Internal structure of the Heisenberg and Robertson-Schrödinger uncertainty relations: Multidimensional generalization

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It is known that the Heisenberg and Robertson-Schrödinger uncertainty relations can be replaced by sharper relations in which the "classical" (depending on the gradient of the phase of the wave function) and "quantum" (depending on the gradient of the envelope of the wave function) parts of the variances $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ are separated. In this paper, multidimensional generalization of these relations is discussed.

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I. INTRODUCTION

The Heisenberg uncertainty relation for the coordinate x and momentum p has the well-known form [1]

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geqslant \frac{\hbar^2}{4}, \tag{1}$$

where

$$\langle(\Delta x)^2\rangle = \int_{-\infty}^{\infty}(x-\langle x\rangle)^2|\psi|^2dx, \tag{2}$$

$$\langle(\Delta p)^2\rangle = \int_{-\infty}^{\infty}\left|(p-\langle p\rangle)\psi\right|^2dx, \tag{3}$$

$\psi=\psi(x,t)$ is the normalized wave function, $p=-i\hbar(\partial/\partial x)$, $\langle\,\rangle$ denotes the usual quantum-mechanical mean value, and $\hbar$ is the reduced Planck constant $\hbar = h/(2\pi)$.

The Robertson-Schrödinger uncertainty relation for the coordinate and momentum has the form [2–6]

$$\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle \geqslant \left[\int_{-\infty}^{\infty}(x-\langle x\rangle)\left(\frac{\partial s_1}{\partial x}-\left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)|\psi|^2dx\right]^2+\frac{\hbar^2}{4}. \tag{4}$$

The Heisenberg relation can be obtained from this relation by neglecting the square of the integral at the right-hand side.

For recent discussion of uncertainty relations see, e.g., [7–22].

It is known that the Heisenberg uncertainty relation and also the Robertson-Schrödinger uncertainty relation can be replaced by a pair of sharper relations in which the "classical" (depending on the gradient of the phase of the wave function) and "quantum" (depending on the gradient of the envelope of the wave function) parts of the variances of the coordinate $\langle(\Delta x)^2\rangle$ and momentum $\langle(\Delta p)^2\rangle$ are separated [5,6,23–27]. This separation is based on the following idea.

The normalized wave function ψ can be always written in terms of its modulus and argument (phase):

$$\psi = |\psi|e^{i\arg(\psi)} = e^{-s_2/\hbar}e^{is_1/\hbar}, \tag{5}$$

where $s_1(x,t)$ and $s_2(x,t)$ are real functions. Then we get

$$p\psi = \frac{\partial s_1}{\partial x}\psi + i\frac{\partial s_2}{\partial x}\psi. \tag{6}$$

The mean momentum can be written as

$$\langle p\rangle = \langle\psi|p\psi\rangle = \int_{-\infty}^{\infty}\frac{\partial s_1}{\partial x}|\psi|^2dx + i\int_{-\infty}^{\infty}\frac{\partial s_2}{\partial x}|\psi|^2dx. \tag{7}$$

Assuming the wave functions with the property $|\psi|^2\to 0$ for $x\to\infty$, the second integral in Eq. (7) does not contribute to the mean momentum:

$$\int_{-\infty}^{\infty}\frac{\partial s_2}{\partial x}|\psi|^2dx = -\frac{\hbar}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}e^{-2s_2/\hbar}dx = -\frac{\hbar}{2}e^{-2s_2/\hbar}\Big|_{x=-\infty}^{\infty} = 0. \tag{8}$$

Therefore, the resulting expression for the mean momentum [5,6,24–27]

$$\langle p\rangle = \int_{-\infty}^{\infty}\frac{\partial s_1}{\partial x}|\psi|^2dx \tag{9}$$

does not depend on $\partial s_2/\partial x$. This formula corresponds to the transition from the point particle in classical mechanics where the probability density has the δ-like character to the particle described by the probability density $|\psi|^2$ in quantum mechanics. At the same time, the expression for the classical momentum $p_{\rm cl}=\partial S/\partial x$, where S is the Hamilton action, is replaced here by the mean value $\langle p\rangle = \langle\partial s_1/\partial x\rangle$, where the function s_1 corresponds to S and the probability density $|\psi|^2$ is introduced.

It follows from Eq. (6) that the mean value $\langle p^2\rangle$ can be written as a sum of two parts [5,26,27]:

$$\langle p^2\rangle = \langle p\psi|p\psi\rangle = \langle p_1^2\rangle + \langle p_2^2\rangle, \tag{10}$$

where

$$\langle p_1^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial s_1}{\partial x}\right)^2|\psi|^2dx \tag{11}$$

and

$$\langle p_2^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial s_2}{\partial x}\right)^2|\psi|^2dx. \tag{12}$$

The first part $\langle p_1^2\rangle$ that can be denoted as "classical" is statistical generalization of the expression $p_{cl}^2 = (\partial S/\partial x)^2$ from classical mechanics, in which the classical momentum $p_{\rm cl}=\partial S/\partial x$ is replaced by $\partial s_1/\partial x$ and the probability density $|\psi|^2$ is introduced. The second "quantum" part $\langle p_2^2\rangle$ is given by $|\psi|^2$ or the envelope of the wave function $|\psi| = \exp(-s_2/\hbar)$ and its derivative. It does not depend on $\partial s_1/\partial x$ and does not have its counterpart in classical mechanics.

Such separation applies not only for $\langle p^2\rangle$ and kinetic energy but also for the variance $\langle(\Delta p)^2\rangle$ appearing in the Heisenberg uncertainty relation [5,26,27]:

$$\langle(\Delta p)^2\rangle = \langle(p-\langle p\rangle)^2\rangle = \langle(\Delta p_1)^2\rangle + \langle(\Delta p_2)^2\rangle, \quad (13)$$

where

$$\langle(\Delta p_1)^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)^2 |\psi|^2 dx, \quad (14)$$

$$\langle(\Delta p_2)^2\rangle = \int_{-\infty}^{\infty}\left(\frac{\partial s_2}{\partial x} - \left\langle\frac{\partial s_2}{\partial x}\right\rangle\right)^2 |\psi|^2 dx = \int_{-\infty}^{\infty}\left(\frac{\partial s_2}{\partial x}\right)^2 |\psi|^2 dx, \quad (15)$$

and Eq. (8) is used.

Using the Schwarz inequality, a few pairs of the one-dimensional uncertainty relations for a different number of classical and quantum parts of $(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ were derived [6]. In this paper, their multidimensional generalization is discussed.

II. MULTIDIMENSIONAL GENERALIZATION OF THE ROBERTSON-SCHRÖDINGER UNCERTAINTY RELATION

Now, we consider the N-dimensional case with the wave function depending on N spatial variables $\psi = \psi(\mathbf{x},t)$, $\mathbf{x} = (x_1,\dots,x_N)$.

In the multidimensional case, the variance $\langle(\Delta x)^2\rangle$ in the Heisenberg uncertainty relation [Eq. (1)] can be generalized to a $N \times N$ matrix:

$$\langle(\Delta X)^2\rangle_{mn} = \int (x_m - \langle x_m\rangle)(x_n - \langle x_n\rangle)|\psi|^2 d\xi, \quad m,n = 1,\dots,N, \quad (16)$$

where $d\xi = dx_1\dots dx_N$ and integration is performed over the whole space. By calculating

$$\sum_{m,n=1}^{N} c_m^* \langle(\Delta X)^2\rangle_{mn} c_n = \int \left|\sum_{m=1}^{N} c_m (x_m - \langle x_m\rangle)\right|^2 |\psi|^2 d\xi \geqslant 0, \quad (17)$$

where c_m are complex numbers and the star denotes the complex conjugate, we see that the matrix $\langle(\Delta X)^2\rangle$ is positive semidefinite.

Analogously, Eqs. (13)–(15) can be generalized as

$$\langle(\Delta P)^2\rangle_{mn} = \langle(\Delta P_1)^2\rangle_{mn} + \langle(\Delta P_2)^2\rangle_{mn}, \quad m,n = 1,\dots,N, \quad (18)$$

where

$$\langle(\Delta P_1)^2\rangle_{mn} = \int \left(\frac{\partial s_1}{\partial x_m} - \left\langle\frac{\partial s_1}{\partial x_m}\right\rangle\right)\left(\frac{\partial s_1}{\partial x_n} - \left\langle\frac{\partial s_1}{\partial x_n}\right\rangle\right)|\psi|^2 d\xi \quad (19)$$

is the classical part of $\langle(\Delta P)^2\rangle$ and

$$\langle(\Delta P_2)^2\rangle_{mn} = \int \frac{\partial s_2}{\partial x_m}\frac{\partial s_2}{\partial x_n}|\psi|^2 d\xi \quad (20)$$

is the quantum part of $\langle(\Delta P)^2\rangle$. Using similar arguments as in the preceding paragraph, it can be shown that the matrices $\langle(\Delta P_1)^2\rangle$ and $\langle(\Delta P_2)^2\rangle$ are positive semidefinite, too.

Following the idea formulated in Eq. (6) we define a correlation matrix G among the coordinates and momentum:

$$G_{mn} = \int (x_m - \langle x_m\rangle)\left(\frac{\partial s_1}{\partial x_n} - \left\langle\frac{\partial s_1}{\partial x_n}\right\rangle + i\frac{\partial s_2}{\partial x_n}\right)|\psi|^2 d\xi, \quad m,n = 1,\dots,N. \quad (21)$$

Using the expression $|\psi|^2 = \exp(-2s_2/\hbar)$, integration by parts, and assuming validity of the conditions $|\psi|^2 \to 0$ and $x_m|\psi|^2 \to 0$ for $x_m \to \pm\infty$ we get [23]

$$\int (x_m - \langle x_m\rangle)\frac{\partial s_2}{\partial x_n}|\psi|^2 d\xi = \frac{\hbar}{2}\delta_{mn} \quad (22)$$

and

$$G_{mn} = \int (x_m - \langle x_m\rangle)\left(\frac{\partial s_1}{\partial x_n} - \left\langle\frac{\partial s_1}{\partial x_n}\right\rangle\right)|\psi|^2 d\xi + i\frac{\hbar}{2}\delta_{mn}, \quad m,n = 1,\dots,N. \quad (23)$$

Then, we create a matrix M of the order $2N$:

$$M = \begin{pmatrix} (\Delta X)^2 & G \\ G^+ & (\Delta P)^2 \end{pmatrix}, \quad (24)$$

where the cross denotes the Hermitian conjugation.

To show that also the matrix M is positive semidefinite we define quantities f_m:

$$f_m = x_m - \langle x_m\rangle, \quad f_{N+m} = \frac{\partial s_1}{\partial x_m} - \left\langle\frac{\partial s_1}{\partial x_m}\right\rangle + i\frac{\partial s_2}{\partial x_m}, \quad m = 1,\dots,N. \quad (25)$$

By analogy with Eq. (17) we get

$$\sum_{m,n=1}^{2N} c_m^* M_{mn} c_n = \int \left|\sum_{m=1}^{2N} c_m f_m\right|^2 |\psi|^2 d\xi \geqslant 0 \quad (26)$$

and see that the matrix M is positive semidefinite, too.

Further, we make use of a general result valid for $N \times N$ matrices A, B, C, and D, where D is a regular matrix [23]:

$$\begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & 0 \\ C & D \end{pmatrix}, \quad (27)$$

leading to

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A - BD^{-1}C)\det(D). \quad (28)$$

Applying this result to the matrix M given by Eq. (24) we get the multidimensional uncertainty relation for $(\Delta X)^2$ and $(\Delta P)^2$:

$$\det\{(\Delta X)^2(\Delta P)^2 - G[(\Delta P)^2]^{-1}G^+(\Delta P)^2\} \geqslant 0. \quad (29)$$

This relation is the multidimensional generalization of the Robertson-Schrödinger uncertainty relation [Eq. (4)].

III. RELATIONS FOR THE QUANTUM AND CLASSICAL PARTS OF THE MOMENTUM

First, we take the matrix M in the form

$$M = \begin{pmatrix} (\Delta X)^2 & G \\ G^+ & (\Delta P_1)^2 \end{pmatrix}, \quad (30)$$

where

$$G_{mn} = \int (x_m - \langle x_m\rangle)\left(\frac{\partial s_1}{\partial x_n} - \left\langle\frac{\partial s_1}{\partial x_n}\right\rangle\right)|\psi|^2 d\xi,\quad m,n = 1,\ldots,N. \tag{31}$$

Applying Eq. (28) to the matrix M given by Eq. (30) we get the multidimensional relation for $(\Delta X)^2$ and the classical part $(\Delta P_1)^2$ [23]:

$$\det\{(\Delta X)^2(\Delta P_1)^2 - G[(\Delta P_1)^2]^{-1}G^+(\Delta P_1)^2\} \geqslant 0. \tag{32}$$

For $N = 1$, the one-dimensional relation for $\langle(\Delta x)^2\rangle$ and the classical part $\langle(\Delta p_1)^2\rangle$ is obtained [23]:

$$\langle(\Delta x)^2\rangle\langle(\Delta p_1)^2\rangle \geqslant \left[\int_{-\infty}^{\infty}(x - \langle x\rangle)\left(\frac{\partial s_1}{\partial x} - \left\langle\frac{\partial s_1}{\partial x}\right\rangle\right)|\psi|^2 dx\right]^2. \tag{33}$$

This relation has the usual meaning known from mathematical statistics: The product of variances of two quantities is greater than or equal to the square of their covariance. Depending on the functions $\partial s_1/\partial x$ and $|\psi|^2$, the square of the covariance of the coordinate and momentum at the right-hand side of this relation can have arbitrary values greater than or equal to zero. If the right-hand side of Eq. (33) equals zero, any of the quantities $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p_1)^2\rangle$ can equal zero independently of the other one. In this sense, this inequality has classical character and is different from the Heisenberg and Robertson-Schrödinger uncertainty relations. Interpretation of Eq. (32) is analogous.

Taking the matrix M in the form

$$M = \begin{pmatrix} (\Delta X)^2 & G \\ G^+ & (\Delta P_2)^2 \end{pmatrix}, \tag{34}$$

where

$$G_{mn} = i\int (x_m - \langle x_m\rangle)\frac{\partial s_2}{\partial x_n}|\psi|^2 d\xi = i\frac{\hbar}{2}\delta_{mn},\quad m,n = 1,\ldots,N, \tag{35}$$

we obtain the multidimensional uncertainty relation for $(\Delta X)^2$ and the quantum part $(\Delta P_2)^2$ [23]:

$$\det\left\{(\Delta X)^2(\Delta P_2)^2 - \frac{\hbar^2}{4}\right\} \geqslant 0. \tag{36}$$

For $N = 1$, the one-dimensional uncertainty relation for $\langle(\Delta x)^2\rangle$ and the quantum part $\langle(\Delta p_2)^2\rangle$ is [23]

$$\langle(\Delta x)^2\rangle\langle(\Delta p_2)^2\rangle \geqslant \frac{\hbar^2}{4}. \tag{37}$$

Similarly to Eq. (33), this relation can be understood as the standard statistical inequality, too. However, the right-hand side of Eq. (37) equals $\hbar^2/4$ and does not depend on the concrete form of the functions s_1 and s_2. Similarly to the Heisenberg uncertainty relation, the left-hand side of this relation cannot be smaller than $\hbar^2/4$. In contrast to Eq. (33), the left-hand side of Eq. (37) does not depend on s_1 and depends only on the envelope $|\psi| = \exp(-s_2/\hbar)$ of the wave function ψ and its derivative. In this sense, Eq. (37) and also Eq. (36) have quantum character.

The sum of Eqs. (33) and (37) leads to the Robertson-Schrödinger uncertainty relation [Eq. (4)]. Therefore, the pair of Eqs. (33) and (37) is sharper than Eqs. (1) and (4).

IV. RELATIONS FOR THE QUANTUM AND CLASSICAL PARTS OF THE COORDINATE

An analogous approach can be used also for the wave function in the momentum representation. By analogy with the coordinate representation, we consider the N-dimensional case with the wave function in the momentum representation $\varphi = \varphi(\mathbf{p},t)$, where $\mathbf{p} = (p_1,\ldots,p_N)$.

The wave function

$$\varphi(\mathbf{p},t) = \frac{1}{(2\pi\hbar)^{N/2}}\int \psi(\mathbf{x},t)\,e^{\mathbf{p}\cdot\mathbf{x}/(i\hbar)}d\xi \tag{38}$$

can be written in the form analogous to Eq. (5):

$$\varphi(\mathbf{p},t) = e^{-r_2/\hbar}e^{ir_1/\hbar}, \tag{39}$$

where $r_1(\mathbf{p},t)$ and $r_2(\mathbf{p},t)$ are real functions.

Analogously to the preceding section we define the matrix $\langle(\Delta P)^2\rangle$ in the momentum representation:

$$\langle(\Delta P)^2\rangle_{mn} = \int (p_m - \langle p_m\rangle)(p_n - \langle p_n\rangle)|\varphi|^2 d\tau, \tag{40}$$

where $d\tau = dp_1\ldots dp_N$ and integration is performed over the whole space. Using the coordinate operator $x_m = i\hbar(\partial/\partial p_m)$ it is possible to derive equations analogous to Eqs. (18)–(20):

$$\langle(\Delta X)^2\rangle_{mn} = \langle(\Delta X_1)^2\rangle_{mn} + \langle(\Delta X_2)^2\rangle_{mn},\quad m,n = 1,\ldots,N, \tag{41}$$

where

$$\langle(\Delta X_1)^2\rangle_{mn} = \int\left(\frac{\partial r_1}{\partial p_m} - \left\langle\frac{\partial r_1}{\partial p_m}\right\rangle\right)\left(\frac{\partial r_1}{\partial p_n} - \left\langle\frac{\partial r_1}{\partial p_n}\right\rangle\right)|\varphi|^2 d\tau \tag{42}$$

is the classical part of $\langle(\Delta X)^2\rangle$ and

$$\langle(\Delta X_2)^2\rangle_{mn} = \int \frac{\partial r_2}{\partial p_m}\frac{\partial r_2}{\partial p_n}|\varphi|^2 d\tau \tag{43}$$

is the quantum part of $\langle(\Delta X)^2\rangle$.

Assuming the matrix M in the form

$$M = \begin{pmatrix} (\Delta P)^2 & G \\ G^+ & (\Delta X_1)^2 \end{pmatrix}, \tag{44}$$

where

$$G_{mn} = -\int (p_m - \langle p_m\rangle)\left(\frac{\partial r_1}{\partial p_n} - \left\langle\frac{\partial r_1}{\partial p_n}\right\rangle\right)|\varphi|^2 d\tau,\quad m,n = 1,\ldots,N \tag{45}$$

and using Eq. (28) we obtain the multidimensional relation for $\langle(\Delta P)^2\rangle$ and the classical part $\langle(\Delta X_1)^2\rangle$:

$$\det\{(\Delta P)^2(\Delta X_1)^2 - G[(\Delta X_1)^2]^{-1}G^+(\Delta X_1)^2\} \geqslant 0. \tag{46}$$

For $N = 1$, it leads to the one-dimensional relation for $\langle(\Delta p)^2\rangle$ and $\langle(\Delta x_1)^2\rangle$ [6]:

$$\langle(\Delta p)^2\rangle\langle(\Delta x_1)^2\rangle \geqslant \left[\int_{-\infty}^{\infty}(p - \langle p\rangle)\left(\frac{\partial r_1}{\partial x} - \left\langle\frac{\partial r_1}{\partial x}\right\rangle\right)|\varphi|^2 dp\right]^2. \tag{47}$$

Taking the matrix M in the form

$$M = \begin{pmatrix} (\Delta P)^2 & G \\ G^+ & (\Delta X_2)^2 \end{pmatrix}, \quad (48)$$

where

$$G_{mn} = -i \int (p_m - \langle p_m \rangle) \frac{\partial r_2}{\partial p_n} |\varphi|^2 d\tau = -i \frac{\hbar}{2} \delta_{mn}, \quad m,n = 1,\ldots,N, \quad (49)$$

we obtain the multidimensional uncertainty relation for $(\Delta P)^2$ and the quantum part $(\Delta X_2)^2$:

$$\det\left\{(\Delta P)^2 (\Delta X_2)^2 - \frac{\hbar^2}{4}\right\} \geqslant 0. \quad (50)$$

For $N = 1$, the one-dimensional uncertainty relation for $\langle(\Delta p)^2\rangle$ and the quantum part $\langle(\Delta x_2)^2\rangle$ is [6]

$$\langle(\Delta p^2)\rangle\langle(\Delta x_2)^2\rangle \geqslant \frac{\hbar^2}{4}. \quad (51)$$

Comments on these relations can be made as in the preceding section and will not be given here.

V. RELATION FOR THE QUANTUM PARTS OF THE COORDINATE AND MOMENTUM

It has been shown in the preceding sections that matrices $(\Delta X)^2 = (\Delta X_1)^2 + (\Delta X_2)^2$ and $(\Delta P)^2 = (\Delta P_1)^2 + (\Delta P_2)^2$ can be written as a sum of two matrices having classical and quantum character. Now we ask if it is possible to derive some relation for quantum parts $(\Delta X_2)^2$ and $(\Delta P_2)^2$ only, without the presence of matrices $(\Delta X_1)^2$ and $(\Delta P_1)^2$. Such relation is discussed in this section.

For this aim, we take the matrix M in the form

$$M = \begin{pmatrix} (\Delta X_2)^2 & G \\ G^+ & (\Delta P_2)^2 \end{pmatrix}, \quad (52)$$

where

$$G_{mn} = \frac{1}{(2\pi\hbar)^{N/2}} \iint \frac{\partial r_2}{\partial p_m} |\varphi| e^{-\mathbf{p}\cdot\mathbf{x}/(i\hbar)} \frac{\partial s_2}{\partial x_n} |\psi| d\xi d\tau \quad (53)$$
$$= \frac{\hbar^2}{(2\pi\hbar)^{N/2}} \iint \frac{\partial|\varphi|}{\partial p_m} e^{-\mathbf{p}\cdot\mathbf{x}/(i\hbar)} \frac{\partial|\psi|}{\partial x_n} d\xi d\tau, \quad m,n = 1,\ldots,N.$$

Then, Eq. (28) yields the relation for the quantum parts of $(\Delta X)^2$ and $(\Delta P)^2$:

$$\det\{(\Delta X_2)^2(\Delta P_2)^2 - G[(\Delta P_2)^2]^{-1}G^+(\Delta P_2)^2\} \geqslant 0. \quad (54)$$

If the wave functions ψ and φ depend on time, all quantities in this relation may be time dependent. For this reason, we do not denote Eq. (54) as an uncertainty relation but as a relation or an inequality.

The wave functions ψ and φ are related by the Fourier transform. In contrast, this is not generally the case for their envelopes $|\psi|$ and $|\varphi|$. For this reason, Eq. (54) has different character than other uncertainty relations discussed in this paper and can lead to interesting results (for detailed discussion in the one-dimensional case see [6]).

For $N = 1$, Eq. (54) becomes

$$\langle(\Delta x_2)^2\rangle\langle(\Delta p_2)^2\rangle \geqslant \hbar^2 |I|^2, \quad (55)$$

where

$$I = \hbar \int_{-\infty}^{\infty} \frac{\partial|\psi|}{\partial x} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\partial|\varphi|}{\partial p} e^{-\mathbf{p}\cdot\mathbf{x}/(i\hbar)} dp dx. \quad (56)$$

If $|\varphi|$ equals the Fourier transform of $|\psi|$, the right-hand side of Eq. (55) has the usual value $\hbar^2/4$. In general, the integral I has to be calculated in each case separately. It can be smaller than $\hbar^2/4$ and go to zero with increasing time [6].

VI. CONCLUSION

In [6], one-dimensional uncertainty relations for the classical and quantum parts of the variances $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ appearing in the Heisenberg and Robertson-Schrödinger uncertainty relations were investigated. In this paper, their multidimensional generalization has been discussed.

Measurement of the coordinate and momentum is characterized by the variances that are the sum of the classical and quantum parts. The quantum parts do not depend on the phase of the wave function and are given by the mean square of the derivative of the probability density or the envelope of the wave function. In contrast, the classical parts depend on the mean square of the derivative of the phase of the wave function. To measure the quantum parts direct measurement of the probability density in the coordinate or momentum representation and its derivative can be made. Depending on the type of measurement, usual Heisenberg [Eq. (1)] and Robertson-Schrödinger [Eq. (4)] relations can be then replaced by Eqs. (29), (32), (33), (36), (37), (46), (47), (50), (51), (54), or (55).

The constant $\hbar^2/4$ appears only in Eqs. (1), (4), (29), (36), (37), (50), and (51) containing at least one quantum part of the variances of the coordinate and momentum. In such cases, attainable accuracy of measurement is limited by the corresponding uncertainty relation.

Constant $\hbar^2/4$ does not appear in Eqs. (32), (33), (46), and (47), which have classical character. In these cases, accuracy of measurement can be in principle arbitrary.

A special case is represented by Eqs. (54) and (55) for two quantum parts of the variances. As shown in [6] for Eq. (55), the constant $\hbar^2/4$ is obtained if $|\varphi|$ equals the Fourier transform of $|\psi|$. In general, the corresponding expression has to be calculated in each case separately. Depending on the result, attainable accuracy of the corresponding measurement can be higher than follows from the Heisenberg and Robertson-Schrödinger uncertainty relations.

Finally, we would like to make a note on the spreading of the wave packets in time. In the one-dimensional case, spreading of the wave packets is given by the increasing value of the left-hand side of the corresponding uncertainty relation [6]. It depends on the number of classical parts appearing at the left-hand side. For the Gaussian wave packet and two classical parts, the left-hand side of Eqs. (1) and (4) is proportional to the square of time. For one classical part, the left-hand side of Eqs. (37) and (51) increases as the first power of time. An interesting case is represented by Eq. (55) with no classical parts, where its right-hand side equals $\hbar^2/4$ in special cases only. For the

Gaussian wave packet, the left-hand side of Eq. (55) goes to zero in time as $1/t^2$ and the right-hand side goes as $1/t^3$ [6]. For more dimensions, similar conclusions can obviously be made.

These results show that the Heisenberg and Robertson-Schrödinger uncertainty relations can be replaced by sharper one-dimensional relations and their multidimensional generalization. It shows also that the Heisenberg and Robertson-Schrödinger uncertainty relations should not be automatically applied to all measurements since it can lead to incorrect conclusions. Depending on the character of measurement, the corresponding relations discussed in this paper and the Heisenberg and Robertson-Schrödinger relations can give different bounds of attainable accuracy of measurement. For this reason, our results are not only of theoretical interest but also important from the experimental point of view.

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