# Non-self-adjoint model of a two-dimensional noncommutative space with an unbound metric

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We demonstrate that a non-self-adjoint Hamiltonian of harmonic-oscillator type defined on a two-dimensional noncommutative space can be diagonalized exactly by making use of pseudobosonic operators. The model admits an antilinear symmetry and is of the type studied in the context of $\mathcal{PT}$-symmetric quantum mechanics. Its eigenvalues are computed to be real for the entire range of the coupling constants and the biorthogonal sets of eigenstates for the Hamiltonian and its adjoint are explicitly constructed. We show that despite the fact that these sets are complete and biorthogonal, they involve an unbounded metric operator and therefore do not constitute (Riesz) bases for the Hilbert space $\mathcal{L}^2(\mathbb{R}^2)$, but instead only $\mathcal{D}$ quasibases. As recently proved by one of us, this is sufficient to deduce several interesting consequences.

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## I. INTRODUCTION

In the past 15 years more and more physicists and mathematicians have developed an interest in non-Hermitian and non-self-adjoint operators possessing real eigenvalues. Such type of models have been investigated before, but the more recent interest has been initiated by the seminal paper [1] in which the complex cubic potential and its close relatives have been studied. The original considerations, focusing mainly on the aspect of the possibility to formulate consistent quantum-mechanical systems, have broadened quickly and are partly replaced by a more general analysis of related aspects. Many experiments [2–5] have now been carried out, mainly for optical analogues to the quantum-mechanical systems, exploiting $\mathcal{PT}$-symmetric phase transitions where real eigenvalues merge into two complex-conjugate pairs, to obtain gain and loss structures. We refer the reader to [6–8] for some reviews on what is commonly named *quasi-Hermitian* [9,10], *pseudo-Hermitian* [11,12], or *$\mathcal{PT}$-symmetric* [1,13] quantum mechanics. However, it was recently pointed out by Krejcirik and Siegl [14] that more mathematically oriented treatments of these type of Hamiltonians are required, as for instance the complex cubic potential lacks possessing a Riesz basis of eigenstates. Therefore, we can still not associate a standard quantum-mechanical interpretation to this model. The purpose of this paper is to shed more light on these issues.

Modifying recent ideas [15], one of us has recently introduced the notion of $\mathcal{D}$ pseudobosons ($\mathcal{D}$-PBs), [16], and used them in connection with several physical systems, whose Hamiltonians are non-self-adjoint operators [17]. Among other aspects, it was shown that $\mathcal{D}$-PBs could be useful in the analysis of a two-dimensional harmonic oscillator described by the Hamiltonian

$$\hat{H} = \tfrac{1}{2}\left(\hat{p}_1^2 + \hat{x}_1^2\right) + \tfrac{1}{2}\left(\hat{p}_2^2 + \hat{x}_2^2\right) + i[A(\hat{x}_1 + \hat{x}_2) + B(\hat{p}_1 + \hat{p}_2)], \qquad (1.1)$$

where $(\hat{x}_j, \hat{p}_j)$ are noncommutative operators satisfying $[\hat{x}_j, \hat{p}_k] = i\delta_{j,k}\mathbb{1}$, $[\hat{x}_j, \hat{x}_k] = i\theta\epsilon_{j,k}\mathbb{1}$, and $[\hat{p}_j, \hat{p}_k] = i\tilde{\theta}\epsilon_{j,k}\mathbb{1}$, where $\theta$ and $\tilde{\theta}$ are two real small parameters, measuring the noncommutativity of the system. In [17] a perturbative expansion in $\theta$ and $\tilde{\theta}$ was set up and it was shown, in particular, that if one neglects all the terms which are at least quadratic in $\theta$ and $\tilde{\theta}$ we can construct explicitly the eigenvectors of (the approximated version of) $\hat{H}$ and deduce the related eigenvalues.

In this paper we show that, if the noncommutativity is restricted to the spatial variables only, i.e., if $\tilde{\theta} = 0$, then $\hat{H}$, and a slightly generalized version of it, can be exactly diagonalized in terms of $\mathcal{D}$-PBs. The corresponding eigenbases are biorthonormal, but involve a metric operator that is unbounded, together with its inverse. Thus we will draw a similar conclusion as reached in [14] and, more recently, in [18].

It is surely worth to underline that these results, all together, suggest that several common beliefs usually taken for granted in the physical literature on these topics require some more care than usually adopted. For instance, in [19] (as well as in many other papers [20]), the biorthogonal sets of eigenstates of a rather general $H$, with $H^\dagger \neq H$, are used to produce a resolution of the identity. In other words, they are used as bases in the Hilbert space. However, the results in [14,18], and those given in this paper, show that this is not always possible, even for extremely simple models. This, we believe, helps to clarify the situation, showing that many claims need to be analyzed in more detail.

This article is organized as follows: in the next section we review the definition and a few central results on $\mathcal{D}$-PBs. In Sec. III we introduce the two-dimensional-harmonic oscillator with linear term in the momenta and position on a noncommutative flat space and we analyze it in terms of $\mathcal{D}$-PBs. We provide the computation of how it may be written in terms of $\mathcal{D}$-PB number operators and subsequently we verify the underlying assumptions, which are needed to have something more than just a formal theory. This will allow for the construction of biorthonormal sets, which are, however, shown not to be Riesz bases and not even bases, but just $\mathcal{D}$ quasibases. Our conclusions are stated in Sec. IV.

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## II. PSEUDOBOSONS, GENERALITIES

We briefly review here a few definitions and central properties of $\mathcal{D}$-PBs. More details can be found in [16].

Let $\mathcal{H}$ be a given Hilbert space with scalar product $\langle .,. \rangle$ and related norm $\|.\|$. Furthermore, let $a$ and $b$ be two operators acting on $\mathcal{H}$, with domains $D(a)$ and $D(b)$, respectively, $a^\dagger$ and $b^\dagger$ their respective adjoints, and let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $a^\sharp\mathcal{D} \subseteq \mathcal{D}$ and $b^\sharp\mathcal{D} \subseteq \mathcal{D}$, where $x^\sharp$ is $x$ or $x^\dagger$. It is worth noticing that we are not requiring here that $\mathcal{D}$ coincides with either $D(a)$ or $D(b)$. Nevertheless, for obvious reasons, $\mathcal{D} \subseteq D(a^\sharp)$ and $\mathcal{D} \subseteq D(b^\sharp)$.

*Definition*. The operators $(a,b)$ are $\mathcal{D}$ pseudobosonic if, for all $f \in \mathcal{D}$, we have

$$abf - baf = f. \tag{2.1}$$

Sometimes, to simplify the notation, instead of (2.1) we will simply write $[a,b] = \mathbb{1}$, having in mind that both sides of this equation have to act on $f \in \mathcal{D}$.

Our working assumptions are the following.

*Assumption $\mathcal{D}$-PB1*. There exists a nonzero $\varphi_0 \in \mathcal{D}$ such that $a\varphi_0 = 0$.

*Assumption $\mathcal{D}$-PB2*. There exists a nonzero $\Psi_0 \in \mathcal{D}$ such that $b^\dagger\Psi_0 = 0$.

Then, if $(a,b)$ satisfy the above definition, it is obvious that $\varphi_0 \in D^\infty(b)$ and that $\Psi_0 \in D^\infty(a^\dagger)$, with $D^\infty(x)$ denoting the common domain of all powers of $x$. Thus we can define the following vectors, all belonging to $\mathcal{D}$:

$$\varphi_n := \frac{1}{\sqrt{n!}} b^n \varphi_0, \quad \Psi_n := \frac{1}{\sqrt{n!}} {a^\dagger}^n \Psi_0, \tag{2.2}$$

for $n \geqslant 0$. As in [16] we introduce the sets $\mathcal{F}_\Psi = \{\Psi_n,\ n \geqslant 0\}$ and $\mathcal{F}_\varphi = \{\varphi_n,\ n \geqslant 0\}$. Once again, since $\mathcal{D}$ is stable under the action of $a^\sharp$ and $b^\sharp$, we deduce that each $\varphi_n$ and each $\Psi_n$ belongs to the domains of $a^\sharp$, $b^\sharp$, and $N^\sharp$, where $N := ba$.

It is now straightforward to deduce the following lowering and raising relations:

$$\begin{aligned} &a\varphi_n = \sqrt{n}\,\varphi_{n-1}, \quad a\varphi_0 = 0,\\ &\quad b^\dagger\Psi_n = \sqrt{n}\,\Psi_{n-1}, \quad b^\dagger\Psi_0 = 0, \quad \text{for } n \geqslant 1,\\ &a^\dagger\Psi_n = \sqrt{n+1}\Psi_{n+1},\\ &\quad b\varphi_n = \sqrt{n+1}\varphi_{n+1}, \quad \text{for } n \geqslant 0, \end{aligned} \tag{2.3}$$

as well as the following eigenvalue equations: $N\varphi_n = n\varphi_n$ and $N^\dagger\Psi_n = n\Psi_n$ for $n \geqslant 0$. As a consequence of these equations, choosing the normalization of $\varphi_0$ and $\Psi_0$ in such a way that $\langle \varphi_0, \Psi_0 \rangle = 1$, we deduce that

$$\langle \varphi_n, \Psi_m \rangle = \delta_{n,m}, \tag{2.4}$$

for all $n,m \geqslant 0$. The third assumption originally introduced in [16] is the following.

*Assumption $\mathcal{D}$-PB3*. $\mathcal{F}_\varphi$ is a basis for $\mathcal{H}$.

This is equivalent to the request that $\mathcal{F}_\Psi$ is a basis for $\mathcal{H}$ as well [16]. In particular, if $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are Riesz bases for $\mathcal{H}$, the $\mathcal{D}$-PBs were called *regular*.

In [16] also a weaker version of Assumption $\mathcal{D}$-PB3 has been introduced, useful for concrete physical applications: for that, let $\mathcal{G}$ be a suitable dense subspace of $\mathcal{H}$. Two biorthogonal sets $\mathcal{F}_\eta = \{\eta_n \in \mathcal{G}, g \geqslant 0\}$ and $\mathcal{F}_\Phi = \{\Phi_n \in \mathcal{G}, g \geqslant 0\}$ were called $\mathcal{G}$ *quasibases* if, for all $f,g \in \mathcal{G}$, the following holds:

$$\langle f,g \rangle = \sum_{n\geqslant 0} \langle f,\eta_n \rangle \langle \Phi_n, g \rangle = \sum_{n\geqslant 0} \langle f,\Phi_n \rangle \langle \eta_n, g \rangle. \tag{2.5}$$

It is clear that, while Assumption $\mathcal{D}$-PB3 implies (2.5), the reverse is false. However, if $\mathcal{F}_\eta$ and $\mathcal{F}_\Phi$ satisfy (2.5), we still have some (weak) form of resolution of the identity. Now Assumption $\mathcal{D}$-PB3 is replaced by the following.

*Assumption $\mathcal{D}$-PBW3*. $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{G}$ quasibases.

Let now assume that Assumption $\mathcal{D}$-PB1, $\mathcal{D}$-PB2, and $\mathcal{D}$-PBW3 are satisfied, with $\mathcal{G} = \mathcal{D}$, and let us consider a self-adjoint, invertible, operator $\Theta$, which leaves, together with $\Theta^{-1}$, $\mathcal{D}$ invariant: $\Theta\mathcal{D} \subseteq \mathcal{D}$, $\Theta^{-1}\mathcal{D} \subseteq \mathcal{D}$. Then, as in [16], we say that $(a,b^\dagger)$ are $\Theta$ conjugate if $af = \Theta^{-1}b^\dagger\Theta f$, for all $f \in \mathcal{D}$. Moreover, we can check that, for instance, $(a,b^\dagger)$ are $\Theta$ conjugate if and only if $(b,a^\dagger)$ are $\Theta$ conjugate and that, assuming that $\langle \varphi_0, \Theta\varphi_0 \rangle = 1$, $(a,b^\dagger)$ are $\Theta$ conjugate if and only if $\Psi_n = \Theta\varphi_n$, for all $n \geqslant 0$. Finally, if $(a,b^\dagger)$ are $\Theta$ conjugate, then $\langle f, \Theta f \rangle > 0$ for all nonzero $f \in \mathcal{D}$. The details of these proofs can be found in [18]. Notice also that, not surprisingly, we also deduce that $Nf = \Theta^{-1}N^\dagger\Theta f$, for all $f \in \mathcal{D}$.

## III. NONCOMMUTATIVE TWO-DIMENSIONAL HARMONIC OSCILLATOR WITH LINEAR TERMS

Let us now consider the non-self-adjoint two-dimensional harmonic oscillator with linear terms in the momenta and positions

$$\begin{aligned} \tilde{H} = {}& \frac{1}{2m}\left(\tilde{p}_1^2 + \tilde{p}_2^2\right) + \frac{m\omega^2}{2}\left(\tilde{x}_1^2 + \tilde{x}_2^2\right) + i\alpha_1\tilde{x}_1\\ &+ \alpha_2\tilde{x}_2 + \alpha_3\tilde{p}_1 + i\alpha_4\tilde{p}_2, \end{aligned} \tag{3.1}$$

on the noncommutative flat space with the nonvanishing commutators $[\tilde{x}_1,\tilde{x}_2] = i\theta$, $[\tilde{x}_j,\tilde{p}_j] = i\hbar$ for $j = 1,2$. Here $\theta$ and $\alpha_i$ for $i = 1,2,3,4$ are real dimensionful parameters. Note that this Hamiltonian is non-self-adjoint even when viewed on a standard space. However, $\tilde{H}$ is constructed in such a way that it is left invariant with respect to the antilinear symmetry $\mathcal{PT}_-$: $\tilde{x}_1 \to -\tilde{x}_1$, $\tilde{x}_2 \to \tilde{x}_2$, $\tilde{p}_1 \to \tilde{p}_1$, $\tilde{p}_2 \to -\tilde{p}_2$, and $i \to -i$ [21]. Thus in the general spirit of $\mathcal{PT}$-symmetric quantum mechanics [1,13] the Hamiltonian is guaranteed to have real eigenvalues provided that its eigenfunctions are eigenstates of $\mathcal{PT}_-$. Evidently, in atomic units, $m = \omega = \hbar = 1$, $\tilde{H}$ reduces to $\hat{H}$ for $\alpha_1 \to A$, $\alpha_2 \to -iA$, $\alpha_3 \to iB$, and $\alpha_4 \to B$. We also notice that $\mathcal{PT}_-$ is no longer a symmetry of $\hat{H}$, i.e., $[\mathcal{PT}_-, \hat{H}] \neq 0$.

Our aim here is to employ $\mathcal{D}$-PBs to diagonalize $\tilde{H}$ exactly, instead of using a perturbative approach as in [17,22] and to determine its spectrum. For this purpose we convert the Hamiltonian first from a flat noncommutative space to one in terms of standard canonical variables $x_i$ and $p_i$ for $i = 1,2$ satisfying the canonical commutation relations $[x_j,p_j] = i\hbar$ and $[x_i,x_j] = [p_i,p_j] = 0$. This is achieved by a standard Bopp shift $\tilde{x}_1 \to x_1 - \frac{\theta}{2\hbar}p_2$, $\tilde{x}_2 \to x_2 + \frac{\theta}{2\hbar}p_1$, $p_1 \to p_1$, and

$p_2 \to p_2$. The Hamiltonian in (3.1) then acquires the form

$$\tilde{H} = \left(\frac{1}{2m} + \frac{m\omega^2\theta^2}{8\hbar^2}\right)(p_1^2 + p_2^2) + \frac{m\omega^2}{2}(x_1^2 + x_2^2) + \frac{m\omega^2\theta}{2\hbar}(x_2 p_1 - x_1 p_2) + i\alpha_1 x_1 + \alpha_2 x_2 + \left(\alpha_3 + \frac{\alpha_2\theta}{2\hbar}\right)p_1 + i\left(\alpha_4 - \frac{\alpha_1\theta}{2\hbar}\right)p_2. \quad (3.2)$$

We now attempt to reexpress this Hamiltonian in terms of pseudobosonic number operators $N_i = b_i a_i$ as

$$\tilde{H} = \gamma_1 N_1 + \gamma_2 N_2 + \gamma_0 \quad \text{for} \quad \gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}, \quad (3.3)$$

where the operators $a_i$ and $b_i$ obey the two-dimensional pseudobosonic commutation relations

$$[a_j, b_k] = i\delta_{jk}, \quad [a_j, a_k] = [b_j, b_k] = 0, \quad \text{for} \quad j,k = 1,2. \quad (3.4)$$

For this purpose we represent the pseudobosonic operators $a_i$ and $b_i$ in terms of standard bosonic creation and annihilation operators $A_i^\dagger$ and $A_i$, respectively,

$$a_1 = \frac{1}{\sqrt{2}}(A_1 + iA_2) + i\beta_1, \quad b_1 = \frac{1}{\sqrt{2}}(A_1^\dagger - iA_2^\dagger) + i\beta_3, \quad (3.5)$$

$$a_2 = -\frac{1}{\sqrt{2}}(iA_1 + A_2) + \beta_2, \quad b_2 = \frac{1}{\sqrt{2}}(iA_1^\dagger - A_2^\dagger) + \beta_4, \quad (3.6)$$

with $[A_j, A_k^\dagger] = i\delta_{jk}$, $[A_j, A_k] = [A_j^\dagger, A_k^\dagger] = 0$ for $j,k = 1,2$ and $\beta_i \in \mathbb{C}$ for $i = 1,2,3,4$. Furthermore, we represent the $A_i^\dagger$ and $A_i$ in terms of the standard canonical variables

$$A_1 = \sqrt{\frac{M\omega}{2\hbar}}x_1 + i\sqrt{\frac{1}{2\hbar M\omega}}p_1, \quad A_2 = \sqrt{\frac{M\omega}{2\hbar}}x_2 + i\sqrt{\frac{1}{2\hbar M\omega}}p_2, \quad (3.7)$$

$$A_1^\dagger = \sqrt{\frac{M\omega}{2\hbar}}x_1 - i\sqrt{\frac{1}{2\hbar M\omega}}p_1, \quad A_2^\dagger = \sqrt{\frac{M\omega}{2\hbar}}x_2 - i\sqrt{\frac{1}{2\hbar M\omega}}p_2. \quad (3.8)$$

We note that the pseudobosonic operators reduce to standard boson operators with $b_i = a_i^\dagger$ if and only if for $\beta_1 = -\bar{\beta}_3$ and $\beta_2 = \bar{\beta}_4$. Upon substitution we compare now (3.3) and (3.2), which become identical subject to the constraints

$$\beta_1 = \frac{\Omega(\alpha_1 + \alpha_2) + 2\hbar m\omega(\alpha_3 - \alpha_4)}{(\Omega + \theta m\omega)\sqrt{2m\Omega\omega^3}}, \quad \beta_2 = \frac{\Omega(\alpha_1 - \alpha_2) + 2\hbar m\omega(\alpha_3 + \alpha_4)}{(\Omega - \theta m\omega)\sqrt{2m\Omega\omega^3}}, \quad (3.9)$$

$$\beta_3 = \frac{\Omega(\alpha_1 - \alpha_2) - 2\hbar m\omega(\alpha_3 + \alpha_4)}{(\Omega + \theta m\omega)\sqrt{2m\Omega\omega^3}}, \quad \beta_4 = \frac{-\Omega(\alpha_1 + \alpha_2) + 2\hbar m\omega(\alpha_3 - \alpha_4)}{(\Omega - \theta m\omega)\sqrt{2m\Omega\omega^3}}, \quad (3.10)$$

$$\gamma_0 = \frac{1}{2}\omega\left[\Omega(1 + \beta_1\beta_3 - \beta_2\beta_4) + \theta m\omega(\beta_1\beta_3 + \beta_2\beta_4)\right], \quad (3.11)$$

$$\gamma_1 = \frac{1}{2}\omega(\Omega + \theta m\omega), \quad \gamma_2 = \frac{1}{2}\omega(\Omega - \theta m\omega), \quad M = \frac{2m\hbar}{\Omega}, \quad (3.12)$$

where $\Omega := \sqrt{4\hbar^2 + \theta^2 m^2\omega^2}$. If we are now able to construct eigenstates $\Psi_{\underline{n}}$ for the pseudobosonic number operators such that $N_i\varphi_{\underline{n}} = \hbar\omega n_i\varphi_{\underline{n}}$, the eigenvalues for $\tilde{H}$ are immediately computed from (3.3) to

$$E_{n_1,n_2} = \gamma_1\hbar\omega n_1 + \gamma_2\hbar\omega n_2 + \gamma_0. \quad (3.13)$$

We observe from (3.9) to (3.12) that the constants $\gamma_i \in \mathbb{R}$ for $i = 0,1,2$ are real and consequently the energy $E_{n_1,n_2}$ is also real. Furthermore, we observe that the presence of the linear terms in (3.1), that is $\alpha_i \neq 0$ for $i = 1,2,3,4$, prevents us from using a standard bosonic oscillator algebra and we are forced to employ pseudobosons. This is seen from the fact that the pseudobosonic operators reduce to standard boson operators if and only if for $\beta_1 = -\bar{\beta}_3$ and $\beta_2 = \bar{\beta}_4$. However, our constraints (3.9) and (3.10) imply that in this boson case some linear terms in our Hamiltonian have to vanish, that is $\alpha_1 = \alpha_4 = 0$.

Furthermore, we notice that for the reduction of $\tilde{H}$ to $\hat{H}$ for $\alpha_1 \to A, \alpha_2 \to iA, \alpha_3 \to iB, \alpha_4 \to B$ we obtain $\beta_1 = \bar{\beta}_3$ and $\beta_2 = -\bar{\beta}_4$, such that $\gamma_0$ and therefore $E_{n_1,n_2}$ remain real. In this case the $\mathcal{PT}_-$ symmetry is broken and it remains unclear which antilinear symmetry, if any, is responsible for keeping the spectrum real.

Let us now verify that eigenstates $\varphi_{\underline{n}}$ and those of the adjoint of the Hamiltonian, $\Psi_{\underline{n}}$ are well defined, really exist, and most crucially whether they constitute a Riesz basis, or even a basis.

### A. Verification of the pseudobosonic assumptions

For simplicity let us now adopt atomic units. We commence by introducing the operators

$$\hat{a}_i := \lim_{\beta_i \to 0} a_i, \quad \hat{a}_i^\dagger := \lim_{\beta_i \to 0} b_i, \quad (3.14)$$

which, from (3.5) and(3.6), satisfy the standard bosonic canonical commutation relations, $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{i,j}\,\mathbb{1}$, $[\hat{a}_i, \hat{a}_j] = 0$, for $i,j = 1,2$. Then, introducing the unitary operators

$$D_i(z) := \exp\{\bar{z}\hat{a}_i - z\hat{a}_i^\dagger\}, \quad D(\underline{z}) := D_1(z_1)D_2(z_2), \quad (3.15)$$

we compute

$$a_i = \hat{a}_i + \nu_i = D(\underline{\nu})\hat{a}_i D^{-1}(\underline{\nu}), \quad b_i = \hat{a}_i^\dagger + \mu_i = D(\underline{\mu})\hat{a}_i^\dagger D^{-1}(\underline{\mu}), \quad (3.16)$$

for $i = 1,2$ with $\underline{\nu} := \{i\beta_1, \beta_2\}$, $\underline{\mu} := \{-i\bar{\beta}_3, \bar{\beta}_4\}$. An orthonormal basis for $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$ is then constructed easily. Let $e_{0,0} = e_{\underline{0}}$ be the vacuum of $\hat{a}_1$ and $\hat{a}_2$, that is $\hat{a}_i e_{\underline{0}} = 0$ for $i = 1,2$. Then, as is common for the purely bosonic case, we introduce

$$e_{n_1,n_2} = e_{\underline{n}} := \frac{1}{\sqrt{n_1!n_2!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}e_{\underline{0}}, \quad (3.17)$$

and the related orthonormal basis $\mathcal{F}_e = \{e_{\underline{n}}, n_1, n_2 \geqslant 0\}$. Of course, for the bosonic number operator $\hat{n}_i := \hat{a}_i^\dagger\hat{a}_i$, we have $\hat{n}_i e_{\underline{n}} = n_i e_{\underline{n}}$.

In order to verify the assumptions of Sec. II, we first seek to construct $\varphi_{\underline{0}}$, i.e., the vacuum of $a_i$ satisfying $a_1\varphi_{\underline{0}} = a_2\varphi_{\underline{0}} = 0$.

Evidently, this holds if, and only if, $\hat{a}_i[D^{-1}(\underline{\nu})\varphi_{\underline{0}}] = 0$ for $i = 1,2$. This implies that $\varphi_{\underline{0}} = D(\underline{\nu})e_{\underline{0}}$, up to a normalization which will be fixed below. Notice that, due to the fact that $D(\underline{\nu})$ is unitary, and therefore everywhere defined, $\varphi_{\underline{0}}$ is well defined.

Similarly, we derive $\Psi_{\underline{0}}$, the vacuum for $b_j^\dagger$. We require $b_1^\dagger\Psi_{\underline{0}} = b_2^\dagger\Psi_{\underline{0}} = 0$, which can be rewritten as $\hat{a}_i[D^{-1}(\underline{\mu})\Psi_{\underline{0}}] = 0$ for $i = 1,2$. These equations are solved by $\Psi_{\underline{0}} = N_\Psi D(\underline{\mu})e_{\underline{0}}$, which, due to the unitarity of $D(\underline{\mu})$, is again well defined. Here $N_\Psi$ is a normalization needed to ensure the normalization $\langle\varphi_{\underline{0}},\Psi_{\underline{0}}\rangle = 1$. It is computed to

$$N_\Psi^2 = \frac{\langle\varphi_{\underline{0}},\varphi_{\underline{0}}\rangle}{\langle\Psi_{\underline{0}},\Psi_{\underline{0}}\rangle} = \exp[|\beta_1|^2 + |\beta_2|^2 - |\beta_3|^2 - |\beta_4|^2 - 2\,\mathrm{Re}(\beta_1\beta_2) - 2\,\mathrm{Re}(\beta_3\beta_4)]. \tag{3.18}$$

Evidently for $\beta_2 = -\beta_3$ and $\beta_1 = \beta_4$ this reduces to the standard bosonic normalization, as is expected.

*Remark*. These results could have also been found quite easily by solving the equations directly in the coordinate representation. For instance, $a_1\varphi_{\underline{0}} = a_2\varphi_{\underline{0}} = 0$ are equivalent to the differential equations

$$\left(x_1 + \partial_{x_1} + ix_2 + i\partial_{x_2} + 2i\beta_1\right)\varphi_{\underline{0}}(x_1,x_2) = \left(-ix_1 - i\partial_{x_1} - x_2 - \partial_{x_2} + 2\beta_2\right)\varphi_{\underline{0}}(x_1,x_2) = 0, \tag{3.19}$$

solved by $\varphi_{\underline{0}}(x_1,x_2) \propto e^{-\frac{1}{2}(x_1^2+x_2^2)-i(\beta_1+\beta_2)x_1-(\beta_1-\beta_2)x_2}$. Similarly, we find $\Psi_{\underline{0}}(x_1,x_2) \propto e^{-\frac{1}{2}(x_1^2+x_2^2)+i(\beta_3-\beta_4)x_1+(\beta_3+\beta_4)x_2}$. We see that both of these functions belong, for instance, to the set $\mathcal{S}(\mathbb{R}^2)$ of $C^\infty$ functions which, together with their derivatives, decrease faster to zero than any inverse power of $x_1$ and $x_2$. However, this property might not be enough for our purposes, since, as we have outlined in Sec. II, we need to identify a set $\mathcal{D}$, dense in $\mathcal{H}$, which not only contains $\varphi_{\underline{0}}$ and $\Psi_{\underline{0}}$, but which is in addition also stable under the action of $a_j^\sharp$, $b_j^\sharp$, and other relevant operators. It is convenient to introduce, therefore, the following set:

$$\mathcal{D} = \{f(x_1,x_2) \in \mathcal{S}(\mathbb{R}^2), \text{ such that } e^{k_1x_1+k_2x_2}f(x_1,x_2) \in \mathcal{S}(\mathbb{R}^2), \quad \forall k_1,k_2 \in \mathbb{C}\}. \tag{3.20}$$

$\mathcal{D}$ is dense in $\mathcal{H}$, since it contains the set $D(\mathbb{R}^2)$ of the $C^\infty$ functions with compact support.

Following Sec. II, we are now interested in deducing the properties of the vectors $\varphi_{\underline{n}} = \frac{1}{\sqrt{n_1!n_2!}}b_1^{n_1}b_2^{n_2}\varphi_{\underline{0}}$ and $\Psi_{\underline{n}} = \frac{1}{\sqrt{n_1!n_2!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\Psi_{\underline{0}}$. We notice that both $\varphi_{\underline{n}}$ and $\Psi_{\underline{n}}$ necessarily belong to $\mathcal{D}$ for all $\underline{n}$, because of the stability of $\mathcal{D}$ under the action of $b_i$ and $a_i^\dagger$, and the previously established fact that $\varphi_{\underline{0}},\Psi_{\underline{0}} \in \mathcal{D}$. The formulas (3.16) state how the pseudobosonic operators $(a_i,b_i)$ are related to the bosonic operators $(\hat{a}_i,\hat{a}_i^\dagger)$ by means of the in general two different unitary operators $D(\underline{\nu})$ and $D(\underline{\mu})$.

A single operator could be used if we introduce the operators:

$$V_i(z,w) := \exp\{\bar{w}\hat{a}_i - z\hat{a}_i^\dagger\}, \qquad V(\underline{\nu},\underline{\mu}) := V_1(\nu_1,\mu_1)V_2(\nu_2,\mu_2). \tag{3.21}$$

Now we compute

$$a_i = V(\underline{\nu},\underline{\mu})\hat{a}_iV^{-1}(\underline{\nu},\underline{\mu}), \quad b_i = V(\underline{\nu},\underline{\mu})\hat{a}_i^\dagger V^{-1}(\underline{\nu},\underline{\mu}), \tag{3.22}$$

which, in contrast to (3.16), only involve a single, albeit in general unbounded, operator to relate the $(a_i,b_i)$ to the $(\hat{a}_i,\hat{a}_i^\dagger)$. We also check directly

$$a_i^\dagger = V(\underline{\mu},\underline{\nu})\hat{a}_iV^{-1}(\underline{\mu},\underline{\nu}), \qquad b_i^\dagger = V(\underline{\mu},\underline{\nu})\hat{a}_iV^{-1}(\underline{\mu},\underline{\nu}). \tag{3.23}$$

An immediate consequence of these formulas are the following relations between the various number operators: $\hat{n}_i = V^{-1}(\underline{\nu},\underline{\mu})N_iV(\underline{\nu},\underline{\mu}) = V^{-1}(\underline{\mu},\underline{\nu})N_i^\dagger V(\underline{\mu},\underline{\nu})$, which in turn implies that

$$N_i = T(\underline{\nu},\underline{\mu})N_i^\dagger T^{-1}(\underline{\nu},\underline{\mu}), \tag{3.24}$$

where $T(\underline{\nu},\underline{\mu}) := V(\underline{\nu},\underline{\mu})V^{-1}(\underline{\mu},\underline{\nu})$. Needless to say, all these equalities and definitions are well defined on $\mathcal{D}$, but not on the whole $\mathcal{H}$.[1] Incidentally, we observe that $T(\underline{\gamma},\underline{\gamma}) = \mathbb{1}$. This is in agreement with the fact that, when $\underline{\mu} = \underline{\nu}$, the operator $V(\underline{\nu},\underline{\mu})$ is bounded with bounded inverse; see below.

By a similar reasoning as above applied for the construction of the vacuum state we now deduce that

$$\varphi_{\underline{n}} = V(\underline{\nu},\underline{\mu})e_{\underline{n}}, \quad \Psi_{\underline{n}} = N_\Psi V(\underline{\mu},\underline{\nu})e_{\underline{n}}. \tag{3.25}$$

In analogy with [18], we see that, while $V(\underline{\nu},\underline{\nu}) = D(\underline{\nu})$ is a unitary operator and as a consequence bounded, the operator $V(\underline{\nu},\underline{\mu})$, as well as its inverse, is unbounded for $\underline{\nu} \neq \underline{\mu}$ . The crucial conclusion from this is that the two sets $\mathcal{F}_\varphi = \{\varphi_{\underline{n}}\}$ and $\mathcal{F}_\Psi = \{\Psi_{\underline{n}}\}$ cannot be Riesz bases. In fact, they are both related to the orthonormal basis $\mathcal{F}_e$ by unbounded operators. Moreover, they are not even a basis, while they are both complete in $\mathcal{H}$. The proofs of these claims do not differ much from those given in [18] and therefore will not be repeated here. We will comment further on the physical meaning of these results in the next subsection.

Similarly as in [18], we can prove that $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are $\mathcal{D}$ quasibases. In fact, repeating almost the same steps, we deduce that, for instance, $\forall\, f,g \in \mathcal{D}$,

$$\langle f,g\rangle = \sum_{\underline{n}}\langle f,\varphi_{\underline{n}}\rangle\langle\Psi_{\underline{n}},g\rangle, \tag{3.26}$$

so that the results listed at the end of Sec. II hold true. In particular, let us introduce the operator $\Theta(\underline{\nu},\underline{\mu}) := T(\underline{\mu},\underline{\nu})$. It is possible to show that $\Theta(\underline{\nu},\underline{\mu})$ is self-adjoint, invertible, and leaves $\mathcal{D}$ invariant. Moreover, $\Theta(\underline{\nu},\underline{\nu}) = 1$, and

$$\Theta(\underline{\nu},\underline{\mu}) = N_\Psi\prod_{i=1}^{2}e^{(\nu_i-\mu_i)\hat{a}_i^\dagger}e^{(\bar{\nu}_i-\bar{\mu}_i)\hat{a}_i}, \tag{3.27}$$

which implies that $\langle f,\Theta(\underline{\nu},\underline{\mu})f\rangle > 0$ for all nonzero vectors $f \in \mathcal{D}$. This is in agreement with the facts that (i) $\Psi_{\underline{n}} = \Theta(\underline{\nu},\underline{\mu})\varphi_{\underline{n}}$, $\forall\underline{n}$ and (ii) $(a_j,b_j^\dagger)$ are $\Theta$ conjugate: $a_jf = \Theta^{-1}(\underline{\nu},\underline{\mu})b_j^\dagger\Theta(\underline{\nu},\underline{\mu})f$, for all $f \in \mathcal{D}$. We conclude also that, again for all $f \in \mathcal{D}$,

$$N_if = \Theta^{-1}(\underline{\nu},\underline{\mu})N_i^\dagger\Theta(\underline{\nu},\underline{\mu})f, \tag{3.28}$$

[1]This aspect is almost never stressed in the physical literature. Unbounded operators never exist *alone*. They exist in connection with some suitable dense subspace of $\mathcal{H}$, their domains.

which is the intertwining relation responsible for the fact that $\tilde{H}$ and $\tilde{H}^\dagger$ have the same eigenvalues and related eigenvectors; see below.

### B. Back to the Hamiltonian

Let us now return to our original problem, i.e., the deduction of the eigenvalues and the eigenvectors for $\tilde{H}$ in (3.2) and $\hat{H}$ in (1.1). As we have shown, we may express them in terms of pseudobosonic number operators. From the above construction is clear that

$$\tilde{H}\varphi_{\underline{n}} = E_{\underline{n}}\varphi_{\underline{n}}, \tag{3.29}$$

with $E_{\underline{n}} \in \mathbb{R}$ given by (3.13). From our results in Sec. II it also follows directly that the eigensystem of the adjoint $\tilde{H}^\dagger = \bar{\gamma}_1 N_1^\dagger + \bar{\gamma}_2 N_2^\dagger + \bar{\gamma}_0$ is computed to

$$\tilde{H}^\dagger \Psi_{\underline{n}} = \bar{E}_{\underline{n}}\Psi_{\underline{n}} = E_{\underline{n}}\Psi_{\underline{n}}. \tag{3.30}$$

The analysis in [18] showed that, as already deduced, two biorthogonal sets of eigenstates of a Hamiltonian and of its adjoint need not to be automatically a Riesz basis, even when they are complete. This is exactly the case here: $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are biorthogonal, complete, eigenstates of $\tilde{H}$ and $\tilde{H}^\dagger$ ($\hat{H}$ and $\hat{H}^\dagger$), respectively, but neither $\mathcal{F}_\varphi$ nor $\mathcal{F}_\Psi$ are bases for $\mathcal{H}$. However, interestingly enough, they are $\mathcal{D}$ quasibases, and this is reflected in the properties we have explicitly verified for our model.

## IV. CONCLUSIONS

We have investigated the properties of a non-self-adjoint model on a noncommutative two-dimensional space. The Hamiltonian $\tilde{H}$ was set up in the standard fashion followed in the literature on $\mathcal{PT}$-symmetric quantum mechanics, by seeking an antilinear symmetry, i.e., $\mathcal{PT}_-$ in this case. From our explicit formulas we observe that $\mathcal{PT}_-$: $\varphi_{\underline{0}} \to \varphi_{\underline{0}}$, $\varphi_{\underline{n}} \to (-1)^{n_1}\varphi_{\underline{n}}$, $\Psi_{\underline{0}} \to \Psi_{\underline{0}}$, $\Psi_{\underline{n}} \to (-1)^{n_1}\Psi_{\underline{n}}$ such that by the standard arguments of Wigner [13] it follows that the eigenvalues of $\tilde{H}$ have to be real. This is confirmed by our explicit computation. The symmetry for the Hamiltonian $\hat{H}$ is not evident from the start, but as demonstrated the overall conclusions are the same as for $\tilde{H}$.

However, despite having well-defined real physical spectrum, we established further that $\tilde{H}$ cannot be considered as a standard quantum-mechanical model, since the corresponding biorthonormal system is not of Riesz type. As already discussed, in many places in the literature, see [19] for instance, it is incorrectly assumed that the eigenvectors of a not self-adjoint Hamiltonian $H$ and $H^\dagger$ automatically form a biorthogonal basis. In fact, this is a rather strong requirement which is quite difficult to find satisfied in concrete models existing in the literature, at least for infinite-dimensional Hilbert spaces. We have shown that even for the simple example presented here this is not the case. This only leaves two of the following options: either this conclusion is wrong for the cases treated, as it would be for the model presented here, or at least some additional analysis is required to justify it. Thus our example supports the suggestion [14,18] that many models, thought to be very interesting quantum-mechanical systems, need to be revisited for further scrutiny.

It is easy to see from our formulas that these conclusions do not rely on the fact that the model is formulated on a noncommutative space and also hold in the limit to the commutative space when setting $\lim_{\theta\to 0}\Omega = 2\hbar$, $\lim_{\theta\to 0} M = m$, etc. In reverse, this also means that the problem of not having automatically a biorthonormal basis cannot be solved by formulating the model on a noncommutative space, which provides more freedom and often removes inconsistencies.

We end this section, and the paper, observing that, even with all the problems we have put in evidence along the paper, we may still make sense of the model presented here, simply because of the role of the quasibases as described above and in more detail in the quoted literature.


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