Comparing quantum cloning: A Fisher-information perspective

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Perfect cloning of an unknown quantum state is impossible. Approximate cloning, which is optimal in various senses, has been found in many cases. Paradigmatic examples are Wootters-Zurek cloning and universal cloning. These cloning machines aim at optimal cloning of the full quantum states. However, in practice, what is important and relevant may only involve partial information in quantum states, rather than quantum states themselves. For example, signals are often encoded as parameters in quantum states, whose information content is well synthesized by quantum Fisher information. This raises the basic issue of evaluating the information transferring capability (e.g., distributing quantum Fisher information) of quantum cloning. We assess and compare Wootters-Zurek cloning and universal cloning from this perspective and show that, on average, Wootters-Zurek cloning performs better than universal cloning for the phase (as well as amplitude) parameter, although they are incomparable individually, and universal cloning has many advantages over Wootters-Zurek cloning in other contexts. Physical insights and related issues are further discussed.

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I. INTRODUCTION

One of the fundamental features of quantum information radically different from those of classical information is no cloning of quantum information [1]. Various extensions and implications of the quantum no-cloning theorem are widely investigated in recent years [2–21]. Specifically, due to the superposition principle of quantum mechanics, nonorthogonal quantum states cannot be cloned perfectly, in stark contrast to the cloning of classical states. Consequently, one has to make some compromise in cloning quantum states, and various approximate or probabilistic cloning schemes have been proposed, with each possessing particular merits in its own context. Paradigmatic examples are Wootters-Zurek cloning [1], universal cloning [2,4], state-dependent cloning [5], phase-covariant cloning [9,10], etc. These cloning machines are important tools for studying a wide spectrum of foundational issues and practical tasks, e.g., the uncertainty principle, quantum state estimation, and eavesdropping in quantum cryptography.

While, theoretically, quantum states encapsulate the complete information of a system, in practice the relevant information is usually encoded in some parameters of the involved quantum states. Thus for the purpose of information processing, it is usually not necessary to clone whole quantum states themselves, but rather the relevant parameter information. In this work, motivated by distributing the information about certain parameters which are encoded in quantum states, rather than distributing the quantum state themselves, to several parties, we study the cloning of partial information in quantum states. More specifically, we consider the scenario of cloning a family of quantum states ρ_θ parameterized by θ (which encodes the signal of interest), such that as much information as possible about parameter θ, rather than the quantum states themselves, is distributed to two parties. We will quantify the information content of θ by virtue of the celebrated quantum Fisher information. Although, due to the subtle noncommutative nature of quantum theory, there are many different (actually infinitely many) useful versions of quantum Fisher information [22–29], here we adopt the most important and significant version, the one based on the symmetric logarithmic derivative [23–25]. This quantum Fisher information is the maximum of the measurement-induced classical Fisher information [25]. By comparing the effects of various quantum cloning machines on transferring and distributing the parameter information synthesized by quantum Fisher information, we will assess their performance and make a comparative study.

Recall that quantum Fisher information of parameterized quantum states ρ_θ (in general, mixed states) is defined as [23–25]

$$F(\rho_\theta) = \mathrm{tr}\rho_\theta L_\theta^2,$$

where L_θ, the symmetric logarithmic derivative, is determined by $\dot{\rho}_\theta = \frac{1}{2}(L_\theta\rho_\theta + \rho_\theta L_\theta)$, with $\dot{\rho}_\theta = \frac{d}{d\theta}\rho_\theta$. With the help of the spectral decomposition $\rho_\theta = \sum_j \lambda_j |j\rangle\langle j|$, the quantum Fisher information can be evaluated as

$$F(\rho_\theta) = \sum_{jk} \frac{2}{\lambda_j + \lambda_k} |\langle j|\dot{\rho}_\theta|k\rangle|^2. \quad (1)$$

Quantum Fisher information quantifies the statistical distinguishability about the parameter encoded in quantum states and is a central concept in quantum detection, estimation, and metrology [30–35].

By evaluating the performance of different quantum cloning machines with respect to quantum Fisher information, we get more insights into the nature of information transferring. In particular, we find that, on average, Wootters-Zurek cloning performs better than universal cloning in distributing quantum Fisher information for both the phase and amplitude parameters.

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The cloning of quantum Fisher information interpolates between the cloning of classical information (which is always possible) and the no cloning of quantum information and is of independent intrinsic interest. Cloning and broadcasting of quantum Fisher information have been recently explored from a general viewpoint in Ref. [36]. Some people distinguish between cloning and broadcasting by assuming that the output copies in a cloning process are uncorrelated, while those for broadcasting may be correlated [3]. In this paper we do not make such a distinction and identify cloning as broadcasting.

This paper is organized as follows. In Sec. II, we briefly describe Wootters-Zurek cloning and universal cloning. Section III is devoted to a generic scenario which illustrates the performance of the two cloning machines in distributing quantum Fisher information. The correlations and entanglement in the output cloning states are also compared. We finish the paper with a conclusion in Sec. IV. We will only consider the $1 \to 2$ symmetric cloning and a single-parameter case; the general cases for asymmetric multiple cloning and several parameters can be studied similarly.

II. CLONING MACHINES

Wootters-Zurek cloning machine on a qubit system, which arises from the first proof of the no-cloning theorem [1], is determined by the following linear transformation:

$$|0\rangle_a|Q\rangle_c \to |00\rangle_{ab}|0\rangle_c, \quad |1\rangle_a|Q\rangle_c \to |11\rangle_{ab}|1\rangle_c,$$

where $\{|0\rangle_a,|1\rangle_a\}$, $\{|0\rangle_b,|1\rangle_b\}$, and $\{|0\rangle_c,|1\rangle_c\}$ are bases of orthonormal states for the system, the clone, and the ancilla, respectively. This cloning machine induces the transformation $\mathcal{W}^{abc}: H^a \to H^a \otimes H^b \otimes H^c$ on a general input state $\alpha|0\rangle_a + \beta|1\rangle_a$:

$$\mathcal{W}^{abc}(\alpha|0\rangle_a + \beta|1\rangle_a) = \alpha|00\rangle_{ab}|0\rangle_c + \beta|11\rangle_{ab}|1\rangle_c,$$

where H^a, H^b, and H^c are the Hilbert spaces for the input system, the clone, and the ancilla, respectively. This cloning machine is clearly input state dependent: The basis states $|0\rangle_a$ and $|1\rangle_a$ are cloned perfectly, while any nontrivial superposition state $\alpha|0\rangle_a + \beta|1\rangle_a$ is altered with the coherence information lost. The various reduced states of Wootters-Zurek cloning will be denoted by $\mathcal{W}^{ab}(|\psi\rangle) = \mathrm{tr}_c\mathcal{W}^{abc}(|\psi\rangle), \mathcal{W}^{a}(|\psi\rangle) = \mathrm{tr}_b\mathcal{W}^{ab}(|\psi\rangle)$, and $\mathcal{W}^{b}(|\psi\rangle) = \mathrm{tr}_a\mathcal{W}^{ab}(|\psi\rangle)$.

Universal cloning machine on a qubit system, which arises from considering approximate cloning in order to circumvent the no-cloning theorem [2], is determined by the following linear transformation:

$$|0\rangle_a|Q\rangle_c \to \sqrt{\frac{2}{3}}|00\rangle_{ab}|0\rangle_c + \sqrt{\frac{1}{3}}|\Psi\rangle_{ab}|1\rangle_c,$$
$$|1\rangle_a|Q\rangle_c \to \sqrt{\frac{2}{3}}|11\rangle_{ab}|1\rangle_c + \sqrt{\frac{1}{3}}|\Psi\rangle_{ab}|0\rangle_c,$$

where $|\Psi\rangle_{ab} = \frac{1}{\sqrt{2}}(|01\rangle_{ab} + |10\rangle_{ab})$. This machine induces the transformation $\mathcal{U}^{abc}: H^a \to H^a \otimes H^b \otimes H^c$ on a general input state $\alpha|0\rangle_a + \beta|1\rangle_a$:

$$\mathcal{U}^{abc}(\alpha|0\rangle_a + \beta|1\rangle_a) = \sqrt{\frac{2}{3}}(\alpha|00\rangle_{ab}|0\rangle_c + \beta|11\rangle_{ab}|1\rangle_c) + \sqrt{\frac{1}{3}}|\Psi\rangle_{ab}(\alpha|1\rangle_c + \beta|0\rangle_c).$$

Universal cloning is input state independent in the sense that all pure states $\alpha|0\rangle_a + \beta|1\rangle_a$ are cloned equally well with respect to fidelity [2]. The various reduced states of universal cloning will be denoted by $\mathcal{U}^{ab}(|\psi\rangle) = \mathrm{tr}_c\mathcal{U}^{abc}(|\psi\rangle)$, $\mathcal{U}^{a}(|\psi\rangle) = \mathrm{tr}_b\mathcal{U}^{ab}(|\psi\rangle)$, and $\mathcal{U}^{b}(|\psi\rangle) = \mathrm{tr}_a\mathcal{U}^{ab}(|\psi\rangle)$.

These cloning machines are symmetric, act on pure states, and yield, in general, correlated clones. They exhibit different merits. It is shown that universal cloning has several advantages over Wootters-Zurek cloning [3]. Here we reveal another aspect of the comparison between these two cloning machines and show that, on average, Wootters-Zurek cloning performs better than universal cloning in distributing quantum Fisher information. For notational simplicity and without loss of clarity, we will omit the subscripts in the states for the system, clone, and ancilla hereafter, without causing any confusion. Thus we will write $|0\rangle$ instead of $|0\rangle_a$, $|00\rangle$ instead of $|00\rangle_{ab}$, etc.

III. COMPARING CLONING IN TERMS OF QUANTUM FISHER INFORMATION

In this section, we make a comparative study of Wootters-Zurek cloning and universal cloning, and illustrate their characteristics in distributing quantum Fisher information through a representative scenario in which the parameter is encoded in phase (as well as in amplitude). It should be emphasized that, in general, quantum Fisher information cannot be cloned when the output states are not correlated [36], just as unknown quantum states cannot be cloned. However, the output states of these two cloning machines are correlated in general.

Suppose the parameter θ is encoded in the phase of state $\sigma_\theta = |\Sigma_\theta\rangle\langle\Sigma_\theta|$ with

$$|\Sigma_\theta\rangle = \frac{1}{\sqrt{2}}(U|0\rangle + e^{i\theta}U|1\rangle), \quad \theta \in [0,2\pi),$$

where $U \in \mathrm{SU}(2)$, which in the computational basis $\{|0\rangle,|1\rangle\}$ can be represented as

$$U = u|0\rangle\langle 0| - \bar{v}|0\rangle\langle 1| + v|1\rangle\langle 0| + \bar{u}|1\rangle\langle 1| = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}, \tag{2}$$

with $|u|^2 + |v|^2 = 1$. The encoding basis, i.e., the unitary U, has to be revealed at some point in order to extract some information on θ from the cloned states.

The present scenario for information encoding is reminiscent of the Bennett-Brassard 1984 (BB84) encoding scheme in which two different orthonormal bases are used for encoding a bit [37]. Here we are using the whole family of orthonormal bases $\{U|0\rangle, U|1\rangle\}$ to encode a continuous parameter θ. We will consider the average effects when we do not have any information about U by assuming that U is uniformly distributed on SU(2). We will derive several informational quantities for any U. If we know more precise information about U, that is, a nontrivial prior distribution of U, then we can integrate with this distribution to yield the corresponding average quantities. A more precise correspondence between the present encoding and the BB84 encoding arises when we take U to be either the identity or the Hadamard matrix (corresponding to the two bases in BB84) with equal *a priori* probability. In BB84, one wants to extract a bit; here one wants to estimate θ. The fundamental information about θ is synthesized by quantum Fisher information, and thus the distribution of quantum Fisher information is of relevance here.

Quantum Fisher information of the original input state can be readily evaluated, by Eq. (1), as

$$F(\sigma_\theta) = 1,$$

which turns out to be independent of the parameter θ.

For Wootters-Zurek cloning, the output clone states are

$$\mathcal{W}^a(\sigma_\theta) = \mathcal{W}^b(\sigma_\theta) = \frac{1-x}{2}|0\rangle\langle 0| + \frac{1+x}{2}|1\rangle\langle 1|,$$

and quantum Fisher information can be evaluated as (see the Appendix)

$$F(\mathcal{W}^a(\sigma_\theta)) = F(\mathcal{W}^b(\sigma_\theta)) = \frac{y^2}{1-x^2},$$

where

$$x = 2\mathrm{Re}(uve^{-i\theta}) \text{ (real part)},$$
$$y = 2\mathrm{Im}(uve^{-i\theta}) \text{ (imaginary part)}.$$

In particular, if $u = 0, v = 1$, then $F(\mathcal{W}^a(\sigma_\theta)) = 0$. On the other hand, if $u = v = 1/\sqrt{2}$, then $F(\mathcal{W}^a(\sigma_\theta)) = 1$. These results are consistent with our intuition since Wootters-Zurek cloning perfectly copies the basis states $|0\rangle$ and $|1\rangle$ but completely destroys the coherence in the superposition state $(|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$.

Noting that $x^2 + y^2 = 4|uv|^2 \leqslant (|u|^2 + |v|^2)^2 = 1$, we readily see that $F(\mathcal{W}^a(\sigma_\theta)) \leqslant F(\sigma_\theta)$, which means that quantum Fisher information is always decreasing for the cloning process.

For universal cloning, the output clone states are

$$\mathcal{U}^a(\sigma_\theta) = \mathcal{U}^b(\sigma_\theta) = \frac{\mathbf{1}}{2} - \frac{1}{3}(x|0\rangle\langle 0| - x|1\rangle\langle 1| - z|0\rangle\langle 1| - \bar{z}|1\rangle\langle 0|),$$

where $\mathbf{1}$ is the identity operator and

$$z = u^2 e^{-i\theta} - \bar{v}^2 e^{i\theta}.$$

Quantum Fisher information can be evaluated as (see the Appendix)

$$F(\mathcal{U}^a(\sigma_\theta)) = F(\mathcal{U}^b(\sigma_\theta)) = \frac{4}{9},$$

which is not only independent of the choice of the input states but also independent of the parameter θ.

In general, $F(\mathcal{W}^a(\sigma_\theta))$ may be larger or smaller than $F(\mathcal{U}^a(\sigma_\theta))$, depending on the basis states $U|0\rangle$ and $U|1\rangle$. For example, for $u = 0, v = 1$, we have $F(\mathcal{W}^a(\sigma_\theta)) = 0 < F(\mathcal{U}^a(\sigma_\theta)) = 4/9$, while for $u = v = 1/\sqrt{2}$, we have $F(\mathcal{W}^a(\sigma_\theta)) = 1 > F(\mathcal{U}^a(\sigma_\theta)) = 4/9$.

To remove the dependence on the input states and to make a fair comparison between Wootters-Zurek cloning and universal cloning, we calculate the average quantum Fisher information (see the Appendix):

$$\bar{F}(\mathcal{W}^a(\sigma_\theta)) = \int_{\mathrm{SU}(2)} F(\mathcal{W}^a(\sigma_\theta))dU = \frac{1}{2},$$
$$\bar{F}(\mathcal{U}^a(\sigma_\theta)) = \int_{\mathrm{SU}(2)} F(\mathcal{U}^a(\sigma_\theta))dU = \frac{4}{9}.$$

The integration is with respect to the normalized Haar measure on SU(2). Here we see that, on average, the performance of Wootters-Zurek cloning is better than that of universal cloning in distributing quantum Fisher information of the phase parameter:

$$\bar{F}(\mathcal{W}^a(\sigma_\theta)) - \bar{F}(\mathcal{U}^a(\sigma_\theta)) = 1/18 \simeq 5.56\%.$$

It might be interesting to analyze the correlations in the cloning, which are related to the distribution of information among the two clones. When Wootters-Zurek cloning is performed on the input state σ_θ, the quantum mutual information (amount of the total correlations) of the joint two-clone state

$$\mathcal{W}^{ab}(\sigma_\theta) = \frac{1-x}{2}|00\rangle\langle 00| + \frac{1+x}{2}|11\rangle\langle 11|$$

is

$$I(\mathcal{W}^{ab}(\sigma_\theta)) = S(\mathcal{W}^a(\sigma_\theta)) + S(\mathcal{W}^b(\sigma_\theta)) - S(\mathcal{W}^{ab}(\sigma_\theta)) = H\left(\frac{1-x}{2}\right),$$

where $S(\rho) = -\mathrm{tr}\rho\log\rho$ is the von Neumann entropy, $H(p) = -p\log p - (1-p)\log(1-p)$ is the binary Shannon entropy function, and the logarithm is with respect to the natural base e. The correlations in $\mathcal{W}^{ab}(\sigma_\theta)$ are purely classical.

For universal cloning, the quantum mutual information of the joint two-clone state

$$\mathcal{U}^{ab}(\sigma_\theta) = \frac{1-x}{3}|00\rangle\langle 00| + \frac{1+x}{3}|11\rangle\langle 11| + \frac{1}{3}|\Psi\rangle\langle\Psi| + \frac{1}{3}\left(z\frac{|00\rangle\langle\Psi| + |\Psi\rangle\langle 11|}{\sqrt{2}} + \bar{z}\frac{|\Psi\rangle\langle 00| + |11\rangle\langle\Psi|}{\sqrt{2}}\right)$$

is

$$I(\mathcal{U}^{ab}(\sigma_\theta)) = 2H\left(\frac{1}{6}\right) - H\left(\frac{1}{3}\right) \simeq 0.2646,$$

which turns out to be independent of θ. Furthermore, the output two-clone state $\mathcal{U}^{ab}(\sigma_\theta)$ is entangled and its concurrence and entanglement of formation can be evaluated as 1/3 and 0.1298, respectively [38].

The amounts of average quantum mutual information are

$$\bar{I}(\mathcal{W}^{ab}(\sigma_\theta)) = \int_{\mathrm{SU}(2)} I(\mathcal{W}^{ab}(\sigma_\theta))dU = \frac{1}{2},$$
$$\bar{I}(\mathcal{U}^{ab}(\sigma_\theta)) = \int_{\mathrm{SU}(2)} I(\mathcal{U}^{ab}(\sigma_\theta))dU \simeq 0.2646.$$

One may wonder what happens if the parameter θ is encoded in the amplitude of the basis states in the superposition

$\gamma_\theta = |\Gamma_\theta\rangle\langle\Gamma_\theta|$, with

$$|\Gamma_\theta\rangle = \cos\frac{\theta}{2}U|0\rangle + \sin\frac{\theta}{2}U|1\rangle, \qquad \theta \in [0,\pi),$$

where $U \in \mathrm{SU}(2)$. It turns out that this case is completely similar to the phase-parameter encoding because phase encoding in the computational basis $\{|0\rangle,|1\rangle\}$ is equivalent to amplitude encoding in a Hadamard-type basis. More precisely, if we define the Hadamard-type basis $\{|+\rangle,|-\rangle\}$ as

$$|+\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}},$$

then

$$|\Gamma_\theta\rangle = e^{-i\theta/2}\frac{1}{\sqrt{2}}(U|+\rangle + e^{i\theta}U|-\rangle),$$

which has a similar structure to $|\Sigma_\theta\rangle$ if we ignore the global phase $e^{-i\theta/2}$. Consequently, all average quantities will be the same as the phase parameter case.

To summarize, for both the phase and amplitude parameters, Wootters-Zurek cloning performs better than universal cloning in distributing quantum Fisher information. The amount of total correlations (quantum mutual information) in the output two-clone state for Wootters-Zurek cloning is larger than that of universal cloning, although the former is purely classical and the latter is entangled.

Now we provide a physical insight into why Wootters-Zurek cloning outperforms universal cloning in the present context. On the one hand, it seems that the best way to extract information about parameter θ from a single copy of state $|\Sigma_\theta\rangle$ is to perform a projective measurement in a suitable basis. If the encoding basis is completely random, i.e., U is uniformly distributed on SU(2), then a projective measurement in any basis is optimal and equally suitable. This is precisely what Wootters-Zurek cloning does and can be interpreted as a measure-and-prepare protocol, where the state to be cloned is measured in the computational basis and the measurement outcomes are broadcast as the two clones prepared either in $|0\rangle$ or $|1\rangle$. The recipients of the clones can recover the measurement outcome by performing a measurement in the computational basis. On the other hand, universal cloning, which maximizes the average fidelity of the two clones, adds some noise to the cloned state. Therefore any projective measurement on a clone produced by universal cloning machine will be equivalent to a noisy measurement on the input state and thus yields less information than the noiseless one does.

IV. CONCLUSION

We have investigated and compared the performance of two cloning machines in distributing quantum Fisher information. For the input states considered in this paper, Wootters-Zurek cloning is still state dependent, and universal cloning remains "universal" in the sense that the performance with respect to distributing quantum Fisher information for the phase and amplitude parameters does not depend on the choice of the basis. While, in general, both of them lose some quantum Fisher information of the parameter during the cloning process, Wootters-Zurek cloning performs better than universal cloning on average for both the phase and the amplitude parameters. These results shed light onto a new aspect of quantum cloning.

We remark that we have only considered cloning machines on pure states; the situation will be more complicated and interesting for mixed states. Finally, the important and natural issue arises as which cloning machines can distribute and transfer quantum Fisher information optimally. Such cloning machines apparently depend strongly on the encoding structure of the parameters and are worthy of further investigations. In particular, if the *a priori* distribution of U would differ from the uniform distribution, then there is no reason to consider only Wootters-Zurek cloning or universal cloning, but rather a cloning which is optimal for a given *a priori* distribution.

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APPENDIX

Here we present detailed calculations of some quantities in the main text.

For Wootters-Zurek cloning, the quantities $F(\sigma_\theta)$ and $F(\mathcal{W}^a(\sigma_\theta))$ can be readily derived from Eq. (1) since σ_θ and $\mathcal{W}^a(\sigma_\theta)$ are already in their spectral decomposition forms.

For universal cloning, the spectral decompositions of the clones are

$$\mathcal{U}^a(\sigma_\theta) = \mathcal{U}^b(\sigma_\theta) = \frac{1}{6}|k_1\rangle\langle k_1| + \frac{5}{6}|k_2\rangle\langle k_2|,$$

with

$$|k_1\rangle = \frac{1}{\sqrt{2(1-x)}}\begin{pmatrix} z \\ x-1 \end{pmatrix},$$
$$|k_2\rangle = \frac{1}{\sqrt{2(1-x)}}\begin{pmatrix} 1-x \\ \bar{z} \end{pmatrix},$$

from which we readily obtain the quantum Fisher information $F(\mathcal{U}^a(\sigma_\theta))$ from Eq. (1).

Next, we evaluate the average quantum Fisher information $\bar{F}(\mathcal{W}^a(\sigma_\theta))$. For our purpose, without loss of generality and up to the local phase degree of basis states, we may simply parametrize $U \in \mathrm{SU}(2)$ defined by Eq. (2) via

$$u = \cos\frac{\phi}{2}, \quad v = e^{i\delta}\sin\frac{\phi}{2}, \quad \phi \in [0,\pi), \quad \delta \in [0,2\pi);$$

then the Haar integration here can be equivalently performed on $[0,\pi) \times [0,2\pi)$ with respect to the measure $\frac{1}{4\pi}\sin\phi d\phi d\delta$.

Noting the periodicity of trigonometric functions, we have

$$\bar{F}(\mathcal{W}^a(\sigma_\theta)) = \int_{\mathrm{SU}(2)} F(\mathcal{W}^a(\sigma_\theta))dU = \int_0^\pi \int_0^{2\pi} F(\mathcal{W}^a(\sigma_\theta))\frac{\sin\phi}{4\pi}d\delta d\phi = \int_0^\pi \frac{\sin^3\phi}{4\pi}\int_0^{2\pi}\frac{\sin^2(\delta-\theta)}{1-\sin^2\phi\cos^2(\delta-\theta)}d\delta d\phi$$
$$= \int_0^\pi \frac{\sin^3\phi}{2\pi}\int_0^\pi \frac{\sin^2\delta}{1-\sin^2\phi\cos^2\delta}d\delta d\phi = \int_0^\pi\left(\frac{\sin\phi}{2} - \frac{\sin\phi\cos^2\phi}{2\pi}\int_0^\pi\frac{1}{1-\sin^2\phi\cos^2\delta}d\delta\right)d\phi.$$

From the indefinite integral ($\frac{b}{a} > -1$)

$$\int \frac{dt}{a+b\cos^2 t} = \frac{-\mathrm{sgn}a}{\sqrt{a(a+b)}}\arctan\left(\sqrt{\frac{a+b}{a}}\cot t\right),$$

we obtain

$$\int_0^\pi \frac{1}{1-\sin^2\phi\cos^2\delta}d\delta = \frac{\pi}{|\cos\phi|},$$

which yields

$$\bar{F}(\mathcal{W}^a(\sigma_\theta)) = \int_0^\pi\left(\frac{\sin\phi}{2} - \frac{\sin\phi\cos^2\phi}{2\pi}\frac{\pi}{|\cos\phi|}\right)d\phi = \frac{1}{2}.$$

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