Quantum state tomography from a sequential measurement of two variables in a single setup

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We demonstrate that the task of determining an unknown quantum state can be accomplished efficiently by making a sequential measurement of two observables, $\hat{A}$ and $\hat{B}$, the eigenstates of which form bases connected by a discrete Fourier transform. The state can be pure or mixed, the dimension of the Hilbert space and the coupling strength are arbitrary, and the experimental setup is fixed. The concept of Moyal quasicharacteristic function is introduced for finite-dimensional Hilbert spaces.

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I. INTRODUCTION

A colleague has challenged you: she has built a black box from which, upon the pressing of a button, a quantum system is released. What is the state of the system? You are not allowed to open the box or to measure any of its properties. You can only measure the quantum system and repeat as many times as you want. This is the essence of quantum state tomography.

The preparation of a quantum system is characterized by a quantum state, which is given by the density operator, a positive-definite operator of trace 1 in a Hilbert space. Often, some information about the system is missing, but it could be recovered, in principle, from the environment and from the preparing apparatus. When all this information is retrieved, which can be done without disturbing the system in any way, the quantum system is described by a pure state, i.e., a density operator of rank 1, which can be written as $\rho_{\rm sys} = |\psi\rangle\langle\psi|$ in terms of a vector $|\psi\rangle$ of the Hilbert space. However, in general, this information is lost for all practical purposes, and the system is to be described by a density operator of higher rank. A fundamental question is, then, How do we determine the unknown state $\rho_{\rm sys}$ of a quantum system?

Reconstructing the unknown quantum state $\rho_{\rm sys}$ is believed to be a difficult task, requiring the separate measurement of several observables. The usual approach is to take the system in the unknown state and measure the statistics of an observable $\hat{A}_1$, then, with a distinct ensemble of identically prepared systems, measure another observable $\hat{A}_2$, etc. The observables $\hat{A}_1,\hat{A}_2,\ldots,\hat{A}_n$ needed to reconstruct the quantum state are known as the *quorum*, and they usually number as d^2, with d the dimension of the Hilbert space, even though some improvement over this number can be achieved [1]. Usually, from each measurement, only the average value is extracted. For instance, to reconstruct the state of a spin-1/2 system, the average values $n_j = \langle\sigma_j\rangle$, $j = x,y,z$, are calculated, and the state $\rho_{\rm sys} = (1 + \mathbf{n}\cdot\boldsymbol{\sigma})/2$ is reconstructed. The noise introduced by the detectors is then a hindrance. However, it is important to note that the full probability distribution of the output is a function (typically, a convolution) both of the initial state of the detector and of $\rho_{\rm sys}$. Thus, extracting only one number, the average, of the many repetitions of a measurement is extremely limitative and a waste of useful information.

Furthermore, the most commonly used statistical tool for the reconstruction of the state is the maximum likelihood estimation, which does not take into account the positive definiteness of the density operator and may give rise to rank-deficient estimates. *Ad hoc* corrections are often devised to overcome this difficulty. The recently introduced Bayesian [2] approach has solved the latter issue, but its adoption is slow. We remark that in the Bayesian approach, the maximum likelihood estimate is justified when uniform priors are assumed and a particular cost function is postulated [3]. In any case, the number of different setups needed for quantum state tomography increases with the dimension of the Hilbert space, making the process time-consuming.

Recently, many schemes based on weak measurement [4–7] have been proposed for quantum state tomography. Experimental realizations have also been demonstrated [8,9]. However, a distinct disadvantage of such schemes is that, on one hand, the formulas for the weak measurement are approximated, introducing a further uncertainty in the reconstruction, and, on the other hand, the weak measurement relies on postselection, which requires that only a fraction of the data be retained, yielding a reduced efficiency.

Haapasalo *et al.* [10] have also pointed out the superiority of phase-space methods over weak measurement methods in order to reconstruct the wave function. This suggests looking for an extension of phase-space methods to finite-dimensional Hilbert spaces. In doing so, we propose a generalization of the Moyal function [11]. The justification for this choice is that the Moyal function has revealed itself to be an extremely useful tool for describing the statistics of joint and sequential measurements of momentum and position [12,13].

A promising avenue for efficient quantum state tomography was opened by considering measurements in mutually unbiased bases [14–16]. All the proposals of which we are aware, however, require many different setups, at least as many as the dimension of the Hilbert space.

Here, instead, we propose a quantum state tomography scheme consisting in a *single* sequential measurement of *arbitrary strength* and relying on an *exact relation* between the initial state of the system and the final output of the measurement. The whole statistics of the measurement is used, and the nonsharpness of the detector is turned into a resource, rather than an obstacle. Our scheme uses a particular pair of mutually unbiased bases, the Fourier conjugated bases. We demonstrate that there are infinitely many pairs of observables

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$\hat{A},\hat{B}$ that allow the reconstruction of an unknown quantum state $\rho_{\rm sys}$, be this pure or mixed. Furthermore, by suitably choosing the first measured observable $\hat{A}$, it is possible to obtain the representation of the state, $\langle m|\rho_{\rm sys}|m'\rangle$, in any basis of choice. We recover the results of Ref. [13] in the limit $d\to\infty$. Furthermore, a sequential measurement of position and momentum may lead to a violation of the Heisenberg noise-disturbance principle [17] if the detectors are initially in a correlated state. The result provided here applies whether or not the detectors are initially correlated.

A related proposal was made by Leonhardt [18,19], who introduced a different quantum characteristic function for discrete systems (see Appendix A for a discussion) and proposed using Ramsey techniques to transform the quadrature observables into energy eigenstates. Furthermore, recently Carmeli *et al.* [20] have demonstrated that sequential measurements of conjugated observables are informationally complete, i.e., for any two density matrices of the system $\rho_1\neq\rho_2$, the probabilities differ, $P(A,B|\rho_1)\neq P(A,B|\rho_2)$. Thus, in principle, there is a one-to-one correspondence between the density matrices and the probabilities $P(A,B|\rho)$. The present article provides this correspondence.

II. PRELIMINARY DEFINITIONS

We report the conventions used throughout this paper:

(i) d integer, dimension of the Hilbert space;

(ii) $S=(d-1)/2$ integer or half-odd "spin";

(iii) m,m' integer or half-odd numbers spaced by 1 in the range $[-S,S]$;

(iv) $\mu=m-m'$ integer in the range $[1-d,d-1]$;

(v) $\bar{M}=\frac{m+m'}{2}$ integer or half-odd in the range $[-S+|\mu|/2,S-|\mu|/2]$ for fixed μ; and

(vi) $\mathcal{I}$ integers or half-odd numbers in the range $[-S,S]$.

Our scheme is based on the quantum version of the characteristic function, the Moyal quasicharacteristic function, or quantum characteristic function. Recall that for a classical probability distribution $\mathcal{P}(\xi)$, one can define its characteristic function as the Fourier transform

$$\mathcal{Z}(\chi)=\int d\xi e^{i\chi\xi}\mathcal{P}(\xi). \tag{1}$$

The derivatives of $\mathcal{Z}$ at $\chi=0$ give the moments of the distribution; its logarithmic derivatives give the cumulants [21]. For a classical pointlike particle in one dimension, $\xi=(p,q)$, momentum and position. In quantum mechanics, however, the momentum and position operators $\hat{p}$ and $\hat{q}$, do not commute, hence it is not possible, in general, to characterize a quantum pointlike particle in one dimension through a non-negative probability $\mathcal{P}$. Instead, we must use the Wigner function $\mathcal{W}(p,q)$, which can take negative values. The quantum characteristic function $\mathcal{M}$ is then defined as the Fourier transform of the Wigner function, $\mathcal{M}(\chi_p,\chi_q)=\int dpdq\exp[i\chi_p p+i\chi_q q]\mathcal{W}(p,q)$. After some straightforward algebra,

$$\mathcal{M}(\chi_p,\chi_q)=\langle\exp[i\chi_p\hat{p}+i\chi_q\hat{q}]\rangle, \tag{2}$$

where the quantum mechanical average is defined as

$$\langle\hat{O}\rangle=\mathrm{Tr}[\hat{O}\rho], \tag{3}$$

with ρ the density operator and Tr the trace. The quantum characteristic function is thus obtained by the inverse Weyl-Wigner transform [22,23]. It solves the question, Given the classical moments $\overline{p^m q^n}$, what is their equivalent expression in terms of quantum mechanical averages (3)? In the simple case $\overline{pq}$ we know that the prescription is to take the symmetric combination $\langle\hat{p}\hat{q}+\hat{q}\hat{p}\rangle/2$, but for higher powers there are several possible combinations. As it turns out, the correct combination of $\langle\hat{p}\ldots\hat{q}\ldots\rangle$ is obtained by differentiating the Moyal function at $\chi_p=0,\chi_q=0$. This is equivalent to taking the average with the Wigner function $\overline{p^m q^n}\to\int dpdqp^m q^n\mathcal{W}(p,q)$.

Now, for a finite-dimensional system, two questions arise:

(i) How do we define two complementary operators $\hat{A}$ and $\hat{B}$? and

(ii) How do we define the quantum characteristic function?

Clearly, we place the restriction that in the limit $d\to\infty$ of an infinite dimension, $\hat{A}\to\hat{q}$, $\hat{B}\to\hat{p}$, and definition (2) is recovered. The answers to the questions above are not unique, since the quantum characteristic function, (2), can be written in several equivalent ways using the Baker-Campbell-Hausdorff formula, making the extension to a finite dimension ambiguous. The sense in which the operators $\hat{A}$ and $\hat{B}$ are complementary cannot be that a relation $[\hat{A},\hat{B}]=i$ is satisfied, since, by taking the trace of this expression, we get the contradiction $0=id$. The canonical commutation relation can be obeyed only in an infinite-dimensional space, where the domain of $\hat{q}$ and $\hat{p}$ is a proper subset of the full Hilbert space. Question ii is strictly related to the generalization of the Wigner function to a finite-dimensional system, a subject of great interest that has spawned many proposals [24].

Here, instead of extolling the virtues of our pet proposal based on aesthetic considerations, we take a pragmatic attitude: we consider the sequential measurement of two arbitrary operators, then define the pair of observables $\hat{A},\hat{B}$ as complementary when they simplify the expression for the measurement and define the discrete characteristic function in such a way that the final characteristic function of the outputs has a simple expression in terms of it as well. The definitions presented below, hence, were not chosen arbitrarily, but were suggested by the physics, as explained in the Methods section.

We answer question 1 following Schwinger [25]: we consider an orthonormal basis $|m\rangle$ labeled by an index $m\in\mathcal{I}=\{-S,-S+1,\ldots,S\}$, with $d=2S+1$ the dimension of the Hilbert space. Thus m is either an integer (a half-even) or a half-odd number, depending which of the two S is. We define the conjugate basis as

$$|\widetilde{m}\rangle=\frac{1}{\sqrt{d}}\sum_{m'}\exp[2\pi imm'/d]|m'\rangle. \tag{4}$$

It is easy to check that $|\widetilde{m}\rangle$ form an orthonormal basis when m ranges in $\mathcal{I}$. Notice that the tilde symbol is associated to the basis, not to the index m.

We define an operator $\hat{A}$ having ma_0 as eigenvalues and $|m\rangle$ as eigenstates, and an operator $\hat{B}$ having the eigenvalues mb_0 but $|\widetilde{m}\rangle$ as eigenstates; the scales are $a_0=l_0/\sqrt{d}$ and $b_0=2\pi/(l_0\sqrt{d})=2\pi/(da_0)$, with l_0 some fundamental length scale. The scaling factors guarantee that $\hat{A}\to\hat{q}$ and $\hat{B}\to\hat{p}$ for $d\to\infty$. We consider a sequential measurement, with a first probe measuring $\hat{A}$, and then a second probe measuring

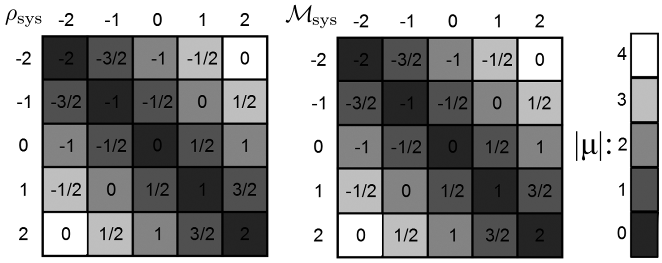

FIG. 1. Allowed values of $\bar{M}$ (numbers) for fixed μ (color) in $d = 5$. Elements of $\mathcal{M}_{\rm sys}$ with a given μ are combinations of the elements of $\rho_{\rm sys}$ with the same μ, i.e., belonging to the same diagonal parallel to the main diagonal of the density matrix.

$\hat{B}$. Here and in the following, we consider the momentum p in units of $\hbar$, so that it has dimensions L^{-1}. We remark that

$$\exp[iza_0\hat{B}]|m\rangle = (-1)^{(d-1)r_{m-z}}|f(m-z)\rangle \tag{5}$$

for any $m \in \mathcal{I}$ and any $z \in \mathbb{Z}$, where $f(m-z)$ is the difference $m - z$ reduced to the interval $\mathcal{I}$ by subtracting an appropriate integer multiple of d, $r_{m-z}d$. In particular, $\exp[ia_0\hat{B}]|-S\rangle = (-1)^{d-1}|S\rangle$ and $\exp[-ia_0\hat{B}]|S\rangle = (-1)^{d-1}|-S\rangle$. Thus $\hat{B}$ is the generator of the modular translations for the basis $|A\rangle$. The viceversa also holds true. As a matter of fact, our conventions differ from the ones used by Schwinger [25], and coincide with the ones introduced by de la Torre and Goyeneche [26].

We answer question 2 defining the Moyal function as

$$\mathcal{M}_{\rm sys}(\phi_A;a) = \sum_{\bar{M}} e^{i\phi_A\bar{M}a_0}\left\langle \bar{M} + \frac{\mu}{2}\right|\rho_{\rm sys}\left|\bar{M} - \frac{\mu}{2}\right\rangle, \tag{6}$$

where μ is an integer of the form $m - m'$, with $m,m' \in \mathcal{I}$, so $\mu \in [1-d,d-1]$, and $a = \mu a_0$. The sum over $\bar{M}$ is restricted by the condition that $\bar{M} \pm \mu/2$ belong to $\mathcal{I}$. See Fig. 1 for an example. This is a fundamental difference from the definition proposed by Leonhardt [18,19]. For instance, if μ takes its maximum value $\mu = 2S = d - 1$, then $\bar{M}$ can only be zero. In general, the values of $\bar{M}$ go from $-S + |\mu|/2$ to $S - |\mu|/2$, and $\bar{M}$ is integer or half-odd depending whether $S - |\mu|/2$ is. While the Moyal function Eq. (6) is defined for any ϕ_A, in order to invert it we need to evaluate only at the finite discrete values $\phi_A = 2\pi\bar{M}_A/[a_0(d-|\mu|)]$, with $\bar{M}_A \in [-S + |\mu|/2, S - |\mu|/2]$,

$$\langle m|\rho_{\rm sys}|m'\rangle = \sum_{\bar{M}_A} \frac{e^{-2\pi i\bar{M}_A\bar{M}/(d-|\mu|)}}{d-|\mu|}\mathcal{M}_{\rm sys}\left(\frac{2\pi\bar{M}_A}{a_0(d-|\mu|)};\mu a_0\right), \tag{7}$$

with $\bar{M} = (m+m')/2$ and $\mu = m - m'$.

As an example, consider a spin-1/2 particle. Then we can take $\hat{A} = \sigma_z/2$, and $\hat{B} = -\pi\sigma_y/2$ as complementary observables, with σ_j Pauli matrices, having chosen $l_0 = \sqrt{2}$ and hence $a_0 = 1$, $b_0 = \pi$. The general state $\rho_{\rm sys} = (1 + \mathbf{n}\cdot\boldsymbol{\sigma})/2$ has the characteristic function $\mathcal{M}_{\rm sys}(\phi_A;0) = \cos(\phi_A/2) + i\sin(\phi_A/2)n_z$, $\mathcal{M}_{\rm sys}(\phi_A;\pm 1) = (n_x \mp in_y)/2$. In this case, the inversion formula (7) gives directly the off-diagonal elements for $\mu = 1$, while for $\mu = 0$ the required values of ϕ_A are $\pm\pi/2$.

Finally, we assume that the initial quantum state of the probes is known, that the pointer variables $\hat{J}_A,\hat{J}_B$, have a continuous spectrum and thus have conjugate variables, $\hat{\Phi}_A,\hat{\Phi}_B$, respectively. Starting from the initial density operator of the two probes $\rho_{\rm pr}$, we infer their initial Moyal function

$$\mathcal{M}_{\rm pr}(\phi;j) = \langle \exp(i\phi\cdot\hat{J} + ij\cdot\hat{\Phi})\rangle = \int dJ\, e^{i\phi\cdot J}\left\langle J + \frac{j}{2}\right|\rho_{\rm pr}\left|J - \frac{j}{2}\right\rangle. \tag{8}$$

For brevity, we indicate by $J = (J_A,J_B)$, $\phi = (\phi_A,\phi_B)$, etc., vectors in an auxiliary two-dimensional Euclidean space.

III. METHODS

Let us consider the probability of observing a readout $J = (J_A,J_B)$ from the two detectors after they have interacted with the system through the von Neumann model,

$$H_{\rm int} = -\delta(t+\tau)\hat{A}\hat{\Phi}_A - \delta(t-\tau)\hat{B}\hat{\Phi}_B, \tag{9}$$

with $\tau \to 0^+$ an infinitesimal time. For now, no relation is assumed between the observables of systems $\hat{A}$ and $\hat{B}$. The variables $\hat{\Phi}$ belong to the detectors, and they are conjugated to the readout variables, $[\hat{\Phi},\hat{J}] = i$. By Born's rule,

$$\mathcal{P}(J) = {\rm Tr}\{[\mathbb{1}\otimes\hat{\Pi}(J)]U_{\rm int}[\rho_{\rm sys}\otimes\rho_{\rm pr}]U_{\rm int}^\dagger\}, \tag{10}$$

with $U_{\rm int} = \exp[i\hat{B}\hat{\Phi}_B]\exp[i\hat{A}\hat{\Phi}_A]$ the time-evolution operator and $\hat{\Pi}(J)$ the projection operator over the eigenstates of $\hat{J}$ with eigenvalues J.

Next, we consider the characteristic function, defined as the Fourier transform of the observable probability:

$$\mathcal{Z}(\phi) = \int dJ e^{i\phi\cdot J}\mathcal{P}(J) = {\rm Tr}\{[\mathbb{1}\otimes e^{i\phi\cdot\hat{J}}]U_{\rm int}[\rho_{\rm sys}\otimes\rho_{\rm pr}]U_{\rm int}^\dagger\}. \tag{11}$$

We write the trace as

$${\rm Tr}[\hat{O}] = \sum_B \int dJ\langle B,J|\hat{O}|B,J\rangle, \tag{12}$$

obtaining

$$\mathcal{Z}(\phi) = \sum_{B,A,A'}\int dJ e^{i\phi\cdot J}\langle J - C|\rho_{\rm pr}|J - C'\rangle \times \langle B|A\rangle\langle A|\rho_{\rm sys}|A'\rangle\langle A'|B\rangle, \tag{13}$$

where we have written the initial state of the system in the basis of eigenstates of $\hat{A}$, $\rho_{\rm sys} = \sum_{A,A'}|A\rangle\langle A|\rho_{\rm sys}|A'\rangle\langle A'|$, exploited the fact that $\hat{\Phi}$ generates the translations in the $|J\rangle$ basis, $\exp[ix\cdot\hat{\Phi}]|J\rangle = |J + x\rangle$, and defined the auxiliary vectors $C = (A,B)$, $C' = (A',B)$. Now, let us define $\bar{A} = (A + A')/2$ and $a = A - A'$ and change the integration variables to $J_A - \bar{A}$ and $J_B - B$. Then

$$\mathcal{Z}(\phi) = \sum_a \mathcal{M}_{\rm pr}(\phi;j_a)N_{\rm sys}(a|\phi), \tag{14}$$

$$N_{\rm sys}(a|\phi) = \sum_{\bar{A}\in D_a} e^{i\phi_A\bar{A}}\left\langle \bar{A} - \frac{a}{2}\right|e^{i\phi_B\hat{B}}\left|\bar{A} + \frac{a}{2}\right\rangle \times \left\langle \bar{A} + \frac{a}{2}\right|\rho_{\rm sys}\left|\bar{A} - \frac{a}{2}\right\rangle, \tag{15}$$

where $j_a = (-a,0)$ and we have introduced the Moyal quasicharacteristic function for the probes, as defined in Eq. (8). Note that the domain of summation in $\bar{A}$ depends on a. In

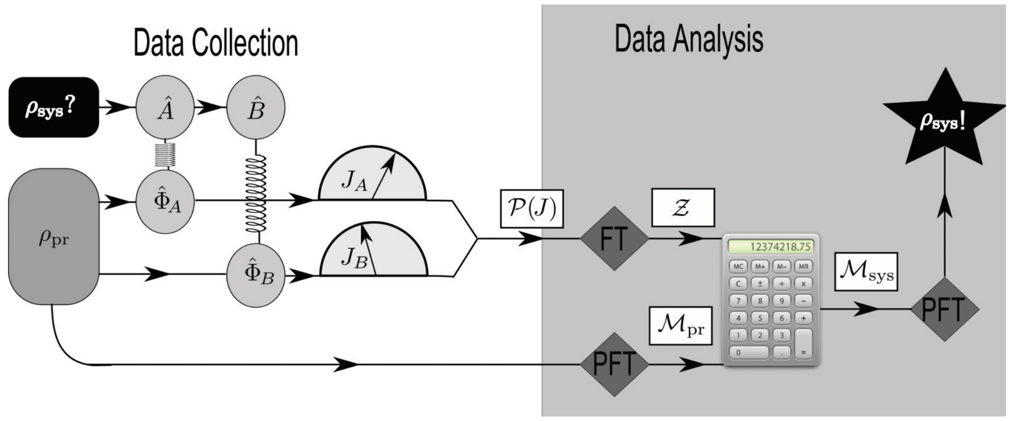

FIG. 2. (Color online) Schematic of the tomography. A system in an unknown state couples sequentially to two detectors through the interactions $\hat{A}\hat{\Phi}_A$ and $\hat{B}\hat{\Phi}_B$. The outputs J_A and J_B of the probes are read, the measurement repeated a large number of times, and the joint probability $\mathcal{P}(J)$ estimated. Then the characteristic function $\mathcal{Z}$ is extracted through a Fourier transform (FT), and the known state of the probes undergoes a partial FT (PFT) to give their Moyal function $\mathcal{M}_{\rm pr}$. Simple algebraic operations (calculator icon) are then applied to $\mathcal{Z}$ and $\mathcal{M}_{\rm pr}$ in order to get the Moyal function of the system $\mathcal{M}_{\rm sys}$. Finally, a PFT yields the target density matrix.

general, Eqs. (14) and (15) are too complicated to invert and be useful in reconstructing the quantum state. For instance, if $\hat{A}$ and $\hat{B}$ commute, only diagonal terms contribute to $N_{\rm sys}(a|\phi)$, so that no reconstruction of the quantum state is possible, as one can only find the the diagonal elements of $\rho_{\rm sys}$, as expected. Furthermore, if $\hat{A}$ and $\hat{B}$ have mutually unbiased eigenbases with a constant relative phase, such that $\langle A|B\rangle = 1/\sqrt{d}$, then $N_{\rm sys}(a|\phi) = g(\phi_B)\mathcal{M}_{\rm sys}(\phi_A;a)$, with $g(\phi_B) = \sum_B \exp(i\phi_B B)/d$, and no actual simplification occurs.

On the other hand, it is clear from Eq. (15) that if, for some ϕ_B, the operator $\exp(i\phi_B\hat{B})$ translates the eigenstates of $\hat{A}$ into each other, then few terms (precisely, two) in a survive. Thus, we exploit the freedom that we have in choosing the bases $|A\rangle$ and $|B\rangle$, and we assume that they are Fourier conjugated, i.e.,

$$\langle A|B\rangle = \frac{\exp[iBA]}{\sqrt{d}}, \tag{16}$$

with the eigenvalues of $\hat{A}$ being of the form $A = ma_0$, and those of $\hat{B}$ being $B = mb_0$, with m an integer or half-odd in the range $[-S,S]$.

We write $\exp[i\phi_B\hat{B}]$ in Eq. (15) as $\sum_B |B\rangle\langle B|\exp[i\phi_B B]$, then substitute Eq. (16) in Eq. (15) so rewritten, obtaining

$$\begin{aligned} N_{\rm sys}(a|\phi) &= \sum_m \sum_M \frac{e^{i\phi_A\bar{A}+i(\phi_B-a)B}}{d}\left\langle \bar{A}+\frac{a}{2}\middle|\rho_{\rm sys}\middle|\bar{A}-\frac{a}{2}\right\rangle \\ &= \frac{\sin[\pi(\phi_B-a)\sqrt{d}]}{d\sin[\pi(\phi_B-a)/\sqrt{d}]}\mathcal{M}_{\rm sys}(\phi_A;a), \end{aligned} \tag{17}$$

with $B = mb_0$, $m\in\mathcal{I}$, $\bar{A} = \bar{M}a_0$, $\bar{M}\in[-S+|\mu|/2,S-|\mu|/2]$, $a=\mu a_0$, and $\mu\in[1-d,d-1]$. We have introduced the Moyal quasicharacteristic function of the system, relative to the $|A\rangle$ basis, defined in Eq. (6). Furthermore, for $\phi_B = \mu' a_0$, $\mu'\in[1-d,d-1]$, $N_{\rm sys}(a|\phi)$ in Eq. (17) simplifies to

$$N_{\rm sys}(a|\phi) = \delta_{a,\phi_B}\mathcal{M}_{\rm sys}(\phi_A;\phi_B) + \delta_{a,\bar{\phi}_B}\mathcal{M}_{\rm sys}(\phi_A;\bar{\phi}_B). \tag{18}$$

For $\phi_B = 0$, instead, only one term survives:

$$N_{\rm sys}(a|(\phi_A,0)) = \delta_{a,0}\mathcal{M}_{\rm sys}(\phi_A;0). \tag{19}$$

Hence, after substituting Eq. (18) into Eq. (14) evaluated at the discrete points $\phi_B = \mu' a_0$, we get the main result, Eq. (20).

IV. RESULTS

After repeating the measurement of $\hat{A}$ and $\hat{B}$ many times, we can estimate $\mathcal{P}(J_A,J_B)$, the joint probability of observing the outputs J_A and J_B in two probes that make a nondemolition measurement of the system. Then we calculate $\mathcal{Z}(\phi_A,\phi_B)$, the final characteristic function, i.e., the Fourier transform of $\mathcal{P}(J_A,J_B)$. The following relation holds between the final characteristic function and the initial Moyal functions,

$$\begin{aligned} \mathcal{Z}(\phi) &= \mathcal{M}_{\rm pr}(\phi;-\phi\sigma_+)\mathcal{M}_{\rm sys}(\phi_A;\phi_B) \\ &\quad + \mathcal{M}_{\rm pr}(\phi;-\bar{\phi}\sigma_+)\mathcal{M}_{\rm sys}(\phi_A;\bar{\phi}_B), \end{aligned} \tag{20}$$

for any ϕ_A and for $\phi_B = \mu a_0$, with μ an integer in the range $[1-d,d-1]$, excluding $\mu = 0$; here, $\sigma_+ = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$, $\bar{\phi} = (\phi_A,\bar{\phi}_B)$, and $\bar{\phi}_B = [\mu - \mathrm{sgn}(\mu)d]a_0$. Equation (20) is the central result of this paper.

Note that $\bar{\bar{\phi}}_B = \phi_B$. Thus, if we take Eq. (20) at $\phi = (\phi_A,\bar{\phi}_B)$, we have a closed system of two linear equations in the two unknowns $x = \mathcal{M}_{\rm sys}(\phi_A;\phi_B)$ and $y = \mathcal{M}_{\rm sys}(\phi_A;\bar{\phi}_B)$. Therefore, we have to solve several decoupled linear equations in two unknowns for different values of ϕ_A. This allows us to finally reconstruct the density matrix in the basis of the eigenstates of $\hat{A}$ by using Eq. (7). Figure 2 illustrates the above procedure. In the limit $d\to\infty$, the second addend in Eq. (20) goes to 0, and the result of Ref. [13] is then recovered.

V. DISCUSSION

An issue to consider is whether assuming the state of the detectors to be known introduces some circularity into the argument. On one hand, we could consider self-consistent calibration and bootstrapping, and on the other hand, the state of the detectors could be determined by means of a standard quantum state tomography scheme for a continuous variable [27]. Then one would know that the detectors prepared in a certain way are in a state ρ_{pr} and could use them to apply the tomographic scheme presented herein to determine the state of any quantum system that couples appropriately to the detectors, leading to an overall increased efficiency.

For simplicity of exposition, we have used the von Neumann model of measurement and assumed that the readout of the detectors had infinite precision. However, the results are valid for any nondemolition sequential measurement, and it can be shown that, under some hypotheses, a finite resolution in the readout introduces a factor $z_0(\phi)$ in front of the right-hand side of Eq. (20).

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APPENDIX A: COMPARISON WITH LEONHARDT'S DEFINITION OF THE MOYAL FUNCTION

Leonhardt [18,19] proposed a tomographic scheme based on a definition of quantum characteristic function for finite-dimensional Hilbert spaces. While Leonhardt's definition differs from ours, the two definitions are related, and in the following we discuss them. We base our discussion on Ref. [19].

First, a remark about the notation is in order. Leonhardt uses an index m that ranges from $-S$ to $S=(d-1)/2$ for odd d and from $1-d/2$ to $d/2$ for even d. We use the letter l, instead of m, for this index, while we keep m to denote an integer or half-odd in the range $[-S,S]$, as in the text. Furthermore, the states $|l\rangle$ coincide with our states $|m\rangle$ for odd d, but for even d there is a difference between our notation and Leonhardt's. Here, we indicate as customary with $|m\rangle$ the states of the tomographic basis, with the proviso that for even d, Leonhardt uses the notation $|m+1/2\rangle_L$. To keep the notation compact, we introduce the number $f=1$ for even d and $f=0$ for odd d, representing the fermionic character of the Hilbert space.

Leonhardt defines the characteristic function as

$$\widetilde{W}(\nu,n) = \sum_{l=-S+f/2}^{S+f/2} \exp\left[-\frac{4\pi i}{d}n(l-\nu)\right]{}_L\langle l|\rho|l-2\nu\rangle_L = \sum_{m=-S}^{S} \exp\left[-\frac{4\pi i}{d}n(m+f/2-\nu)\right]\langle m|\rho|m-2\nu\rangle, \tag{A1}$$

with the convention that whenever $m-2\nu$ is outside the range $[-S,S]$, it is reduced back to it by adding or subtracting an appropriate multiple of d. Note that 2ν is limited to integer values, but n can be arbitrary. Thus, for ease of comparison, we put $-4\pi n/d=\phi$ and $2\nu=\mu$, substitute ϕ for n as the first argument, and substitute μ for ν as the second argument, so that $\widetilde{\mathcal{W}}(\phi;\mu)\equiv \widetilde{W}(\mu/2,-\phi d/(4\pi))$. Furthermore, noting that $\widetilde{\mathcal{W}}(\phi;\mu+d)=\exp[-i\phi d/2]\widetilde{\mathcal{W}}(\phi;\mu)$, the values of μ can be restricted to the range $[0,d-1]$.

In the text, we defined the Moyal function as

$$\mathcal{M}(\phi;\mu) = \sum_{\bar{M}=-S+|\mu|/2}^{S-|\mu|/2} e^{i\phi\bar{M}}\left\langle \bar{M}+\frac{\mu}{2}\middle|\rho\middle|\bar{M}-\frac{\mu}{2}\right\rangle. \tag{A2}$$

With the position $m\to m-\mu/2$, we can rewrite Eq. (A1) as

$$\widetilde{\mathcal{W}}(\phi;\mu) = \sum_{m=-S-\mu/2}^{S-\mu/2} \exp[i\phi(m+f)] \times \langle m+\mu/2|\rho|m-\mu/2\rangle. \tag{A3}$$

For $\mu=0$, we have that the two definitions coincide, apart from a phase factor for the even-dimensional case:

$$\widetilde{\mathcal{W}}(\phi;0) = e^{i\phi f/2}\mathcal{M}(\phi;0). \tag{A4}$$

For $\mu>0$, we can split the sum in Eq. (A3) as

$$\widetilde{\mathcal{W}}(\phi;\mu) = \left[\sum_{m=-S+\mu/2}^{S-\mu/2} + \sum_{-S-\mu/2}^{-S+\mu/2-1}\right]\exp[i\phi(m+f/2)] \times \langle m+\mu/2|\rho|m-\mu/2\rangle. \tag{A5}$$

The first sum yields $\exp[i\phi f/2]\mathcal{M}(\phi;\mu)$. In the second sum, we put $\mu=\widetilde{\mu}+d$ (note that $\widetilde{\mu}<0$), $m=\widetilde{m}-d/2$, obtaining

$$\sum_{\widetilde{m}=-S-\widetilde{\mu}/2}^{S+\widetilde{\mu}/2} \exp\{i\phi[\widetilde{m}+(f-d)/2]\} \times \langle \widetilde{m}+\widetilde{\mu}/2+d|\rho|\widetilde{m}-\widetilde{\mu}/2\rangle = \exp[i\phi(f-d)/2]\mathcal{M}(\phi;\widetilde{\mu}), \tag{A6}$$

where we have exploited the convention that $|\widetilde{m}+\widetilde{\mu}/2+d\rangle = |\widetilde{m}+\widetilde{\mu}/2\rangle$. Thus, we find that, for $\mu>0$,

$$\widetilde{\mathcal{W}}(\phi;\mu) = \exp[i\phi f/2]\mathcal{M}(\phi;\mu) + \exp[i\phi(f-d)/2]\mathcal{M}(\phi;\mu-d). \tag{A7}$$

In particular, for the discrete values considered by Leonhardt, $\phi=-4\pi n/d$,

$$\widetilde{W}(\nu,n) = \exp[-2\pi i n f/d][\mathcal{M}(-4\pi n/d;2\nu) + \mathcal{M}(-4\pi n/d;2\nu-d)]. \tag{A8}$$

APPENDIX B: FURTHER DISCUSSION

As both $\hat{A}$ and $\hat{B}$ have the same eigenvalues, except for a trivial rescaling, we can write $\hat{B}=(b_0/a_0)U\hat{A}U^\dagger$, with U a unitary operator. Precisely, $U=\sum_m|\widetilde{m}\rangle\langle m|$. Let us say, for definiteness, that the Hilbert space represents an angular momentum S and that $\hat{A}=a_0\hat{S}_z$ is proportional to an angular momentum operator, in the sense that, upon rotation, it transforms accordingly. The natural question arises: Is $\hat{B}/b_0$ an angular momentum operator as well? i.e., Is there a unit vector $\mathbf{n}$ such that $\hat{B}=b_0\mathbf{n}\cdot\hat{\mathbf{S}}$? The answer is no, unless $d=2$, since in the latter case any unitary operator corresponds to a rotation. In general, however, the distinct unitary operators, *modulo* a global phase, are characterized by d^2-1 real parameters, while there are only three independent rotations [28]. The proof that, for $d>2$, none of these rotations yields $\hat{B}/b_0=\hat{S}_{\mathbf{n}}$ is as follows: since $\exp(-(i/\sqrt{d})\hat{B})|S\rangle=\pm|-S\rangle$, $\hat{B}$ must be $\hat{B}=(2z+1)\pi\sqrt{d}\hat{S}_\perp$, with $z\in\mathbb{Z}$ and the $\perp$ symbol indicating an appropriate direction in the plane orthogonal to Z. Thus, $\hat{S}_{\mathbf{n}} = [(2z+1)d/2]\hat{S}_{\perp}$. This equation implies necessarily that $\perp = \pm\mathbf{n}$, $d=2$, and either $z=0$ or $z=-1$.

At any rate, Reck *et al.* [29] have proved that any unitary operator U in a finite-dimensional Hilbert space can be realized by a suitable combination of elementary unitary operators that act nontrivially only in a two–dimensional subspace. Furthermore, in quantum computation, it is well known that if the Hilbert space is made up of N distinguishable qubits, any unitary operator can be approximated at will by a sequence of controlled nots and of elementary unitary operations on each qubit.

The main problem consists, then, in constructing the operator $\hat{A}$, in the worst-case scenario that this is not provided to us by Nature. For a system composed of n distinguishable qubits, the operator $\hat{A}$ can be constructed, apart from a trivial shift and rescaling as $\hat{A} = \sum_{p=1}^{N} 2^{p-1}\sigma_{z,p}$, with $\sigma_{z,p}$ a spin operator on the pth qubit.

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[28] An important auxiliary question is, then, What physical operations do the remaining d^2-4 generators represent? If the angular momentum is obtained by adding several spin-1/2 systems, we know that these additional operations are individual spin rotations and entangling transformations. This consideration implies that all elementary particles are either spin-1/2 fermions, spin-0 bosons, or spin-1 massless bosons.
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