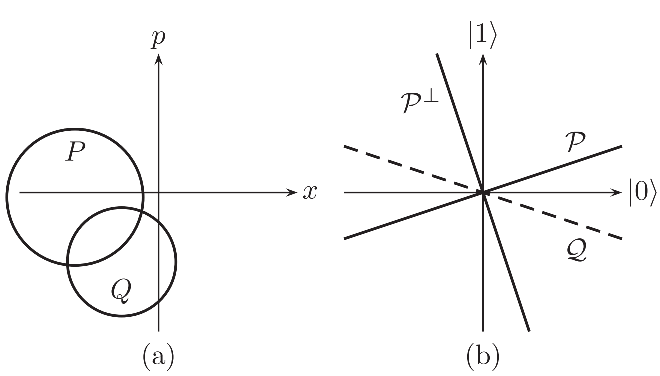

FIG. 1. (a) Phase space Γ. Properties are indicated by collections of points such as $\mathcal{P}$ and $\mathcal{Q}$. (b) Hilbert space with properties represented by rays such as $\mathcal{P}$, $\mathcal{P}^\perp$, and $\mathcal{Q}$.

The situation is shown schematically for a two-dimensional Hilbert space in Fig. 1(b), where the subspace or ray consisting of all multiples of some nonzero $|\psi\rangle$ is labeled $\mathcal{P}$ and its orthogonal complement is the ray $\mathcal{P}^\perp$. In the classical case any point that is outside $\mathcal{P}$ is inside $\mathcal{P}^c$, corresponding to the two possibilities that this property is either true or false. By contrast, in the quantum case there are rays, such as $\mathcal{Q}$ in Fig. 1(b), which are different from both $\mathcal{P}$ and $\mathcal{P}^\perp$. Thus once one accepts von Neumann's prescription for quantum properties and their negations, a prescription which lies behind but is seldom clearly explained in textbook discussions, it is clear that the move from classical to quantum mechanics must include some changes in ideas about logical reasoning and truth.

This becomes even clearer when considering the conjunction, "$\mathcal{P}$ AND $\mathcal{Q}$" or $\mathcal{P} \wedge \mathcal{Q}$, of two distinct physical properties. In the classical case this corresponds to the intersection $\mathcal{P} \cap \mathcal{Q}$ of the two sets, or the product PQ of their indicators. For the quantum case Birkhoff and von Neumann [15] proposed using the intersection of two subspaces, which is itself a subspace, to represent the conjunction of quantum properties, with the disjunction, "P or Q or both," $P \vee Q$, defined as the span of the set-theoretic union of P and Q, consistent with the usual rule that $\neg(P \vee Q) = (\neg P) \wedge (\neg Q)$. While this seems plausible, the resulting logical structure promptly leads to paradoxes—see Sec. 4.6 of [13] for a simple example—if one employs the usual rules for reasoning about properties. Birkhoff and von Neumann were aware of this, and their remedy was to abandon the distributive law $P \wedge (Q \vee R) = (P \vee Q) \wedge (P \vee R)$ as a rule of reasoning in the scheme of *quantum logic* they proposed. Despite a great deal of effort, quantum logic has not turned out to be a useful tool for understanding quantum mechanics and resolving its conceptual difficulties [16,17].[1]

The problematic nature of quantum conjunctions is also evident when one uses projectors. The product PQ of two projectors, corresponding to $P \wedge Q$, is itself a projector if and only if $PQ = QP$, in which case it projects onto the intersection $\mathcal{P} \cap \mathcal{Q}$ of the corresponding subspaces. But if the two projectors do not commute, neither PQ nor QP is a projector, and there is no simple relationship between either of them and the projector onto $\mathcal{P} \cap \mathcal{Q}$. In the histories approach, unlike quantum logic, this is dealt with by introducing a syntactical rule, an instance of the single framework rule, that says that the conjunction $P \wedge Q$ is defined only if the projectors commute; otherwise, when $PQ \neq QP$, the conjunction of these properties is undefined or meaningless (in the sense that this interpretation of quantum mechanics assigns it no meaning). Note the distinction between a false statement and a meaningless statement, such as $P \wedge \vee Q$ in ordinary logic. The negation of a false but meaningful statement is a true statement, whereas the negation of a meaningless statement is equally meaningless.

[1]This may simply reflect the fact that physicists are not smart enough. I tell my students that perhaps superintelligent robots, when they make their appearance, may be able to solve the quantum mysteries using quantum logic. But if they do, will they be able (or even want) to explain it to us?

To better understand what is and is not implied by the single framework rule and its relation to textbook quantum mechanics, consider the Hilbert space of a spin-1/2 particle, with the orthonormal basis $\{|z^+\rangle, |z^-\rangle\}$ corresponding to $S_z = +1/2$ and $-1/2$ in units of $\hbar$, and the projectors—we use $[\psi]$ for $|\psi\rangle\langle\psi|$ if $|\psi\rangle$ is normalized—$[z^+]$ and $[z^-]$. These projectors commute; in fact $[z^+][z^-] = [z^-][z^+] = 0$, the zero operator, which in quantum mechanics represents the proposition that is always false. On the other hand, $S_x = +1/2$ and $-1/2$ correspond to the projectors $[x^+]$ and $[x^-]$, projecting on the rays containing $|x^\pm\rangle = (|z^+\rangle \pm |z^-\rangle)/\sqrt{2}$, neither of which commutes with either $[z^+]$ or $[z^-]$. The single framework rule says that it is meaningless to simultaneously assign values to S_x and S_z, e.g., "$S_z = +1/2$ AND $S_x = -1/2$" makes no sense. Note that the single framework rule does not at all forbid a quantum discussion or description using either S_x or S_z, both of which could be individually meaningful or useful; what it says is that the *combination* lacks physical meaning.

One way to see that the combination "$S_z = +1/2$ AND $S_x = -1/2$" cannot be defined is to note that *every* ray in a (complex) two-dimensional Hilbert space can be interpreted to mean that the some component of spin angular momentum, corresponding to some direction in space, has the value $+1/2$. There are no rays left over, and thus no room in the Hilbert space, for a property representing such a (supposed) conjunction.[2] That it does not make sense is also implicit in the assertion found in textbooks that there is no way to simultaneously *measure* S_x and S_z. This is correct, and we shall say more about measurements in Sec. III. However, students would be less confused were they given the fundamental reason behind this: it is impossible to measure what is not there.

In histories quantum mechanics the single framework rule is a basic tool for resolving all manner of quantum paradoxes, or at least taming them in the sense of changing them from unresolved conceptual difficulties into interesting examples of how the quantum world differs from that of everyday experience. The reader will find numerous examples in Chaps. 19 to 25 of [13].

B. Probabilities

The standard (Kolmogorov) probability theory used in the histories approach requires three things: a *sample space* $\mathcal{S}$ of mutually exclusive possibilities, an *event algebra* $\mathcal{E}$, and a probability measure $\mathcal{M}$ that assigns probabilities to the elements of $\mathcal{E}$. Classical statistical mechanics employs the phase space Γ as the sample space. However, a more useful analogy for discussing the quantum case is that of a coarse graining of Γ formed by dividing it up into a finite number of nonoverlapping cells which together cover the entire space. With cell j is associated an indicator function $P_j(\gamma)$ equal to 1 if the point γ lies in cell j, and 0 otherwise. Since the cells do not overlap, the product P_jP_k of two indicators is 0 if $j \neq k$. This corresponds to the fact that the different cells which form the sample space represent *mutually exclusive* possibilities. In addition, $\sum_j P_j = I$ corresponds to the fact that at any given time the phase point γ representing the system must be in one of the cells; thus one and only one of these mutually exclusive possibilities is true. The simplest choice for the event algebra $\mathcal{E}$ is the collection of all subsets of Γ formed by unions of some of the cells which make up the sample space, with the indicator function of the union equal to the sum of the indicators of the cells of which it is composed. Including the empty set, whose indicator 0 is the function everywhere equal to zero, results in a Boolean algebra in that the negation $I - P$ of any $P \in \mathcal{E}$ and the conjunction PQ of any two of its elements are also members of $\mathcal{E}$.

By analogy, in the quantum case a sample space $\mathcal{S}$ is obtained by choosing a collection of projectors $\{P_j\}$ which sum to the identity I—which implies that the projectors are orthogonal to each other, $P_jP_k = 0 = P_kP_j$ for $j \neq k$—as a quantum sample space $\mathcal{S}$. The set of all projectors formed by taking sums of some of the projectors in $\{P_j\}$, plus the 0 operator, is the corresponding quantum event algebra $\mathcal{E}$. The event algebra is called a *framework*, a term also used for the projective decomposition that generates it. (As there is a one-to-one correspondence between $\mathcal{S}$ and $\mathcal{E}$, this double usage should not cause confusion.) The same physical interpretation can be used as in the classical case: the $\{P_j\}$ constitute a collection of mutually exclusive properties, one and only one of which is true at a particular time. Thus in the sample space employed in (1), the observable A will possess one and only one of its eigenvalues. The event algebra allows more general things; e.g., "A has either the value a_2 or a_3" is represented by $P_2 + P_3$.

An important difference between the classical and the quantum case is that in the former if one uses two different coarse grainings, two different collections of cells, each of which covers the entire phase space, there is always a common refinement, a coarse graining using cells made up of intersections of cells from the two collections. Its event algebra includes among its members all the members of the event algebras of the two coarse grainings from which it is derived. Exactly the same is possible in the quantum case *if and only if* each of the projectors in one decomposition commutes with every projector in the other; that is, if the two decompositions are *compatible*. Otherwise there is no common refinement, and the single framework rule prevents putting the frameworks together in a common probabilistic model. For example, with specific reference to the observable A in (1), let Q be a projector that does not commute with one of the P_j. Then the framework $\{Q, I - Q\}$ is incompatible with $\{P_j\}$, and according to the single framework rule the question "What is the value of A given that the quantum system has the property Q?" has no meaning. [The situation is different if Q is understood as a preprobability; e.g., the role played by $[\Psi_2]$ in (12) in Sec. III C.]

Since the single framework rule lacks any exact classical analog, it is easily misunderstood. The following principles of Liberty, Equality, Incompatibility, and Utility may help prevent such misunderstanding. First, the single framework rule allows the physicist perfect Liberty to construct different, perhaps incompatible, frameworks when analyzing and describing a quantum system. No law of nature singles out a particular quantum framework as the "correct" description of a quantum system; there is, from a fundamental

[2] Quantum logic gets around this by assigning to $(S_z = +1/2) \wedge (S_x = -1/2)$ the 0 projector, the property that is always false. The difficulties this leads to are discussed in Sec. 4.6 of [13].

point of view, perfect Equality among different possibilities. The key principle of Incompatibility prohibits *combining* incompatible frameworks into a single description, or employing them for a single logical argument leading from premisses to conclusions. Finally comes Utility: not every framework is useful for understanding a particular physical situation. It is also important to avoid the mistake of thinking that the physicist's choice of framework somehow influences reality. Instead, quantum reality allows a variety of alternative descriptions, useful for different purposes, which when they are incompatible cannot be combined. Different coarse grainings of a classical phase space, or different views of a mountain from the north and from the south, are classical analogies which may help in understanding Liberty, Equality, and Utility. But Incompatibility requires a quantum example, as provided by the S_x and S_z descriptions of a spin-1/2 particle discussed in Sec. II A above.

The probability measure $\mathcal{M}$ in standard probability theory is a non-negative function μ on the event algebra $\mathcal{E}$. It is additive over disjoint sets, and normalized, $\mu(I) = 1$. For our purposes it suffices to assume that a non-negative number μ_j is attached to each indicator P_j in the sample space in such a way that $\sum_j \mu_j = 1$. From this the probability of elements of the event algebra is determined in the usual way, e.g., $\Pr(P_2 + P_3) = \mu_2 + \mu_3$. The same procedure works in the quantum case: to each projector P_j in the decomposition of the identity (quantum sample space) under consideration one assigns a probability $\mu_j \geqslant 0$ satisfying $\sum_j \mu_j = 1$ and then sums of these to the projectors making up the corresponding event algebra. It is important to note that, aside from positivity, additivity, and normalization, mathematical probability theory imposes no restrictions on the μ_j. The same is true for quantum theory, except that under certain conditions one can use the Born rule and its extensions in order to generate probabilities for a closed system, as discussed below.

Probability theory can be understood as an extension of propositional logic, where probability 1 corresponds to a proposition that is true, and probability 0 to one that is false. In order to maintain the same connection in quantum theory, it follows that "true" and "false" must, like probabilities, be framework-dependent concepts. This dependence has sometimes been thought to imply that the histories approach leads to contradictions, propositions which are both true and false [18–20]. However, the single framework rule prevents contradictions from arising [21–26], and one can show [see Chap. 16 of [13]] that the histories approach provides a consistent scheme for probabilistic inference.

C. Time development

In textbook quantum mechanics Schrödinger's equation provides a deterministic unitary time development of "the wave function" until an external measurement causes a mysterious wave function collapse. This approach, found in [14], is widely (and properly) regarded as unsatisfactory. In the histories approach quantum dynamics is *always* a stochastic process, whether or not a measurement occurs, and solutions to Schrödinger's equation are used to compute probabilities by means of the Born rule and its extensions. Here we summarize the essentials needed for the discussion of measurements in Sec. III.

Quantum stochastic time development can be described using a *history Hilbert space* $\breve{\mathcal{H}}$, which for a sequence of events at times $t_0 < t_1 < \cdots < t_f$ is a tensor product

$$\breve{\mathcal{H}} = \mathcal{H} \odot \mathcal{H} \odot \cdots \odot \mathcal{H}, \tag{2}$$

of $f+1$ copies of the Hilbert space $\mathcal{H}$ used for the system at a single time, where the customary tensor product symbol $\otimes$ has been replaced by $\odot$ as a matter of convenience, to have a distinctive symbol separating events at different times. An individual quantum history of the simplest sort is a tensor product of projectors

$$Y = F_0 \odot F_1 \odot \cdots \odot F_f, \tag{3}$$

and thus itself a projector on the history Hilbert space. Its physical interpretation is "property F_0 at time t_0, then property F_1 at time t_1, then...", where "then" could be replaced by "and." In general, successive events are *not* connected with each other in any way related to Schrödinger's equation.

Rather than the most general case we restrict ourselves to the situation in which at time t_0 the projector $F_0 = [\Psi_0]$ projects onto a specific initial state $|\Psi_0\rangle$, and at each later time t_m, F_m belongs to the event algebra generated by a specific decomposition $\{P_m^{\alpha_m}\}$ of the identity, $\sum_{\alpha_m} P_m^{\alpha_m} = I$. Here the α_m are labels, not exponents, the subscript m indicates the time, and different decompositions may be used at different times. The sample space of histories corresponds to a collection $\{Y^\alpha\}$, where $\alpha = (\alpha_1, \ldots, \alpha_f)$ is a vector of labels, and

$$Y^\alpha = [\Psi_0] \odot P_1^{\alpha_1} \odot P_2^{\alpha_2} \odot \cdots \odot P_f^{\alpha_f}. \tag{4}$$

If in addition one includes the special history $(I - [\Psi_0]) \odot I \odot I \odot \cdots \odot I$, which is assigned a probability of 0 and hence plays no role in the following discussion, the history projectors in the sample space sum to the history space identity $\breve{I} = I \odot I \odot \cdots \odot I$, and thus constitute a set of mutually exclusive possibilities, one and only one of which can be said to occur. The collection of all projectors which are sums of some of the Y^α forms the event algebra.

For a closed system that does not interact with an external environment, solving Schrödinger's equation yields a unitary time development operator $T(t,t')$ for the time interval from t' to t; it is equal to $\exp[-i(t-t')H/\hbar]$ in the case of a time-independent Hamiltonian H. Using this time development operator we define a *chain ket*

$$|\alpha\rangle = P_f^{\alpha_f} T(t_f,t_{f-1}) P_{f-1}^{\alpha_{f-1}} T(t_{f-1},t_{f-2}) \cdots P_1^{\alpha_1} T(t_1,t_0) |\Psi_0\rangle \tag{5}$$

for every history Y^α in the sample space. A family of histories satisfies the *consistency condition*, and is called a *consistent family*, provided the inner product of two chain kets for distinct elements of the history sample space vanishes,

$$\langle\alpha|\alpha'\rangle = \mu_\alpha \delta(\alpha,\alpha'), \tag{6}$$

where $\delta(\alpha,\alpha')$ is 1 if $\alpha_m = \alpha'_m$ for every m, and is 0 otherwise. When (6) is satisfied the μ_α are the (extended) Born probabilities for histories in the sample space, and determine the probabilities for histories in the event algebra in the usual way. Condition (6) is needed to ensure that the probabilities defined in this way satisfy the usual rules of probability theory;

e.g., if one sums the joint distribution over all possibilities at a particular time t_j the result should be the joint distribution for the events at the remaining times as calculated omitting all mention of t_j from the family of histories. For a more detailed discussion of this point see Sec. 10.2 of [13].

In the case $f = 1$, histories involving only two times t_0 and t_1, the consistency condition is automatically satisfied, since the projectors at time t_1 are orthogonal to each other, and the probabilities are exactly those given by the usual Born rule. Note, however, that these probabilities refer to states of affairs inside a closed quantum system, not to outcomes of measurements carried out on that system by some external apparatus. The overall consistency of this approach is shown in Sec. III below, where measurements themselves are treated as quantum mechanical processes occurring within a (large) closed system.

When $f = 2$ or more, a family of histories involving three or more times, the consistency condition (6) on the orthogonality of chain kets for $\alpha \neq \alpha'$ is quite restrictive. Families of histories for which it is not satisfied cannot be assigned probabilities in a consistent manner. It may be that even when the history projectors of two different consistent families commute with each other, so that there is a common refinement, this refinement does not satisfy the consistency conditions, and so cannot be assigned probabilities. It is then natural to extend the single framework rule to include a prohibition of such combinations.[3]

There are various extensions of the type of analysis given above to more general situations. In place of an initial pure state $[\Psi_0]$ at t_0 one can use a more general projector or a density operator; in that case chain operators are employed in place of chain kets. Sometimes the weaker requirement that the real part of $\langle\alpha|\alpha'\rangle$ vanish for $\alpha \neq \alpha'$, allowing the imaginary part to be nonzero, is used in place of (6), though there are reasons [27] for preferring the stronger condition.[4] Rather than assuming an initial state at t_0 one can use a final state at t_f, or indeed some property at an intermediate time, as the "initial" condition. In constructing a history family the choice of projectors at a particular time can be made dependent on which event occurred at an earlier (or at a later) time. Numerous examples illustrating some of these points will be found in [13]. (Using the Heisenberg representation for projectors that enter chain kets or chain operators results in more compact expressions that lead to the same consistency conditions and probabilities as in the more intuitive Schrödinger picture employed here.)

III. MEASUREMENTS

A. Two measurement problems

It is clear that any claim to know something about a microscopic quantum system must go beyond elementary human sense impressions and make use of data provided by suitable instruments that amplify quantum effects and, so to speak, make them visible; in particular we need to understand physical measurements as genuine quantum processes. But this is the infamous *measurement problem* of quantum foundations, which has two parts. The *first measurement problem* is that of understanding the macroscopic outcome—we adopt the picturesque though outdated language of the position of a visible pointer—in proper quantum mechanical terms. The *second measurement problem* is to relate the pointer position to the prior microscopic property the instrument was designed to measure. Here "prior" means *earlier in time*, since very often the measurement either destroys or radically alters the system being measured: think of the detection of a gamma ray, or the scattering process by which it is inferred that a neutrino came from the sun or from a supernova. We need a quantum theory of *retrodiction*, inferring something about the past from present data (not to be confused with *retrocausation*, the notion that the future can influence the past.) Obviously, analyzing measurements as physical processes cannot employ measurement as some sort of primitive concept, as in textbooks. Hence the need for a fully consistent description of microscopic quantum properties, one constructed without using measurement as a primitive concept or axiom, as summarized in Sec. II above. We shall now show how these principles can be used to resolve both measurement problems. The result will then be used in Sec. IV to argue that Hilbert space quantum mechanics itself gives rise, in a rather natural way, to an epistemic restriction which does not need to be added as an extra axiom.

B. Quasiclassical frameworks

Describing ordinary macroscopic objects in a consistent, fully quantum-mechanical fashion is a nontrivial problem, and it would be premature to claim that every detail has been worked out. Nevertheless, the work of Omnès [28,29] and Gell-Mann and Hartle [30–32] provides a general procedure which seems adequate to the task. We will briefly describe the strategy used by Gell-Mann and Hartle (also see Chap. 26 of [13]). The first idea is that classical properties can be usefully described using a *quasiclassical* quantum framework employing coarse-grained projectors that project onto Hilbert subspaces of enormous, albeit finite, dimension, suitably chosen so as to be counterparts of classical properties such as those used in macroscopic hydrodynamics. Next one argues that the stochastic quantum dynamics associated with a family of histories constructed using these coarse-grained projectors gives rise, in suitable circumstances, to individual histories which occur with high probability and are quantum counterparts of the trajectories in phase space predicted by classical Hamiltonian mechanics. There are exceptions; for example, in a system whose classical dynamics is chaotic with sensitive dependence upon initial conditions one does not expect the quantum histories to be close to deterministic, but then in practice one also has to replace the deterministic classical description with something probabilistic in order to obtain useful results.

A quasiclassical family can hardly be unique given the enormous size of the corresponding Hilbert subspaces, but this is of no great concern provided classical mechanics is reproduced to a good approximation, in the sense just

[3]The three-box paradox discussed in Sec. 22.5 of [13] provides a simple example.

[4]Whereas this weaker requirement is mentioned in Ch. 10 of [13], all the examples in that book conform to the stronger condition, i.e., (6).

discussed, by any of them. Therefore all discussions which involve nothing but classical physics can, from the quantum perspective, be carried out using only a single quasiclassical framework. And as long as reasoning and descriptions are restricted to this one framework there is no need for the single framework rule, which explains why a central principle needed to understand quantum mechanics is completely absent from classical physics.

Sometimes the objection has been raised [33,34] that quasiclassical frameworks are not the unique possibilities allowed by the histories approach to quantum mechanics. In particular, a consistent family can be constructed in which the projectors are quasiclassical up to some time, and then followed by a completely different type of projector at later times, and there is no reason from the perspective of fundamental quantum mechanics to disallow this. However, there is also no reason to prefer it. The histories approach does not deny that other incompatible consistent families can be constructed; it simply insists that this possibility does not invalidate a description employing a quasiclassical framework, which is what is needed for thinking about pointer positions. See the discussion of Liberty, etc. in Sec. II B. By analogy, the possibility of using S_x to describe a spin-1/2 particle does not invalidate a description based on S_z; what cannot be done is to combine them.

C. A measurement model

To see how the histories approach resolves both measurement problems we consider a simple model of the measuring process that, apart from minor changes, goes back to Chap. VI of [14]. Let $\mathcal{H}_S$ be the Hilbert space of the system to be measured, henceforth referred to as a *particle*, and $\mathcal{H}_M$ that of the measuring device. For example, $\mathcal{H}_S$ could be the two-dimensional Hilbert space of the spin of a spin-1/2 particle, while the quantum description of its position might be among the variables included in $\mathcal{H}_M$. Let $\{|s^j\rangle\}$ be an orthonormal basis for $\mathcal{H}_S$, with states labeled by a superscript so that the subscript position can refer to time. At t_0 let $|M_0\rangle$ be the initial (normalized) state of the apparatus, while

$$|\psi_0\rangle = \sum_j c_j |s^j\rangle, \quad |\Psi_0\rangle = |\psi_0\rangle \otimes |M_0\rangle \tag{7}$$

are the initial states of the particle and of the total closed system that includes both particle and measuring device. The c_j are complex numbers satisfying $\sum_j |c_j|^2 = 1$, so both $|\psi_0\rangle$ and $|\Psi_0\rangle$ are normalized.

Let $T(t,t')$, as in Sec. II C, be the unitary time development operator for the total system, and assume it is trivial, equal to the identity operator $I = I_S \otimes I_M$, for t and t' both less than some t_1 or both greater than t_2, and that for the interval from t_1 to t_2,

$$T(t_2,t_1)(|s^j\rangle \otimes |M_0\rangle) = |\hat{s}^j\rangle \otimes |M^j\rangle. \tag{8}$$

Here the $\{|M^j\rangle\}$ are orthonormal states of the apparatus, $\langle M^j|M^k\rangle = \delta_{jk}$, corresponding to different pointer positions, and the $\{|\hat{s}^j\rangle\}$ on the right side of the equation—note they are *not* assumed to be the same as the $\{|s^j\rangle\}$ on the left side—are normalized $\langle \hat{s}^j|\hat{s}^j\rangle = 1$, though (unlike in von Neumann's original model) not necessarily orthogonal. The transformation (8) is unitary, or, to be more precise, it can be extended to a unitary transformation on $\mathcal{H}_S \otimes \mathcal{H}_M$, because the orthogonality of the $|M^j\rangle$ ensures that the states on the right side of (8) are mutually orthogonal, even though this may not be true of the $\{|\hat{s}^j\rangle\}$. Noting that $T(t_2,t_0) = T(t_2,t_1) \cdot I$ and applying (8) to (7), one sees that unitary time development leads to states

$$|\Psi_1\rangle = T(t_1,t_0)|\Psi_0\rangle = |\Psi_0\rangle,$$
$$|\Psi_2\rangle = T(t_2,t_0)|\Psi_0\rangle = \sum_j c_j |\hat{s}^j\rangle \otimes |M^j\rangle \tag{9}$$

for the total system at times t_1 and t_2.

We now wish to consider various consistent families that begin with $[\Psi_0] = |\Psi_0\rangle\langle\Psi_0|$ at time t_0. One possibility is unitary time development:

$$\mathcal{F}_0 : [\Psi_0] \odot \{[\Psi_1], I - [\Psi_1]\} \odot \{[\Psi_2], I - [\Psi_2]\} \tag{10}$$

for times $t_0 < t_1 < t_2$, where the different histories in the sample space are made up by choosing one of the projectors inside the curly brackets at each of the later times. Since the events $I - [\Psi_1]$ and $I - [\Psi_2]$ occur with zero probability they could actually be omitted without causing confusion; only the history $[\Psi_0] \odot [\Psi_1] \odot [\Psi_2]$ occurs, and it is assigned a probability of 1. While $\mathcal{F}_0$ is perfectly acceptable as a family of quantum histories, it cannot be used to discuss possible outcomes of the measurement because it does not include the projectors $\{[M^k]\}$ for the pointer positions at time t_2, and it cannot be refined to include them because $[\Psi_2]$ does not commute with some of the $[M^k]$, provided at least two of the c_j in (7) are nonzero. Thus the first measurement problem cannot be solved if one insists that all time development is unitary and not stochastic.

The histories approach can solve the first measurement problem by using the family

$$\mathcal{F}_1 : [\Psi_0] \odot [\Psi_1] \odot \{[M^k]\} \tag{11}$$

in place of $\mathcal{F}_0$. Here the alternative $I - [\Psi_1]$ at t_1, which occurs with zero probability, has been omitted, and we employ the usual physicist's convention that $[M^k] = |M^k\rangle\langle M^k|$ means $I_S \otimes [M^k]$ on the full Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_M$. An additional projector $R' = I - \sum_k [M^k]$ should be included at the final time in (11) so that the total sum is the I, but again it is omitted since its probability is zero. Note that there is no reference to the later particle states $|\hat{s}^k\rangle$ in (8); they are in fact irrelevant for discussing the macroscopic outcomes of the measurement. While $[\Psi_2]$ cannot be one of the properties at time t_2 in family $\mathcal{F}_1$ (see the discussion of $\mathcal{F}_0$ above), it can very well be used as a mathematical device (a pre-probability in the terminology of Sec. 9.4 of [13]) to calculate probabilities of the different pointer positions:

$$\Pr([M^k]_2) = \mathrm{Tr}([\Psi_2][M^k]) = \langle\Psi_2|\,[M^k]\,|\Psi_2\rangle. \tag{12}$$

Note that this is a perfectly legitimate and consistent "epistemic" use of $|\Psi_2\rangle$, since it, like a probability distribution, provides some information about the system, even when it does not represent a physical property.

In order to relate the measurement outcome to a prior property of the measured particle and thus solve the second measurement problem, one needs still another family:

$$\mathcal{F}_2 : [\Psi_0] \odot \{[s^j]\} \odot \{[M^k]\}. \tag{13}$$

Here the decomposition of the identity $\{[s^j]\}$ at t_1 refers to properties of the particle ($[s^j]$ means $[s^j] \otimes I_M$ on the full Hilbert space) without reference to the apparatus. It is straightforward to show that the family $\mathcal{F}_2$ is consistent and leads to a joint probability distribution, where the subscripts on $[s^j]$ and $[M^k]$ refer to the time,

$$\Pr([s^j]_1,[M^k]_2) = |c_j|^2 \delta_{jk}, \tag{14}$$

with marginals given by

$$\Pr([s^j]_1) = \Pr([M^j]_2) = |c_j|^2. \tag{15}$$

If $|c_k|^2 > 0$ the conditional probability for the earlier property $[s^j]_1$ given the later pointer position $[M^k]_2$ is

$$\Pr([s^j]_1 | [M^k]_2) = \delta_{jk}. \tag{16}$$

That is to say, from the (macroscopic) measurement outcome or pointer position k at time t_2, standard statistical inference allows one to infer, to retrodict, that the particle had the property $[s^k]$ at the earlier time t_1. Also note that, (15), the probabilities for particle properties just before the measurement are identical to those of the later pointer positions. This is what one would expect for ideal measurements. It shows that textbooks (which follow the lead of [14]) in which students are taught to calculate $|c_j|^2$ for the particle alone and then ascribe the resulting probability to the outcome of a measurement, are not wrong, just confusing. It is worth remarking once again that the later particle states, the $|\hat{s}^j\rangle$ in (8), play no role in the discussion. In von Neumann's model these were set equal to the $|s^j\rangle$, and while this is of course a perfectly legitimate choice, it has given rise to considerable confusion in that "measurements" are often interpreted as involving a correlation between the pointer position and the *later* particle state, something that should properly be called a *preparation*, not a measurement.

The measurement model discussed above is in some ways rather artificial. However, once one understands its basic features it is possible to construct more realistic models in which the particle states to be measured form a general (projective) decomposition of I_S not limited to pure states, the pointer positions are represented not by pure states but by quasiclassical projectors onto appropriate Hilbert spaces, the initial state of the apparatus is a quasiclassical projector or a density operator, and allowance is made for thermodynamic irreversibility in the measurement process. See the relevant sections of Chap. 17 in [13] for details. None of these extensions alters the basic conclusions reached on the basis of the simple model used above.

IV. INFORMATION

Given a system with a d-dimensional Hilbert space, a projective decomposition of the identity can contain at most d projectors, one for every element of some orthonormal basis. And since a probabilistic quantum description must use only a single framework, the maximum amount of Shannon information that can be associated with a sample space of a d-dimensional quantum system, the Shannon entropy H if a probability $1/d$ is assigned to each possibility, is $\log d$. Thus a qudit of dimension d (using the terminology of quantum information theory) cannot receive or contain or carry more than $\log_2 d$ bits of information. This is a fundamental epistemic restriction arising directly from the Hilbert space structure of quantum mechanics.

This restriction is confirmed by, or is at least consistent with, various results in quantum information theory. A common scenario is one in which Alice prepares a qudit of dimension d in a known quantum state, chosen from a specified set of possibilities according to some probability distribution, and sends it to Bob, who is allowed to carry out a generalized measurement [positive-operator valued measured or POVM], or perhaps a collective measurement on several qudits at the same time, with the aim of determining which states Alice prepared. We assume the protocol is repeated N times, and each time Alice records what she prepares. Similarly, Bob records his measurement outcomes. As both of their records belong to the quasiclassical world they can be analyzed using classical (Shannon) information theory, and a well-known bound due to Holevo (see, e.g., Sec. 12.1.1 of [35]) shows that the Shannon mutual information $H(A:B)$ cannot exceed $N \log_2 d$. This upper bound is actually achieved if Alice randomly (with equal probability) prepares states corresponding to some orthonormal basis, and Bob measures in the same basis. Thus the limitation implied by the analysis of properties and measurements in Sec. III C can, with the help of some not altogether trivial mathematics, be shown to be quite general. Neither sophisticated encoding schemes nor the most general of generalized measurements can do any better than what is implied by the analysis in Secs. II and III: at most d^N distinct messages, corresponding to $\log_2 d^N = N \log_2 d$ bits, can be constructed using N letters chosen from an alphabet of size d.

The reader could be concerned that this epistemic limit might be violated by schemes for *dense coding*, or is somehow inconsistent with *teleportation*, or maybe does not apply to quantum, in contrast to classical, information. Let us briefly discuss each of these, starting with the last. Both "quantum" and "classical" information remain ill-defined terms in the quantum information literature, despite (or perhaps because of?) an enormous number of publications. However, one can understand the basic issue by means of a simple example, a $d = 2$ perfect quantum channel, constructed from a magnetic-field-free pipe into which Alice sends a spin-1/2 particle, which is measured by Bob when it emerges at the other end. If Alice prepares the state $S_w = +1/2$, where w is some specific direction in space: z or $-z$ or x or whatever, and Bob measures in the S_w basis, the result will always (probability 1) be $S_w = +1/2$ and not $S_w = -1/2$. The basis must be specified, as there is no way to prepare (or measure) a particle with, say, $S_z = +1/2$ AND $S_x = 1/2$. Consequently this quantum channel with capacity one (qu)bit cannot actually transmit information at a rate greater than a perfect classical channel that can only transmit quasiclassical states corresponding to a bit which is either 0 or 1, and always yields the same output as the input. More generally, the quantum capacity cannot exceed the classical capacity (see Secs. 12.3 and 12.4 of [35] for technical discussions), and talking about quantum information (whatever it might be) does not alter the $\log d$ upper limit for one qudit.

One can be misled by the very useful but somewhat dangerous visualization of a spin-$1/2$ particle as a little top spinning about a particular axis. The direction w which Alice uses to produce the state $S_w = +1/2$ might be specified very precisely by some macroscopic setting on her apparatus, so it is tempting to suppose that information about this precise setting, which might amount to many bits (depending on the precision), is then carried away by the particle. But, as noted in Sec. II A, there is no room in the quantum Hilbert space for this kind of information, the distinction one might want to make between w and a direction w' which is nearby, or even between w and a w' which is perpendicular to it (e.g., x instead of z).[5] A less misleading, albeit still classical, visualization is to think of the $S_w = +1/2$ state as a spinning top whose axis is oriented at random, but with the constraint that the w component of its angular momentum be positive.

Dense coding is a process by which d^2 messages can be transmitted from Alice to Bob by sending a single d-dimensional qudit through a quantum channel, provided a fully entangled state of two qudits, one in Alice's laboratory and the other in Bob's, is initially available. This might seem to violate the epistemic limit, since $\log_2 d^2 = 2\log_2 d$ is clearly larger than $\log_2 d$. The solution to this apparent paradox is a proper microscopic analysis of where information is located at the intermediate time between Alice's preparation and Bob's measurement [36], a type of analysis which is not easy to do using the tools provided in typical textbooks (including those devoted to quantum information), because they provide no systematic way of thinking about microscopic quantum properties during the time interval between the (quasi)classical preparation and the (quasi)classical measurement. To discuss the quantum state of the two qudits, which are initially in a fully entangled state, one has to use the tensor product of their Hilbert spaces, of dimension d^2. There is then a projective decomposition of its identity corresponding to an orthonormal basis containing d^2 fully entangled states, which are used to encode the d^2 messages. Thus there is no violation of the epistemic limit, for two particles are involved. We refer the reader to [36] for a similar discussion of teleportation; once again, all information can be properly accounted for, and the fact that two uses of a d-dimensional classical channel are required to complete the protocol does not imply that a qudit or the corresponding quantum channel has an information capacity of $2\log_2 d$. (Also see the discussion in [37] for further insight into the need for a double usage of a classical channel.)

In addition, quantum information theory contains various epistemic restrictions on the information that can be obtained from a quantum system or transmitted from one place to another in the form of inequalities, sometimes referred to as epistemic uncertainty relations. Because some of them are obtained using sophisticated mathematics, and tend to be expressed in terms of (quasiclassical) preparations and measurements, it is not always made clear that these, too, arise from the fundamental Hilbert space structure of quantum theory. So far as is known at present, information inequalities derived in this way, such as in [38], are entirely consistent with experiment, in contrast to Bell inequalities and the like, which do not agree with experiment, and whose derivation is based in a fundamental way upon classical physics (see, e.g., [1]).

[5]The claim sometimes made that the precise information about w is "really" present in the particle but cannot be measured reminds one of the student who, having just failed the examination, tells the professor he understood the subject perfectly, but simply could not recall it during the test.

V. CONCLUSION

If one assumes, following von Neumann, and consistent with textbook quantum theory where the issue is not always clearly discussed, that properties of a quantum system correspond to subspaces of its Hilbert space, then there is a very natural epistemic restriction on what an external observer can know about a microscopic quantum system. The Hilbert space does not contain, has no room for, combinations of incompatible properties, such as $S_x = +1/2$ AND $S_z = -1/2$ for a spin-$1/2$ particle, and as a consequence these must be excluded from a consistent quantum ontology, as discussed in detail in [12]. And of course what does not exist cannot be known; such an ontological restriction leads automatically to an epistemic restriction.

Quantum textbooks already contain a version of this epistemic restriction. Students are told that incompatible quantum properties cannot be simultaneously measured. However, because textbooks treat measurements as a sort of axiom which is incapable of further analysis, a black box which cannot be pried open to see what is going on inside, the nature of this restriction remains clouded in a dense conceptual fog. One needs a consistent quantum analysis of measurements, one capable of resolving both the first and the second quantum measurement problems, to relate this restriction on measurements to mathematical properties of the Hilbert space used in quantum mechanics to represent physical properties. And, as noted in Sec. IV, some of the rigorous inequalities developed by quantum information theorists using the quantum Hilbert space are also epistemic restrictions.

In addition to resolving the measurement problems, the histories approach provides a foundation for understanding all the other strange, i.e., nonclassical, quantum phenomena, including those which cannot at present be obtained from a classical model by adding an epistemic restriction (see the discussion in Sec. V of [5]). This is because it has a consistent formulation of the fundamental principles of Hilbert space quantum mechanics, the principles that underlie the generally accepted calculational techniques taught in textbooks. It may be that applying still more restrictions, or perhaps additions, to classical models will eventually yield the correct quantum outcomes. But one can ask whether such a circuitous route, somewhat analogous to tweaking Bohr's semiclassical quantization condition, would be worthwhile, given the availability of a more direct path to understanding the phenomena in question.

This is not to say that the study of classical models of the sort considered in [4,5] is without value. It is surely of interest to understand the limits of classical physics when it is pushed as far as possible into the quantum domain. Not least because in the domain where classical physics functions very well, the quasiclassical regime of macroscopic phenomena, it provides a much simpler and easier calculational scheme than any full-scale quantum counterpart. Who would ever want to compute an earth satellite orbit starting with a wave function? However, such studies of classical models will, we believe, be most effective when combined with a consistent and complete microscopic quantum theory, one that takes full account of the noncommutation of quantum projectors, is not dependent upon a vague concept of "measurement," and does not require any additional epistemic restrictions beyond those implied by the formalism itself. It is hoped that the work presented here will contribute to that end.

ACKNOWLEDGMENTS

This article had its genesis in a very helpful conversation with Robert Spekkens, who pointed out to the author the significance of [5]. The research described here has been supported by the National Science Foundation through Grant No. PHY-1068331.

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