QED in a momentum-cutoff vacuum

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We consider a vacuum in which all of the wavelengths of vacuum fluctuations in a preferred inertial frame are longer than a given minimum length $\hbar/\Lambda$. This paper studies spinor QED in such momentum-cutoff vacuums, in particular, the Lorentz anomalies which appear in the radiative corrections that result from vacuum fluctuations in the continuum limit $\Lambda \to \infty$. A gauge-invariant momentum-cutoff generating functional $Z_\Lambda[A,J]$ in the background-field formalism is given, from which well-defined radiative correction terms as well as renormalization constants can be derived, at least in the form of a loop-expansion series. Using the conventional Lorentz-covariant renormalized perturbation procedure, one-loop and two-loop (for photon self-energy only) calculations are carried out in detail. We find that the non-Lorentz-covariant terms in one-loop and two-loop radiative corrections converge to nonvanishing terms in the limit $\Lambda \to \infty$. The physical meaning as well as some of the phenomenological consequences of these Lorentz anomalous terms are discussed.

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I. INTRODUCTION

In quantum field theory, the perturbation calculations of radiative corrections lead to divergent loop integrals. To make these divergent integrals mathematically meaningful, we are required to devise a suitable regularization scheme to treat the divergent integrals as limits of convergent ones. The most intuitive regularization method is to introduce a cutoff Λ in the momentum integrals. This method was employed in earlier literature for QED and was abandoned in practical loop-integral computations after 1972, mainly because the gauge invariance cannot be maintained explicitly in this regularization procedure [1]. In an earlier paper [2], we proposed a gauge-invariant momentum-cutoff regularization scheme for QED and carried out one-loop calculations. This paper will show that the scheme presented in Ref. [2] can be generalized to multiloop calculations in QED and also to calculations in non-Abelian gauge theories.

The key step of our momentum-cutoff regularization scheme is to write the gauge-invariant loop integrals in the form of momentum integrals with gauge-invariant integrands, which are obtained by following a special parallel-transportation procedure. In comparison with the commonly used gauge-invariant regularization schemes (Pauli-Villars or dimensional scheme), which are devised specifically for treating divergent integrals appearing in loop expansions, this cutoff scheme has the advantage that, in the background-field approach, it can be implemented directly on the gauge-invariant generating functional $Z[A,J]$. In other words, we can construct a regularized version of $Z[A,J]$ (denoted as $Z_\Lambda[A,J]$) which preserves the gauge-invariant character, and all of the Green functions derived from $Z_\Lambda[A,J]$ are mathematically well defined. Furthermore, based on the observation of the difference between the gauge-covariant free propagator and the gauge-invariant free propagator, we shall propose a renormalization procedure for the electron propagator in which the renormalization is not only gauge invariant, but also independent of the gauge fixing in covariant gauges.

Another subject of great concern in this paper is whether the Lorentz symmetry of the original theory would be preserved after implementing a momentum-cutoff scheme. In the case of a sharp momentum cutoff, this issue is related to a proper designation of a momentum-integration domain $\mathcal{D}(\Lambda)$ in the momentum space. In Ref. [2], we have shown that all Lorentz-invariant $\mathcal{D}(\Lambda)$ in the Minkowski four-momentum space are noncompact and are not suitable for rendering divergent integrals finite, while a Lorentz-invariant compact $\mathcal{D}(\Lambda)$ can be easily constructed in a Euclidean four-momentum space. Thus, we have two alternatives to make the divergent integrals finite by using the cutoff method, i.e., either to use a Lorentz-invariant cutoff scheme, which is a purely mathematical procedure and has no physical consequences, or to use a non-Lorentz-covariant cutoff scheme, which may provide us with a finite but non-Lorentz-covariant quantum field theory.

The main purpose of this paper is to examine a finite QED theory based on the assumption that all of the wavelengths of vacuum quantum fluctuations in a preferred reference frame are longer than a minimum length $\hbar/\Lambda$. In comparison with the conventional QED theory, we would like to emphasize the following aspects of this momentum-cutoff QED theory (denoted in the following as Λ-QED).

(i) Λ-QED preserves the gauge-invariant character of the conventional QED and can be constructed based on a gauge-invariant momentum-cutoff generating functional $Z_\Lambda[A,J]$, which is expected to be a well-defined functional.

(ii) Since the effects from high-momentum virtual quanta have been replaced with a simple boundary condition in the momentum space, Λ-QED is not assumed to be able to give a good description of the extremely high-energy physics.

(iii) For studying the low-energy physics described with Λ-QED, we shall use the conventional renormalized perturbation procedure (i.e., adding Lorentz-invariant counterterms to eliminate divergent outcomes). But now the renormalization will no longer play the role of the rescuer of the theory from inconsistency, unless the continuum limit $\Lambda \to \infty$ is actually taken.

(iv) Noting that the cutoff is implemented on the quantum fluctuation part of the field while its classical part is held unchanged, the difference between the Λ-QED and the

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conventional QED appears only in the radiative-correction calculations. The same consideration also illustrates the usefulness of the background-field formalism in studying the Λ-QED.

(v) In the description of the effects from high-momentum virtual quanta on the low-energy regime, a major difference between the conventional QED and the Λ-QED is that the latter may exhibit extra non-Lorentz-covariant effects. In this respect, we shall face one of the following three cases when the continuum limit $\Lambda \rightarrow \infty$ is taken:

(a) All Lorentz anomalies, i.e., non-Lorentz-covariant terms which show up at this limit in the renormalized radiative corrections, vanish.

(b) All Lorentz anomalies are finite and some of them are nonvanishing.

(c) Some of the Lorentz anomalies diverge.

In case (a), the conventional QED and the Λ-QED give essentially the same description of the low-energy physics. In case (b), their descriptions of the low-energy physics will be different unless all Lorentz anomalies appearing in the Λ-QED can be canceled out with a finite renormalization of the light velocity. Last, if case (c) happens, one would not regard the Λ-QED as a physically sound theory, at least in its present form.

Accordingly, in order to establish a quantum field theory based on a momentum-cutoff vacuum, as well as to construct a gauge-invariant momentum-cutoff generating functional $Z_\Lambda[A,J]$, it is equally important to know the asymptotic behavior of the non-Lorentz-covariant consequences deduced from this generating functional. In this paper, we present one-loop and two-loop (for photon self-energy only) calculations of Lorentz anomalies, and the result indicates that the Λ-QED we are trying to construct probably belongs to the case (b).

Finally, we make several remarks, as follows, concerning the meaning and the implications of the minimum length $\hbar/\Lambda$ appearing in our momentum-cutoff theory, as the recent studies in the field of quantum gravity have shown that the problems of violation of causality and/or unitarity are usually accompanied with a quantum field theory of a fundamental length [3].

(i) Unlike the fundamental length which manifests itself at Planck's scale in the regime of quantum gravity, the minimum length $\hbar/\Lambda$ of a vacuum is introduced artificially into quantum field theory (QFT) primarily for suppressing the effects from high-momentum virtual quanta. Thus, in contrast with the fact that the existence of the Planck's scale implies a drastic change of the very concept of the spacetime beyond this scale [4], the implications of the minimum length $\hbar/\Lambda$ are comparatively simple and limited only within the regime of radiative corrections. On the other hand, it is important to remark that, in Λ-QED, the minimum length of the vacuum is allowed to be removed by using the conventional renormalization technique while its effects (Lorentz anomalies) remain.

(ii) Unitarity is indispensable for a workable Λ-QFT and will be preserved when the Λ cutoff is implemented directly on all quantum fluctuation fields. However, as discussed in this paper, such a cutoff scheme may break the gauge symmetry and should be replaced with a more sophisticated scheme which is applied on the trace of a gauge-covariant operator. In Sec. II, we shall propose a cutoff scheme which ensures that both the gauge symmetry and the unitarity of the original theory will be preserved.

(iii) It is obvious that when we introduce a minimum length in a relativistic field theory, we have to reconsider the notion of causality. This problem exists undeniably in Λ-QED because the Lorentz anomaly would certainly lead to a breakdown of microcausality. However, if the Lorentz violation is small, we may anticipate that the Λ-QED will still describe a perfectly local and causal physics in some reference frames [5].

The paper is organized as follows. In Sec. II, we illustrate the notations used in this paper. Then, for realizing the gauge-invariant momentum-cutoff project, we propose a parallel-transportation scheme so that one can construct a gauge-invariant p representation for the trace of the gauge-covariant operator used in (Abelian or non-Abelian) gauge-field theories. In Sec. III, we construct a gauge-invariant momentum-cutoff version of the generating functional of QED in the background-field formalism and discuss the issue of gauge dependence. Section IV presents the one-loop calculations for photon and electron self-energy, respectively. There we also show how to define a dressed electron propagator in a gauge-independent way. In Sec. V, we calculate the two-loop photon self-energy by the renormalized perturbation. We discuss the physical meaning as well as the phenomenological consequences of the Lorentz anomaly in Sec. IV.

II. GAUGE-INVARIANT *p* REPRESENTATION OF THE TRACE OF A GAUGE-COVARIANT OPERATOR

We begin by specifying some basic notations which will be used throughout this paper. Let $\mathcal{H}\{|\varphi\rangle\}$ be a field space. In x representation, a c number (or Grassmann number) field is represented as $\varphi_r(x) = \langle r,x|\varphi\rangle$, where r denotes the labels other than spacetime coordinates. Local gauge transformations are represented by local unitary operators U_θ on the field space $\mathcal{H}$ with

$$\langle r',x'|U_\theta|r,x\rangle = \delta_{\alpha'\alpha}\delta^{(4)}(x'-x)(e^{-i\theta^a(x)T^a})_{i'i}, \quad r=(\alpha,i), \tag{2.1}$$

where i denotes the index of representation space of the gauge group G, and $T^a,\theta^a(x)$, $a=1,\ldots,n$, are generators of G and local gauge functions, respectively.

We shall represent a vector gauge field as an element $|A_\mu\rangle$ of the field space $\mathcal{H}_{ad}$ with $A^a_\mu(x) = \langle a,x|A_\mu\rangle$, where a denotes the index of the adjoint representation of G. Sometimes we shall also represent the vector gauge field as a local Hermitian operator A_μ on a field space $\mathcal{H}$, defined by

$$\begin{aligned}\langle r',x'|A_\mu|r,x\rangle &= \delta_{\alpha'\alpha}\delta^{(4)}(x'-x)[A_\mu(x)]_{i'i},\\ A_\mu(x) &= A^a_\mu(x)T^a.\end{aligned} \tag{2.2}$$

Under a local gauge transformation, the vector gauge field transforms according to

$$A_\mu(x) \longrightarrow A^\theta_\mu(x) = U_\theta(x)\left[A_\mu(x) - \frac{i}{g}\partial_\mu\right]U^{-1}_\theta(x). \tag{2.3}$$

Let $L[A]$ be an operator on $\mathcal{H}$ which depends on the gauge field $A^a_\mu(x)$. $L[A]$ is gauge covariant if it satisfies $L[A^\theta] = U_\theta L[A]U^{-1}_\theta$. The trace of $L[A]$ in the x representation can be written as $\mathrm{Tr}\{L[A]\} = \int d^4x \mathcal{L}(x|A)$, with $\mathcal{L}(x|A) = \langle x|\mathrm{tr}\{L[A]\}|x\rangle$, where tr denotes the trace in r space. Similarly, in p representation, we have $\mathrm{Tr}\{L[A]\} = \int \frac{d^4p}{(2\pi)^4}\mathcal{L}(p|A)$, with $\mathcal{L}(p|A) = \langle p|\mathrm{tr}\{L[A]\}|p\rangle$. We note that, for a gauge-covariant $L[A]$, the x-space density $\mathcal{L}(x|A)$ is gauge invariant, while the transformation law of the density $\mathcal{L}(p|A)$ is nonlocal in p space, under local gauge transformations.

When $L[A]$ is a trace-class gauge-covariant operator, there will be no problem of gauge symmetry breaking to use p representation for a calculation of $\mathrm{Tr}\{L[A]\}$. However, when $L[A]$ yields a well-defined p-space density $\mathcal{L}(p|A)$ but the integral of $\mathcal{L}(p|A)$ exhibits ultraviolet divergence, a momentum-cutoff regularization will usually break the gauge symmetry of the trace of $L[A]$. In order to obtain a gauge-invariant momentum-cutoff regularization of $\mathrm{Tr}\{L[A]\}$, it is advisable to replace $L[A]$ with a new operator $\tilde{L}[A]$ such that (i) $\mathrm{Tr}\{\tilde{L}[A]\} = \mathrm{Tr}\{L[A]\}$ and (ii) $\tilde{\mathcal{L}}(p|A) = \langle p|\mathrm{tr}\{\tilde{L}[A]\}|p\rangle$ is a gauge-invariant p-space density. In fact, such an operator can be constructed as follows.

First, we introduce a parallel-transportation operator $T[A]$ on $\mathcal{H}$, which is defined by

$$\langle x'|T[A]|x\rangle = P_s \exp\left\{-ig\int_0^1 ds A_\mu^a[x+s(x'-x)](x'-x)^\mu T^a\right\}, \tag{2.4}$$

where P_s denotes the path ordering with the ordering parameter s.

Second, for any two operators $\mathbf{A}$ and $\mathbf{B}$ on $\mathcal{H}$, we define (i) the x transposition $\mathbf{A}^{\mathbf{t}}$ of $\mathbf{A}$ by

$$\langle r',x'|\mathbf{A}^{\mathbf{t}}|r,x\rangle = \langle r',x|\mathbf{A}|r,x'\rangle; \tag{2.5}$$

(ii) the x product $\mathbf{A}\circ\mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ by

$$\langle r',x'|\mathbf{A}\circ\mathbf{B}|r,x\rangle = \sum_{r''}\langle r',x'|\mathbf{A}|r'',x\rangle\langle r'',x'|\mathbf{B}|r,x\rangle, \tag{2.6}$$

with $(\mathbf{A}^{\mathbf{t}})^\dagger = (\mathbf{A}^\dagger)^{\mathbf{t}}$ and $(\mathbf{A}\circ\mathbf{B})^\dagger = \mathbf{B}^\dagger\circ\mathbf{A}^\dagger$.

Let the operator $L'[A]$ be defined by

$$L'[A] = L[A]\circ T^{\mathbf{t}}[A]. \tag{2.7}$$

It is obvious that we have $\mathrm{Tr}\{L'[A]\} = \mathrm{Tr}\{L[A]\}$ because $\langle x|T[A]|x\rangle = 1$ implies $\mathcal{L}'(x|A) = \mathcal{L}(x|A)$. Furthermore, noting that $T[A]$ is gauge covariant, the transformation law of $L'[A]$ has the form

$$\begin{aligned}\langle x'|L'[A^\theta]|x\rangle &= \langle x'|U_\theta(x')L[A]U_\theta^{-1}(x)|x\rangle\langle x|U_\theta(x)T[A]U_\theta^{-1}(x')|x'\rangle \\ &= \langle x'|U_\theta(x')L'[A]U_\theta^{-1}(x')|x\rangle.\end{aligned} \tag{2.8}$$

From Eq. (2.8), we know that $\langle x'|\mathrm{tr}\{L'[A]\}|x\rangle$ is gauge invariant for all (x',x). Hence $\mathcal{L}'(p|A) = \langle p|\mathrm{tr}\{L'[A]\}|p\rangle$ is a gauge-invariant p-space density. So that the mapping $L[A]\to\tilde{L}[A]$ is Hermiticity preserving, we introduce another operator

$$L''[A] = (T^{\mathbf{t}}[A])^\dagger\circ L[A], \tag{2.9}$$

which also satisfies our requirements. Then we have

$$\tilde{L}[A] = \tfrac{1}{2}\{L'[A] + L''[A]\}. \tag{2.10}$$

In the following, we present explicit expressions of the gauge-invariant p density $\tilde{\mathcal{L}}(p|A)$ by expanding the matrix elements of $L[A]$ and $\tilde{L}[A]$ in p representation in powers of A,

$$\begin{aligned}\langle p'|L[A]|p\rangle &= \sum_{n=0}^\infty \frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n \\ &\quad\times \mathfrak{L}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p',p;k_1,\ldots,k_n)A^{a_1\mu_1}(k_1)\cdots A^{a_n\mu_n}(k_n),\end{aligned} \tag{2.11}$$

$$\begin{aligned}\langle p'|\tilde{L}[A]|p\rangle &= \sum_{n=0}^\infty \frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n \\ &\quad\times \tilde{\mathfrak{L}}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p',p;k_1,\ldots,k_n)A^{a_1\mu_1}(k_1)\cdots A^{a_n\mu_n}(k_n).\end{aligned} \tag{2.12}$$

Making use of (2.4) and noting that, in p representation, Eqs. (2.5) and (2.6) have the form

$$\langle r',p'|\mathbf{A}^{\mathbf{t}}|r,p\rangle = \langle r',-p|\mathbf{A}|r,-p'\rangle \tag{2.13}$$

and

$$\langle r',p'|\mathbf{A}\circ\mathbf{B}|r,p\rangle = \int\frac{d^4q}{(2\pi)^4}\int\frac{d^4q'}{(2\pi)^4}\sum_{r''}\langle r',q'|\mathbf{A}|r'',q\rangle\langle r'',p'-q'|\mathbf{B}|r,p-q\rangle, \tag{2.14}$$

we obtain

$$\begin{aligned}\tilde{\mathfrak{L}}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p',p;k_1,\ldots,k_n) &= \sum_{m=0}^n\frac{i^m}{m!}\int_0^1 ds_1\cdots\int_0^1 ds_m(\partial'+\partial)_{\mu_1}\cdots(\partial'+\partial)_{\mu_m} \\ &\quad\times\frac{1}{2}\left\{\mathfrak{L}^{(n-m)a_{m+1}\cdots a_n}_{\mu_{m+1}\cdots\mu_n}\left[p'-\sum_{j=1}^m(1-s_j)k_j,p+\sum_{j=1}^m s_jk_j;k_{m+1},\ldots,k_n\right],P_s\left(T^{a_1}_{s_1}\cdots T^{a_m}_{s_m}\right)\right\},\end{aligned} \tag{2.15}$$

where ∂'_μ and ∂_μ denote $\partial/\partial p'^\mu$ and $\partial/\partial p^\mu$, respectively.

When the operator $L[A]$ is covariant under spacetime translations, its expansion coefficients have the form

$$\mathfrak{L}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p',p;k_1,\ldots,k_n) = \delta^{(4)}\left[p'-p-\sum_{j=1}^{n}k_j\right]L^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n). \tag{2.16}$$

Then, we have

$$\langle p'|\tilde{L}[A]|p\rangle = \sum_{n=0}^{\infty}\frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n\delta^{(4)}\left(p'-p-\sum_{j=1}^{n}k_j\right)\tilde{L}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)A^{a_1\mu_1}(k_1)\cdots A^{a_n\mu_n}(k_n), \tag{2.17}$$

with

$$\tilde{L}^{(n)a_1\cdots a_n}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) = \sum_{m=0}^{n}\frac{i^m}{m!}\int_0^1 ds_1\cdots\int_0^1 ds_m\partial_{\mu_1}\cdots\partial_{\mu_m} \times\frac{1}{2}\left\{L^{(n-m)a_{m+1}\cdots a_n}_{\mu_{m+1}\cdots\mu_n}\left(p+\sum_{j=1}^{m}s_jk_j;k_{m+1},\ldots,k_n\right),\ P_s\left(T^{a_1}_{s_1}\cdots T^{a_m}_{s_m}\right)\right\}. \tag{2.18}$$

In practical calculations of radiative corrections in Λ-QED, one would frequently use the p-representation vertex expansion of the gauge-covariant electron propagator $S_F[A]$ and the gauge-invariant electron propagator $G[A]=S_F[A]\circ T^{\mathbf{t}}[A]$ in the presence of a classical electromagnetic field A. For $S_F[A]$, we have

$$\langle p'|S_F[A]|p\rangle = \sum_{n=0}^{\infty}\frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(p'-p-\sum_{j=1}^{n}k_j\right), \tag{2.19}$$

where

$$S^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) \equiv S_F\left(p+\sum_{j=1}^{n}k_j\right)\gamma_{\mu_n}\cdots S_F(p+k_1)\gamma_{\mu_1}S_F(p), \tag{2.20}$$

with $S_F(p)=\frac{i(\gamma^\mu p_\mu+m)}{p^2-m^2+i\epsilon}$. For the gauge-invariant propagator $G[A]$, we have

$$\langle p'|G[A]|p\rangle = \sum_{n=0}^{\infty}\frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n)A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(p'-p-\sum_{j=1}^{n}k_j\right), \tag{2.21}$$

where

$$G^{(n)}_{\mu_1\cdots\mu_n}(p;k_1,\ldots,k_n) = \sum_{m=0}^{n}\frac{i^m}{m!}\int_0^1 ds_1\cdots\int_0^1 ds_m\partial_{\mu_1}\cdots\partial_{\mu_m}S^{(n-m)}_{\mu_{m+1}\cdots\mu_n}\left(p+\sum_{j=1}^{m}s_jk_j;k_{m+1},\ldots,k_n\right). \tag{2.22}$$

III. A GAUGE-INVARIANT MOMENTUM-CUTOFF GENERATING FUNCTIONAL IN THE BACKGROUND-FIELD FORMALISM

In this section, we shall use the technique illustrated in Sec. II to construct a gauge-invariant momentum-cutoff generating functional for QED in the background-field formalism [6,7]. According to this formalism, a quantum electromagnetic field is divided into a background field $A_\mu(x)$ plus a quantum-fluctuation field $Q_\mu(x)$ which is the variable of integration in the functional integral. The generating functional for Green functions of Q in the presence of the background field A and in a Lorentz gauge is then given by [7]

$$Z[A,J] = \int[dQ]\exp\left\{i\left(S[A+Q]+W[A+Q] + \frac{1}{2\xi}\langle Q^\mu|\partial_\mu\partial_\nu|Q^\nu\rangle + \langle Q^\mu|J_\mu\rangle\right)\right\}, \tag{3.1}$$

where

$$S[A] = -\tfrac{1}{4}\langle F_{\mu\nu}|F^{\mu\nu}\rangle,\ \ |F^{\mu\nu}\rangle = \partial^\mu|A^\nu\rangle - \partial^\nu|A^\mu\rangle, \tag{3.2}$$

and

$$\exp\{iW[A]\} = \int [d\psi d\bar{\psi}] \exp\{i\langle\bar{\psi}|i\gamma^\mu(\partial_\mu + igA_\mu) - m|\psi\rangle\}, \quad (3.3)$$

with g (or, alternatively, $-e$ in Ref. [2]) denoting the charge of electron. In addition, we assume that under a local gauge transformation, $A_\mu(x)$ and $\psi(x)$ transform according to $A_\mu(x) \to A_\mu(x) + \frac{1}{g}\partial_\mu\theta(x)$ and $\psi(x) \to e^{-i\theta(x)}\psi(x)$, respectively, while $Q_\mu(x)$ and $J_\mu(x)$ are held invariant. Thus, $Z[A,J]$ is a gauge-invariant functional.

It is crucial for the following discussions to carry out formally the functional integration for all quantum-fluctuation field variables. For the integration of the Q variable, we separate out the terms quadratic in Q contained in $S[A+Q]$ and write

$$S[A+Q] + \frac{1}{2\xi}\langle Q^\mu|\partial_\mu\partial_\nu|Q^\nu\rangle = S[A] + \frac{1}{2}\langle Q^\mu|(D^{\mu\nu})^{-1}|Q^\nu\rangle + \langle Q^\mu|\partial^\nu|F_{\nu\mu}\rangle, \quad (3.4)$$

where

$$D^{\mu\nu} = \left[g_{\mu\nu}\partial^\lambda\partial_\lambda - \left(1 - \frac{1}{\xi}\right)\partial_\mu\partial_\nu\right]^{-1} \quad (3.5)$$

is the photon propagator. Letting

$$I[A,Q] = \langle Q^\mu|\partial^\nu|F_{\nu\mu}\rangle + W[A+Q] - W[A] \quad (3.6)$$

and performing the functional integration of Q, Eq. (3.1) can be written as

$$Z[A,J] = \exp\{i(S[A] + W[A])\} \exp\left\{iI\left[A, \frac{\delta}{i\delta J}\right]\right\} \times \exp\left\{-\frac{i}{2}\langle J^\mu|D_{\mu\nu}|J^\nu\rangle\right\}. \quad (3.7)$$

On the other hand, according to Refs. [2] and [6], the effective action $W[A]$ can be formally written in the form

$$W[A] = g\mathrm{Tr}\left\{\int_0^1 d\lambda S_F[\lambda A]\gamma^\mu A_\mu\right\}, \quad (3.8)$$

where

$$S_F[A] = [\gamma^\mu(\partial_\mu + igA_\mu) + im]^{-1} \quad (3.9)$$

is the electron propagator in the background field A.

Now we introduce an auxiliary field

$$|\phi_\mu\rangle = -D_{\mu\nu}|J^\nu\rangle. \quad (3.10)$$

By making use of the identity

$$\exp\left\{\frac{i}{2}\langle J^\mu|D_{\mu\nu}|J^\nu\rangle\right\}\frac{\delta}{i\delta J_\lambda}\exp\left\{-\frac{i}{2}\langle J^\mu|D_{\mu\nu}|J^\nu\rangle\right\} = \exp\left\{\frac{i}{2}\left\langle\frac{\delta}{\delta\phi_\mu}\right|D_{\mu\nu}\left|\frac{\delta}{\delta\phi_\nu}\right\rangle\right\}\phi^\lambda\exp\left\{-\frac{i}{2}\left\langle\frac{\delta}{\delta\phi_\mu}\right|D_{\mu\nu}\left|\frac{\delta}{\delta\phi_\nu}\right\rangle\right\}, \quad (3.11)$$

Eq. (3.7) may be rewritten as

$$Z[A,J] = \exp\left\{i\left(S[A] + W[A] - \tfrac{1}{2}\langle J^\mu|D_{\mu\nu}|J^\nu\rangle\right)\right\} \times (\exp\{\mathbf{D}_\phi\}\exp\{iI[A,\phi]\})_{\phi_\mu=-D_{\mu\nu}J^\nu}, \quad (3.12)$$

where $\mathbf{D}_\phi$ denotes a bilinear functional derivative operator which, in p representation, has the form

$$\mathbf{D}_\phi = \frac{i}{2}\int d^4k\int d^4k'\langle k'|D_{\mu\nu}|k\rangle\frac{\delta^2}{\delta\phi_\mu(k')\delta\phi_\nu(-k)}. \quad (3.13)$$

A serious problem to notice here is that the right-hand side of Eq. (3.12) is actually not mathematically well defined. The problem arises because (i) the traces taken in the Dirac field space $\mathcal{H}_\psi$ in the definitions of $W[A]$ and $I[A,\phi]$ do not exist, and (ii) $\mathbf{D}_\phi F[\phi]$ is not well defined when $F[\phi]$ is a local functional of $\phi(x)$. In order to construct a regularized version of $Z[A,J]$, we implement a momentum-cutoff scheme both on the trace taken in $\mathcal{H}_\psi$ and on the derivative operator $\mathbf{D}_\phi$ defined on the photon-field space $\mathcal{H}_\phi$ as follows.

Let $\mathbf{O}$ be an operator on $\mathcal{H}_\psi$. We define a Λ trace of $\mathbf{O}$,

$$\mathrm{Tr}_\Lambda\{\mathbf{O}\} = \int_{\mathcal{D}(\Lambda)}\frac{d^4p}{(2\pi)^4}\mathrm{tr}\{\langle p|\mathbf{O}|p\rangle\}, \quad (3.14)$$

where $\mathcal{D}(\Lambda)$ is a momentum-integration domain in p space. In addition, we also define a momentum-cutoff functional derivative operator on $\mathcal{H}_\phi$,

$$\mathbf{D}_{\phi_\Lambda} = \frac{i}{2}\int_{\mathcal{D}(\Lambda)} d^4k\int_{\mathcal{D}(\Lambda)} d^4k'\langle k'|D_{\mu\nu}|k\rangle\frac{\delta^2}{\delta\phi_\mu(k')\delta\phi_\nu(-k)}. \quad (3.15)$$

With a properly chosen momentum-integration domain $\mathcal{D}(\Lambda)$, it is not difficult to ensure that (i) the regularized version of Eq. (3.8),

$$W_{S_\Lambda}[A] = g\mathrm{Tr}_\Lambda\left\{\int_0^1 d\lambda S_F[\lambda A]\gamma^\mu A_\mu\right\}, \quad (3.16)$$

is finite, and (ii) $\mathbf{D}_{\phi_\Lambda}F[\phi]$ is well defined for most of the local functional $F[\phi]$. As discussed in the previous section of this paper and Ref. [2], a Λ trace of a gauge-covariant operator usually breaks the gauge symmetry, while a gauge-invariant momentum-cutoff version of $W[A]$ can be defined as

$$W_\Lambda[A] = g\mathrm{Tr}_\Lambda\left\{\int_0^1 d\lambda G[\lambda A]\gamma^\mu A_\mu\right\}, \quad (3.17)$$

where $G[A]$ denotes a gauge-invariant electron propagator given by

$$G[A] = S_F[A]\circ T^{\mathbf{t}}[A]. \quad (3.18)$$

Also, when the background field A_μ satisfies $\partial^\nu|F_{\nu\mu}\rangle = 0$, the gauge-invariant momentum-cutoff version of $I[A,\phi]$ can be written in the form (see Appendix)

$$I_\Lambda[A,\phi] = W_\Lambda[A+\phi] - W_\Lambda[A] = g\mathrm{Tr}_\Lambda\left\{\int_0^1 d\lambda G[A+\lambda\phi]\gamma^\mu\phi_\mu\right\}. \quad (3.19)$$

Hence, we define the gauge-invariant momentum-cutoff generating functional for QED as

$$Z_\Lambda[A,J] = \exp\left\{i\left(S[A] + W_\Lambda[A] - \tfrac{1}{2}\langle J^\mu|D_{\mu\nu}|J^\nu\rangle\right)\right\} \times (\exp\{\mathbf{D}_{\phi_\Lambda}\}\exp\{iI_\Lambda[A,\phi]\})_{\phi_\mu=-D_{\mu\nu}J^\nu}, \quad (3.20)$$

which is expected to be well defined mathematically and will be served as the starting point of our discussions of the QED in a momentum-cutoff vacuum. Noting that the gauge-invariant effective action $\Gamma[A]$ can be computed by evaluating $Z[A,0]$ and summing all connected one-particle-irreducible graphs with A fields on external legs [8], the momentum-cutoff gauge-invariant effective action should be given by

$$\Gamma_\Lambda[A] = S[A] + W_\Lambda[A] - i(\exp\{\mathbf{D}_{\phi_\Lambda}\}\exp\{iI_\Lambda[A,\phi]\})^{\text{connected}1PI}_{\phi=0}. \quad (3.21)$$

In addition, the quantum expectation value of any functional $F[A]$ in a momentum-cutoff vacuum will be given by

$$\bar{F}_\Lambda[A] = \left.\frac{\exp\{\mathbf{D}_{\phi_\Lambda}\}(F[A+\phi]\exp\{iI_\Lambda[A,\phi]\})}{\exp\{\mathbf{D}_{\phi_\Lambda}\}\exp\{iI_\Lambda[A,\phi]\}}\right|_{\phi=0}. \quad (3.22)$$

Finally, we should also consider the issue of gauge dependence of our momentum-cutoff theory. Let $\mathbf{D}^F_{\phi_\Lambda}$ denote the derivative operator $\mathbf{D}_{\phi_\Lambda}$ in the Feynman gauge (i.e., $\xi = 1$), $F[A]$ be a gauge-invariant functional, and ϕ^T be the transversal part of ϕ. Then we have $F[A+\phi] = F[A+\phi^T]$, which implies

$$\mathbf{D}_{\phi_\Lambda}F[A+\phi] = \mathbf{D}^F_{\phi_\Lambda}F[A+\phi]. \quad (3.23)$$

Consequently, from Eq. (3.19), we obtain

$$\mathbf{D}_{\phi_\Lambda}I_\Lambda[A,\phi] = \mathbf{D}^F_{\phi_\Lambda}I_\Lambda[A,\phi]. \quad (3.24)$$

Hence, the contribution of each graph to the effective action $\Gamma_\Lambda[A]$ is independent of the gauge-fixing parameter ξ, while $\bar{F}_\Lambda[A]$ may depend on ξ only if $F[A]$ is a non-gauge-invariant functional.

IV. ONE-LOOP CALCULATION

A. Photon self-energy

From Eq. (3.21), we see that a loop expansion of the gauge-invariant effective action $\Gamma_\Lambda[A]$ in a momentum-cutoff vacuum can be achieved by expanding $I_\Lambda[A,\phi]$ in a power series of ϕ. Letting

$$\Gamma_\Lambda[A] = S[A] + \sum_{n=1}^{\infty}\Gamma^{n\text{-loop}}_\Lambda[A], \quad (4.1)$$

and expanding $G[A+\lambda\phi]$ on the right-hand side of Eq. (3.19) in a power series of λ,

$$G[A+\lambda\phi] = G[A] + \sum_{n=1}^{\infty}\lambda^n G^{(n)}[A,\phi], \quad (4.2)$$

we obtain immediately

$$\Gamma^{\text{one-loop}}_\Lambda[A] = W_\Lambda[A] \quad (4.3)$$

and

$$\Gamma^{\text{two-loop}}_\Lambda[A] = \frac{g}{2}\mathbf{D}_{\phi_\Lambda}\mathrm{Tr}_\Lambda\{G^{(1)}[A,\phi]\gamma^\mu\phi_\mu\}. \quad (4.4)$$

On the other hand, in order to obtain the vacuum polarization function $\Pi_{\mu\nu}(k|\Lambda)$ in a momentum-cutoff vacuum, we need a vertex expansion of $\Gamma_\Lambda[A]$,

$$\Gamma_\Lambda[A] = \sum_{n=0}^{\infty}\frac{(-ig)^n}{(2\pi)^{4(n-1)}}\int d^4k_1\cdots\int d^4k_n\Gamma^{(n)}_{\mu_1\cdots\mu_n} \times(k_1,\ldots,k_n|\Lambda)A^{\mu_1}(k_1)\cdots A^{\mu_n}(k_n)\delta^{(4)}\left(\sum_{j=1}^{n}k_j\right), \quad (4.5)$$

which yields

$$\Pi_{\mu\nu}(k|\Lambda) = -g^2\left[\Gamma^{(2)}_{\mu\nu}(k,-k|\Lambda) + \Gamma^{(2)}_{\nu\mu}(-k,k|\Lambda)\right]. \quad (4.6)$$

Thus, according to Eqs. (4.3), (3.17), (2.22), and (2.20), the one-loop contribution to $\Pi_{\mu\nu}(k|\Lambda)$ is given by

$$\Pi^{\text{one-loop}}_{\mu\nu}(k|\Lambda) = -\frac{ig^2}{2}\int_{\mathcal{D}(\Lambda)}\frac{d^4p}{(2\pi)^4}\mathrm{tr}\{G^{(1)}_\mu(p;k)\gamma_\nu + G^{(1)}_\nu(p;-k)\gamma_\mu\}, \quad (4.7)$$

where

$$G^{(1)}_\mu(p;k) = S_F(p+k)\gamma_\mu S_F(p) + i\int_0^1 ds\,\partial_\mu S_F(p+sk). \quad (4.8)$$

Separating out the divergent part of $\Pi^{\text{one-loop}}_{\mu\nu}(k|\Lambda)$ by expanding it in a power series of k,

$$\Pi^{\text{one-loop}}_{\mu\nu}(k|\Lambda) = \Pi^{\text{div}}_{\mu\nu}(k|\Lambda) + \Pi^{(2)R}(k^2)d_{\mu\nu}(k) + O(\Lambda^{-1}), \quad (4.9)$$

we have

$$\Pi^{\text{div}}_{\mu\nu}(k|\Lambda) = \frac{8ig^2}{3}\int_{\mathcal{D}(\Lambda)}\frac{d^4p}{(2\pi)^4}\left[\frac{d_{\mu\nu}(k)}{(p^2-m^2)^2} - \frac{\kappa_{\mu\nu}(p;k)}{(p^2-m^2)^3}\right], \quad (4.10)$$

$$\Pi^{(2)R}(0) = 0, \quad (4.11)$$

with

$$d_{\mu\nu}(k) = k^2g_{\mu\nu} - k_\mu k_\nu, \quad (4.12)$$

$$\kappa_{\mu\nu}(p;k) = (pk)^2g_{\mu\nu} - (pk)(p_\mu k_\nu + k_\mu p_\nu) + k^2p_\mu p_\nu. \quad (4.13)$$

We shall use two types of the momentum-cutoff domain $\mathcal{D}(\Lambda)$ to evaluate divergent integrals such as $\Pi^{\text{div}}_{\mu\nu}(k|\Lambda)$ given by Eq. (4.10):

(i) a Lorentz-invariant four-dimensional (4D) cutoff in the Euclidian four-momentum space $\mathcal{R}^4\{p_\alpha;\alpha = 1,2,3,4\}$ with $p_4 = -ip_0$, which we shall denote by

$$\mathcal{D}_4(\Lambda) = \mathcal{R}^4\{p_\alpha \mid |p|^2 < \Lambda^2\}, \quad (4.14)$$

where $|p|^2 = -p^2$ is a Lorentz invariant; and

(ii) a 3D cutoff in the Minkowski four-momentum space $\mathcal{R}^4\{p_\mu;\mu = 0,1,2,3\}$ with rotational symmetry in a preferred inertial frame specified by a timelike unit vector $n^\mu = \{1,0,0,0\}$. For an arbitrary inertial reference frame, we have

$$\mathcal{D}_3(\Lambda) = \mathcal{R}^4\{p_\mu \mid (np)^2 - p^2 < \Lambda^2\}. \quad (4.15)$$

Thus, under $\mathcal{D}_4(\Lambda)$ cutoff, we obtain

$$\Pi^{\rm div}_{\mu\nu}(\Lambda,k) = \Pi(\Lambda)d_{\mu\nu}(k), \tag{4.16}$$

where

$$\Pi(\Lambda) = -\frac{g^2}{12\pi^2}\left[\ln\frac{\Lambda^2}{m^2} - \frac{1}{2}\right] + O(\Lambda^{-1}), \tag{4.17}$$

while under $\mathcal{D}_3(\Lambda)$ cutoff, we obtain

$$\Pi^{\rm div}_{\mu\nu}(\Lambda,k) = \Pi'(\Lambda)d_{\mu\nu}(k) + \Pi^a(\Lambda)\kappa_{\mu\nu}(n;k), \tag{4.18}$$

where

$$\Pi'(\Lambda) = -\frac{g^2}{12\pi^2}\left[\ln\frac{4\Lambda^2}{m^2} - \frac{4}{3}\right] + O(\Lambda^{-1}) \tag{4.19}$$

and

$$\Pi^a(\Lambda) = \frac{g^2}{36\pi^2} + O(\Lambda^{-1}). \tag{4.20}$$

On the other hand, when we use a $\mathcal{D}_3(\Lambda)$ cutoff in studying higher-order contributions, the polarization function given by Eq. (4.6) can be generally written in the form

$$\Pi_{\mu\nu}(k|\Lambda) = \Pi(\Lambda,k^2)d_{\mu\nu}(k) + \Pi^a_{\mu\nu}[\Lambda,(nk)^2,k^2], \tag{4.21}$$

where

$$\Pi^a_{\mu\nu}[\Lambda,(nk)^2,k^2] = \Pi^{a1}[\Lambda,(nk)^2,k^2]d_{\mu\nu}(k) + \Pi^{a2}[\Lambda,(nk)^2,k^2]\kappa_{\mu\nu}(n;k) \tag{4.22}$$

is the Lorentz anomalous term in $\Pi_{\mu\nu}(k|\Lambda)$, with

$$\Pi^{a1}(\Lambda,0,k^2) = 0. \tag{4.23}$$

Let $\Pi^{(ct)a}_{\mu\nu}[\Lambda,(nk)^2,k^2]$ be the corresponding Lorentz anomalous term contributed from the counterterm (see Sec. V). Then the Lorentz anomaly of $\Pi_{\mu\nu}(k|\Lambda)$ is defined as

$$\Pi^{\rm anom}_{\mu\nu}(n,k) = \lim_{\Lambda\to\infty}\left\{\Pi^a_{\mu\nu}[\Lambda,(nk)^2,k^2] + \Pi^{(ct)a}_{\mu\nu}[\Lambda,(nk)^2,k^2]\right\}. \tag{4.24}$$

In particular, from Eq. (4.20), the order-g^2 Lorenz anomaly of $\Pi_{\mu\nu}(k|\Lambda)$ is given by

$$\Pi^{(2)\rm anom}_{\mu\nu}(n,k) = \frac{g^2}{36\pi^2}\kappa_{\mu\nu}(n;k). \tag{4.25}$$

Furthermore, if we denote the renormalized photon propagator as $\bar{D}^{\mu\nu}$ and write its p representation in the form

$$\langle k'|\bar{D}^{\mu\nu}|k\rangle = -i(2\pi)^4\delta^{(4)}(k'-k)\bar{D}^{\mu\nu}(k), \tag{4.26}$$

then, in the Λ-QED, we have

$$\bar{D}^{\mu\nu}k = -i\left\{[1-\Pi^R(k^2)]d_{\mu\nu}(k) + \frac{1}{\xi}k_\mu k_\nu - \Pi^{\rm anom}_{\mu\nu}(n,k)\right\}^{-1}. \tag{4.27}$$

B. Electron self-energy

In computing the radiative corrections to a free-electron propagator, we should take notice of the fact that there are two different versions of free-electron propagator in the background-field formalism: a gauge-covariant one, which is defined by $S_F[A]|_{F_{\mu\nu}=0}$, and a gauge-invariant one, which is defined by $G[A]|_{F_{\mu\nu}=0} = S_F = (\partial_\mu\gamma^\mu + im)^{-1}$, where $G[A]$ is the gauge-invariant electron propagator given by Eqs. (2.21) and (2.22). In Ref. [2], we have computed the one-loop renormalization constant Z_2 and the Lorentz anomaly by using the Feynman gauge and $S_F[A]|_{F_{\mu\nu}=0}$ as the free-electron propagator. It is easy to verify that, in an arbitrary covariant gauge, Z_2 will depend on the gauge-fixing parameter ξ, while the one-loop Lorentz anomaly of the electron self-energy is independent of ξ. Since the dependence of Z_2 on ξ can be completely avoided when we use $G[A]|_{F_{\mu\nu}=0}$ as the free-electron propagator, it is worthwhile here to carry out the calculations of the radiative corrections to the gauge-invariant electron propagator $G[A]$.

According to Eq. (3.22), the dressed gauge-invariant electron propagator has the form

$$\bar{G}_\Lambda = \bar{G}_\Lambda[A]\big|_{A=0} = \frac{1}{\partial_\mu\gamma^\mu + im + i\Sigma_\Lambda} = \left.\frac{\exp\{\mathbf{D}_{\phi_\Lambda}\}(G[\phi]\exp\{iI_\Lambda[0,\phi]\})}{\exp\{\mathbf{D}_{\phi_\Lambda}\}\exp\{iI_\Lambda[0,\phi]\}}\right|_{\phi=0}, \tag{4.28}$$

where Σ_Λ denotes the unrenormalized electron self-energy in a momentum-cutoff vacuum. Let $\bar{G}_\Lambda|^{{\rm connected}1PI}$ denote the sum of all connected one-particle-irreducible graphs in the perturbation expansion of the right-hand side of Eq. (4.28) and rewrite it as

$$\bar{G}_\Lambda|^{{\rm connected}1PI} = S_F + \sum_{n=1}^{\infty}\bar{G}^{n\text{-loop}}_\Lambda. \tag{4.29}$$

Then the n-loop contribution to the electron self-energy has the form

$$\Sigma^{n\text{-loop}}_\Lambda = iS_F^{-1}\bar{G}^{n\text{-loop}}_\Lambda S_F^{-1}. \tag{4.30}$$

For a calculation of $\Sigma^{\text{one-loop}}_\Lambda$, we use the expansion (4.2) and obtain

$$\bar{G}^{\text{one-loop}}_\Lambda = \mathbf{D}_{\phi_\Lambda}\{G^{(2)}[0,\phi]\}. \tag{4.31}$$

Writing $\Sigma^{\text{one-loop}}_\Lambda$ in p representation,

$$\langle p'|\Sigma^{\text{one-loop}}_\Lambda|p\rangle = (2\pi)^4\delta^{(4)}(p'-p)\Sigma(p|\Lambda), \tag{4.32}$$

from Eqs. (2.21), (4.30), and (4.31), and making use of the transversality of $G^{(2)}_{\mu\nu}(p;k,-k)$, we obtain

$$\Sigma(p|\Lambda) = -g^2\int_{\mathcal{D}(\Lambda)}\frac{d^4k S_F^{-1}(p)g^{\mu\nu}G^{(2)}_{\mu\nu}(p;k,-k)S_F^{-1}(p)}{(2\pi)^4(k^2-\mu^2)}, \tag{4.33}$$

where μ is the photon mass. Furthermore, by making use of Eqs. (2.22) and (2.20), we have

$$g^{\mu\nu}G^{(2)}_{\mu\nu}(p;k,-k) = \mathcal{L}^{(a)}(p;k) + \mathcal{L}^{(b)}(p;k) + \mathcal{L}^{(c)}(p;k), \tag{4.34}$$

where

$$\mathcal{L}^{(a)}(p;k) = S_F(p)\gamma^\mu S_F(p+k)\gamma_\mu S_F(p), \tag{4.35}$$

$$\mathcal{L}^{(b)}(p;k) = i\int_0^1 ds\,\partial_\mu\{S_F[p+(s-1)k]\gamma^\mu S_F(p+sk)\}, \tag{4.36}$$

$$\mathcal{L}^{(c)}(p;k) = -\frac{1}{2}\int_0^1 ds_1 \int_0^1 ds_2 \partial^\mu \partial_\mu S_F[p+(s_1-s_2)k]. \tag{4.37}$$

We note that the contribution to $\Sigma(p|\Lambda)$ from $\mathcal{L}^{(a)}(p;k)$ is just the conventional one-loop electron self-energy in the Feynman gauge,

$$\Sigma^{(a)}(p|\Lambda) = 2ig^2 \int_{\mathcal{D}(\Lambda)} \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(p_\mu + k_\mu) - 2m}{(k^2-\mu^2)[(p+k)^2 - m^2]}. \tag{4.38}$$

Now we write

$$\Sigma^{(a)}(p|\Lambda) = \Sigma^{\rm div}(p|\Lambda) + \Sigma^R(p|\mu) + O(\mu) + O(\Lambda^{-1}) \tag{4.39}$$

and require that $\Sigma^R(p|\mu)|_{(\gamma p)=m} = 0$. Then, under 4D cutoff, we have

$$\Sigma^{\rm div}(p|\Lambda) = \frac{g^2}{8\pi^2}\left\{\frac{1}{4}[(\gamma p)+2m] + \left[2m - \frac{(\gamma p)}{2}\right]\ln\frac{\Lambda^2}{m^2}\right\}, \tag{4.40}$$

$$\Sigma^R(p|\mu) = \frac{g^2}{8\pi^2}\left\{\int_0^1 ds[2m-(1-s)(\gamma p)]\ln\frac{m^2}{w_s} - \frac{5m}{2}\right\}, \tag{4.41}$$

while under 3D cutoff, we have

$$\Sigma^{\rm div}(p|\Lambda) = \frac{g^2}{8\pi^2}\left\{\frac{2}{3}(\gamma p) - \frac{3}{2}m + \left[2m - \frac{(\gamma p)}{2}\right]\ln\frac{4\Lambda^2}{m^2} + \frac{1}{3}(np)(\gamma n)\right\}, \tag{4.42}$$

$$\Sigma^R(p|\mu) = \frac{g^2}{8\pi^2}\left\{\int_0^1 ds[2m-(1-s)(\gamma p)]\ln\frac{m^2}{w_s} - \frac{5m}{2}\right\}, \tag{4.43}$$

where

$$w_s = sm^2 - s(1-s)p^2 + (1-s)\mu^2 \tag{4.44}$$

and

$$\Sigma^{(2)\rm anom} = \frac{g^2}{24\pi^2}(np)(\gamma n) \tag{4.45}$$

gives the Lorentz anomaly of the electron self-energy $\Sigma^{(a)}(p|\Lambda)$.

The contribution to $\Sigma(p|\Lambda)$ from $\mathcal{L}^{(b)}(p;k) + \mathcal{L}^{(c)}(p;k)$ is given by

$$\Sigma^{(b+c)}(p|\Lambda) = g^2 \int_{\mathcal{D}(\Lambda)} \frac{d^4k}{(2\pi)^4}\frac{\mathcal{L}(p;k)}{(k^2-\mu^2)}, \tag{4.46}$$

where

$$\mathcal{L}(p;k) = [(\gamma p) - m][\mathcal{L}^{(b)}(p;k) + \mathcal{L}^{(c)}(p;k)][(\gamma p) - m]. \tag{4.47}$$

In order to evaluate $\Sigma^{(b+c)}(p|\Lambda)$, it is convenient to carry out first the integration of s variables included in $\mathcal{L}(p;k)$. A careful analysis shows that we have $\mathcal{L}(p;k)|_{(\gamma p)=m} = 0$, but the on-shell point is, in fact, a branch point of $\mathcal{L}(p;k)$ as it contains the terms with the factor $\ln(m^2-p^2)$. Since further integrations on the cutoff k space would be very complicated, we shall separate out the ultraviolet divergent part of $\Sigma^{(b+c)}(p|\Lambda)$ just as we have done in Eq. (4.39) for $\Sigma^{(a)}(p|\Lambda)$,

$$\Sigma^{(b+c)}(p|\Lambda) = \Delta^{\rm div}(p|\Lambda) + \Delta^R(p|\mu) + O(\mu) + O(\Lambda^{-1}). \tag{4.48}$$

Now the divergent part $\Delta^{\rm div}(p|\Lambda)$ can be obtained by expanding the integrand in a power series of k. Our calculation shows that the divergent part under both 4D and 3D cutoff are the same, which is given by

$$\Delta^{\rm div}(p|\Lambda) = \frac{g^2}{16\pi^2}[(\gamma p) - m]\left[\frac{2m + (\gamma p)\eta(z)|_{z=1-p^2/m^2}}{p^2-m^2}\right] \times [(\gamma p) - m]\ln\frac{\Lambda^2}{m^2}, \tag{4.49}$$

with $\eta(0) = 1$ and $\eta(z)$ is nonanalytic at $z = 0$.

Thus, the self-energy $\Sigma^{(b+c)}(p|\Lambda)$ has no contribution to the mass renormalization constant Δ_m and the Lorentz anomaly, while its contribution to the on-shell wave-function renormalization constant Z_2 has the form

$$\delta Z_2^{(b+c)} = -\frac{3g^2}{32\pi^2}\ln\frac{\Lambda^2}{m^2} + Z^R\left(\frac{\mu}{m}\right) + O(\mu) + O(\Lambda^{-1}). \tag{4.50}$$

V. RENORMALIZATION AND MULTILOOP CALCULATION

The purpose of this section is to study the renormalized two-loop photon self-energy in a momentum-cutoff vacuum by making use of the renormalized perturbation method [9].

After adding an order-g^2 counterterm to the Lagrangian, the order-g^4 momentum-cutoff vacuum polarization function can be written as

$$\Pi^{(4)}_{\mu\nu}(k|\Lambda) = \Pi^{\text{two-loop}}_{\mu\nu}(k|\Lambda) + (\delta^{(1)}\Pi)^{(4)}_{\mu\nu}(k|\Lambda), \tag{5.1}$$

where $\Pi^{\text{two-loop}}_{\mu\nu}(k|\Lambda)$ is the polarization function derived from Eqs. (4.4)–(4.6) and $(\delta^{(1)}\Pi)^{(4)}_{\mu\nu}(k|\Lambda)$ denotes the contribution from the order-g^2 counterterm,

$$\mathcal{L}^{(1)}_{ct} = -\frac{\delta^{(1)}_3}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi\left[\delta^{(1)}_2 i\gamma^\mu(\partial_\mu + igA_\mu) - \delta^{(1)}_m\right]\psi, \tag{5.2}$$

which is to be added to the renormalized Lagrangian,

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar\psi[i\gamma^\mu(\partial_\mu + igA_\mu) - m]\psi + \mathcal{L}_{gf}, \tag{5.3}$$

for a cancellation of the ultraviolet divergences in $\Pi^{\text{one-loop}}_{\mu\nu}(k|\Lambda)$. The renormalization constants $\delta^{(1)}_3, \delta^{(1)}_2$, and $\delta^{(1)}_m$ may be determined by the one-loop calculations carried out in Sec. VI.

Let us calculate $(\delta^{(1)}\Pi)^{(4)}_{\mu\nu}(k|\Lambda)$ at first. This can be done by recalculating the one-loop polarization function $\Pi^{\text{one-loop}}_{\mu\nu}(k|\Lambda)$ starting from the modified Lagrangian $\mathcal{L} + \mathcal{L}^{(1)}_{ct}$ and focus on its order-g^4 term. The electron propagator in the background field A is now given by

$$S_F^{(1)}[A] = \left[\left(1+\delta_2^{(1)}\right)\gamma^\mu(\partial_\mu + igA_\mu) + i\left(m+\delta_m^{(1)}\right)\right]^{-1}. \tag{5.4}$$

According to Eq. (3.8), the modified one-loop effective action has the form

$$\begin{aligned} W^{(1)}[A] &= \left(1+\delta_2^{(1)}\right) g \mathrm{Tr}\left\{\int_0^1 d\lambda S_F^{(1)}[\lambda A]\gamma^\mu A_\mu\right\} \\ &= W[A] - ig\Delta_m^{(1)} \mathrm{Tr}\left\{\int_0^1 d\lambda (S_F[\lambda A])^2 \gamma^\mu A_\mu\right\} \\ &\quad + O(g^5), \end{aligned} \tag{5.5}$$

where $\Delta_m^{(1)} = \delta_m^{(1)} - m\delta_2^{(1)}$. Noting that $(S_F[A])^2 = i\partial S_F[A]/\partial m$, Eq. (5.5) can be rewritten as

$$W^{(1)}[A] = W[A] + \Delta_m^{(1)} \frac{\partial W[A]}{\partial m} + O(g^5). \tag{5.6}$$

Thus, by making use of the results given in Eqs. (4.9)–(4.20), we obtain the order-g^4 term of the polarization function derived from $W_\Lambda^{(1)}[A]$,

$$\begin{aligned} &\left(\delta^{(1)}\Pi\right)_{\mu\nu}^{(4)}(k|\Lambda) \\ &\quad = \Delta_m^{(1)}\left[\frac{g^2}{6\pi^2 m} + \frac{\partial \Pi^{(2)R}(k^2)}{\partial m} + O(\Lambda^{-1})\right] d_{\mu\nu}(k), \end{aligned} \tag{5.7}$$

with

$$\begin{aligned} \Delta_m^{(1)} &= -\frac{3g^2 m}{16\pi^2}\left[\ln\frac{\Lambda^2}{m^2} + \frac{1}{2}\right], \quad \text{under 4D cutoff,} \\ \Delta_m^{(1)} &= -\frac{3g^2 m}{16\pi^2}\left[\ln\frac{4\Lambda^2}{m^2} - \frac{5}{9}\right], \quad \text{under 3D cutoff.} \end{aligned} \tag{5.8}$$

Particularly, since the Lorentz anomaly of $\Pi_{\mu\nu}^{\text{one-loop}}(k|\Lambda)$ does not depend on m, the contribution to $\Pi_{\mu\nu}^{(4)}(k|\Lambda)$ from $\mathcal{L}_{ct}^{(1)}$ will not contain any Lorentz anomalous term.

Now we calculate the polarization function $\Pi_{\mu\nu}^{\text{two-loop}}(k|\Lambda)$ by expanding the right-hand side of Eq. (4.4) in powers of A. Assuming the invariance of the integration domain $\mathcal{D}(\Lambda)$ under the reflection $p_\mu \to -p_\mu$ and doing a straightforward calculation, we obtain

$$\begin{aligned} \Pi_{\mu\nu}^{\text{two-loop}}(k|\Lambda) = &\frac{g^4}{2(2\pi)^8}\int_{\mathcal{D}(\Lambda)} d^4q \\ &\times \int_{\mathcal{D}(\Lambda)} d^4p \frac{3!\mathrm{tr}\{\bar{G}_{\lambda\mu\nu}^{(3)}(p;q,k,-k)\gamma^\lambda\}}{q^2}, \end{aligned} \tag{5.9}$$

where $\bar{G}_{\mu_1\cdots\mu_n}^{(n)}(p;k_1,\ldots,k_n)$ denotes the symmetrized version of the functions $G_{\mu_1\cdots\mu_n}^{(n)}(p;k_1,\ldots,k_n)$ given by Eq. (2.22) [2]. Expand $3!\mathrm{tr}\{\bar{G}_{\lambda\mu\nu}^{(3)}(p;q,k,-k)\gamma^\lambda\}$ in a power series of k and write

$$3!\mathrm{tr}\left\{\bar{G}_{\lambda\mu\nu}^{(3)}(p;q,k,-k)\gamma^\lambda\right\} = Q_{\mu\nu}(p,q;k) + R_{\mu\nu}(p,q;k), \tag{5.10}$$

where $Q_{\mu\nu}(p,q;k)$ is a second-degree polynomial of k and $R_{\mu\nu}(p,q;k)$ denotes the second-degree remainders of the power series. Rewrite Eq. (5.1) as

$$\Pi_{\mu\nu}^{(4)}(k|\Lambda) = \Pi_{\mu\nu}^{(4)\text{div}}(k|\Lambda) + \Pi_{\mu\nu}^{(4)R}(k|\Lambda) + O(\Lambda^{-1}), \tag{5.11}$$

where

$$\begin{aligned} \Pi_{\mu\nu}^{(4)\text{div}}(k|\Lambda) = &\frac{g^4}{2(2\pi)^8}\int_{\mathcal{D}(\Lambda)} d^4q \int_{\mathcal{D}(\Lambda)} d^4p \frac{Q_{\mu\nu}(p,q;k)}{q^2} \\ &+ \frac{g^2\Delta_m^{(1)}}{6\pi^2 m} d_{\mu\nu}(k), \end{aligned} \tag{5.12}$$

$$\begin{aligned} \Pi_{\mu\nu}^{(4)R}(k|\Lambda) = &\frac{g^4}{2(2\pi)^8}\int_{\mathcal{D}(\Lambda)} d^4q \int_{\mathcal{D}(\Lambda)} d^4p \frac{R_{\mu\nu}(p,q;k)}{q^2} \\ &+ \Delta_m^{(1)} \frac{\partial \Pi^{(2)R}(k^2)}{\partial m} d_{\mu\nu}(k). \end{aligned} \tag{5.13}$$

In the following, we shall focus on the calculation of $\Pi_{\mu\nu}^{(4)\text{div}}(k|\Lambda)$ [$\Pi_{\mu\nu}^{(4)R}(k|\Lambda)$ is expected to be convergent when $\Lambda \to \infty$]. By making use of the transversality of $Q_{\mu\nu}(p,q;k)$, we have

$$\frac{g^4}{2(2\pi)^8}\int_{\mathcal{D}_4(\Lambda)} d^4q \int_{\mathcal{D}_4(\Lambda)} d^4p \frac{Q_{\mu\nu}(p,q;k)}{q^2} = \Pi^{(4)}(\Lambda) d_{\mu\nu}(k), \tag{5.14}$$

$$\begin{aligned} &\frac{g^4}{2(2\pi)^8}\int_{\mathcal{D}_3(\Lambda)} d^4q \int_{\mathcal{D}_3(\Lambda)} d^4p \frac{Q_{\mu\nu}(p,q;k)}{q^2} \\ &\quad = \Pi^{(4)\prime}(\Lambda) d_{\mu\nu}(k) + \Pi^{(4)a}(\Lambda)\kappa_{\mu\nu}(n;k). \end{aligned} \tag{5.15}$$

Let $n^\mu = \{1,0,0,0\}$ and

$$Q(p,q) = \frac{\partial^2}{\partial k^\lambda \partial k_\lambda} g^{\mu\nu} Q_{\mu\nu}(p,q;k), \tag{5.16}$$

$$Q^{(0)}(p,q) = \frac{\partial^2}{\partial k^0 \partial k_0} g^{\mu\nu} Q_{\mu\nu}(p,q;k). \tag{5.17}$$

Then, from Eqs. (5.14)–(5.17), we obtain

$$\Pi^{(4)}(\Lambda) = \frac{g^4}{48(2\pi)^8}\int_{\mathcal{D}_4(\Lambda)} d^4q \int_{\mathcal{D}_4(\Lambda)} d^4p \frac{Q(p,q)}{q^2}, \tag{5.18}$$

$$\begin{aligned} \Pi^{(4)\prime}(\Lambda) = &\frac{g^4}{24(2\pi)^8}\int_{\mathcal{D}_3(\Lambda)} d^4q \\ &\times \int_{\mathcal{D}_3(\Lambda)} d^4p \frac{Q(p,q) - 2Q^{(0)}(p,q)}{q^2}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} \Pi^{(4)a}(\Lambda) = &\frac{g^4}{24(2\pi)^8}\int_{\mathcal{D}_3(\Lambda)} d^4q \\ &\times \int_{\mathcal{D}_3(\Lambda)} d^4p \frac{4Q^{(0)}(p,q) - Q(p,q)}{q^2}. \end{aligned} \tag{5.20}$$

When doing calculations of these two-loop integrals, we tried various different cutoffs of the eight-dimensional (p,q) space and investigated the asymptotic behavior of the integrals under the limit $\Lambda \to \infty$. Our result indicates that only the cutoff which corresponds to an universal 3D-momentum cutoff of each kind of virtual particle yielded a definite Lorentz anomaly. In other words, it would be more appropriate to interpret the Lorentz anomaly as a typical feature of the momentum-cutoff vacuum rather than as a general consequence of a non-Lorentz-covariant cutoff scheme. As an illustration, let us consider the case where the cutoff Λ in q space can be different from the cutoff $\Lambda_p = z\Lambda$ in p space.

A straightforward calculation shows that, under such cutoffs, the Lorentz anomalous term $\Pi^{(4)a}(\Lambda)$ given by Eq. (5.20) can be written in the form $a(z)\ln\frac{\Lambda^2}{m^2}+b(z)+O(\Lambda^{-1})$. Moreover, we found that $a(z)=0$ if and only if $z=1$ and both $a(z)$ and $b(z)$ are discontinuous at $z=1$. With $z=1$, we obtain

$$\Pi^{(4)}(\Lambda)=\frac{g^4}{64\pi^4}\left[\ln\frac{\Lambda^2}{m^2}-\frac{3}{2}+O(\Lambda^{-1})\right], \qquad (5.21)$$

$$\Pi^{(4)\prime}(\Lambda)=\frac{g^4}{64\pi^4}\left[\ln\frac{\Lambda^2}{m^2}+\frac{5}{3}\ln 2-\frac{65}{54}+O(\Lambda^{-1})\right], \qquad (5.22)$$

$$\Pi^{(4)a}(\Lambda)=\frac{g^4}{12\pi^4}\left[\ln 2-\frac{133}{144}+O(\Lambda^{-1})\right]. \qquad (5.23)$$

By making use of these results as well as Eqs. (5.12) and (5.8), we obtain

$$\Pi^{(4)\text{div}}_{\mu\upsilon}(\Lambda,k)=-\frac{g^4}{64\pi^4}\left[\ln\frac{\Lambda^2}{m^2}+\frac{5}{2}+O(\Lambda^{-1})\right]d_{\mu\nu}(k), \qquad (5.24)$$

and

$$\Pi^{(4)\text{div}}_{\mu\upsilon}(\Lambda,k) = -\frac{g^4}{64\pi^4}\left[\ln\frac{\Lambda^2}{m^2}+\frac{7}{3}\ln 2+\frac{5}{54}+O(\Lambda^{-1})\right]d_{\mu\nu}(k) + \Pi^{(4)a}(\Lambda)\kappa_{\mu\nu}(n;k), \qquad (5.25)$$

under 4D and 3D cutoff, respectively. Hence for a cancellation of the divergent term included in $\Pi^{(4)\text{div}}_{\mu\upsilon}(\Lambda,k)$, we need an order-$g^4$ counterterm $\mathcal{L}^{(2)}_{ct}$ with

$$\delta^{(2)}_3=-\frac{g^4}{64\pi^4}\ln\frac{\Lambda^2}{m^2}+\text{ finite constant.} \qquad (5.26)$$

Combining this with what has been given in Sec. IV, we conclude that in the Λ-QED, the β function up to order g^4 is given by

$$\beta(g)=\frac{g}{2}m\frac{\partial}{\partial m}\left[\delta^{(1)}_3+\delta^{(2)}_3\right]=\frac{g^3}{12\pi^2}+\frac{g^5}{64\pi^4}, \qquad (5.27)$$

which coincides with that given by the conventional QED. On the other hand, the order-g^4 Lorentz anomaly has the form

$$\Pi^{(4)\text{anom}}_{\mu\upsilon}(n,k)=\frac{g^4}{12\pi^4}\left[\ln 2-\frac{133}{144}\right]\kappa_{\mu\nu}(n;k)+O(k^4). \qquad (5.28)$$

VI. DISCUSSION AND CONCLUSION

In previous sections, we have proposed a scheme for realizing gauge-invariant momentum-cutoff regularization on quantum gauge-field theories. By making use of this scheme, we construct a gauge-invariant momentum-cutoff generating functional $Z_\Lambda[A,J]$ for QED and carry out one- and two-loop calculations. It is evident, when the cutoff is imposed on the Euclidean four-momentum space and the integration domain is chosen to be $\mathcal{D}_4(\Lambda)$, that the generating functional $Z_\Lambda[A,J]$ becomes a Lorentz invariant and all of the Green functions deduced from it are both gauge invariant and Lorentz covariant. Thus, after renormalization and removing the 4D Λ cutoff, the physical results deduced from these Green functions are the same as those deduced by using other conventional regularization schemes, as demonstrated in our one- and two-loop calculations. Accordingly, we may think of this work as providing a workable cutoff regularization scheme for the conventional quantum gauge-field theory [10]. However, our investigations on the consequences of 3D Λ cutoff demonstrate that, if we treat the cutoff as a physical process rather than as a mathematical procedure, we can do more interesting works, i.e., constructing quantum field theories in a momentum-cutoff vacuum and investigating their unusual behavior in the continuum limit [11].

In constructing such quantum field theories, a challenge we have to face is the violation of Lorentz symmetry. In fact, in order to have the theory make sense, we require that after a renormalization procedure, all Lorentz anomalous terms in radiative corrections converge to finite terms in the continuum limit. Since the generating functional $Z_\Lambda[A,J]$ is well defined and can be constructed according to a definite procedure (see Sec. III), whether the theory thus constructed meets this requirement reduces to a well-defined mathematical problem. We tackle this problem by perturbation calculations and anticipate that the problem can be solved eventually with nonperturbative techniques.

Finally, we would like to present a short discussion about the physical meaning of the Lorentz anomalies as well as some of the phenomenological consequences of the momentum-cutoff vacuum in Λ-QED.

Let us consider first the propagation of the electromagnetic field in a momentum-cutoff vacuum. After renormalization and taking the continuum limit, the Maxwell equation in the momentum-cutoff vacuum, according to Eq. (4.27), can be written as

$$\left\{[1-\Pi^R(-\partial^\lambda\partial_\lambda)]d^{\mu\nu}(i\partial)-\frac{1}{\xi}\partial^\mu\partial^\nu-\Pi^{\mu\nu}_{\text{anom}}(n,i\partial)\right\}A_\nu(x) = 0. \qquad (6.1)$$

For simplifying our discussion, we shall use the Landau gauge (i.e., let $\xi\to\infty$) and consider only the approximation of order g^2. In this approximation, Eq. (6.1) has the form

$$\{[1-\Pi^{(2)R}(-\partial^\lambda\partial_\lambda)]d^{\mu\nu}(i\partial)-\Pi^{\text{anom}}\kappa^{\mu\nu}(n,i\partial)\}A_\nu(x)=0, \qquad (6.2)$$

where $\Pi^{\text{anom}}=N_f\frac{g^2}{36\pi^2}$, with N_f denoting the number of the species of the charged fermion with charge g. In comparison with the corresponding equation in the conventional QED,

$$[1-\Pi^{(2)R}(-\partial^\lambda\partial_\lambda)]d^{\mu\nu}(i\partial)A_\nu(x)=0, \qquad (6.3)$$

Eq. (6.2) involves an extra Lorentz anomalous term, which breaks explicitly the Lorentz symmetry exhibited in Eq. (6.3). However, we can show that Eq. (6.2) is actually covariant under a Lorentz transformation with a light velocity that is different from that which has been used in Eq. (6.3). In other words, if we change the light velocity from c to $\bar{c}=e^{-\alpha_p}c$ in the preferred inertial frame with $n^\mu=\{1,0,0,0\}$ by invoking an alternative definition of the interval

$$d\bar{s}^2=ds^2-(1-e^{-2\alpha_p})n_\mu n_\nu dx^\mu dx^\nu, \qquad (6.4)$$

Eq. (6.2) would have the same form as Eq. (6.3). Certainly, provided that the spacetime coordinate x^μ as well as the covariant vector fields A_ν and the spinor fields ψ and $\bar{\psi}$ are regarded as independent of the metric, a change of the light velocity in QED can be achieved by changing the Minkovski metric as follows:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = g_{\mu\nu} - (1 - e^{-2\alpha_p}) n_\mu n_\nu. \tag{6.5}$$

Thus, we obtain

$$\begin{aligned}\bar{d}^{\mu\nu}(i\partial) &\equiv (-\bar{g}^{\mu\nu}\bar{g}^{\lambda\varpi} + \bar{g}^{\mu\lambda}\bar{g}^{\nu\varpi})\partial_\lambda\partial_\varpi \\ &= d^{\mu\nu}(i\partial) - (1 - e^{2\alpha_p})\kappa^{\mu\nu}(n,i\partial).\end{aligned} \tag{6.6}$$

Letting $\alpha_p = \frac{1}{2}\ln(1 - N_f\frac{g^2}{36\pi^2})$ and noting that $\Pi^{(2)R}(-\bar{g}^{\lambda\varpi}\partial_\varpi\partial_\lambda) = \Pi^{(2)R}(-g^{\lambda\varpi}\partial_\varpi\partial_\lambda) + O(g^4)$, Eq. (6.2) can be rewritten as

$$[1 - \Pi^{(2)R}(-\bar{g}^{\lambda\varpi}\partial_\varpi\partial_\lambda)]\bar{d}^{\mu\nu}(i\partial)A_\nu(x) = 0, \tag{6.7}$$

which is covariant under the Lorentz transformation with the light velocity $\bar{c} = e^{-\alpha_p}c$.

Second, we consider the Dirac equation in a momentum-cutoff vacuum. In the approximation of order g^2, this equation has the form

$$[\gamma^\mu\partial_\mu + im + i\Sigma^{(2)R}(i\gamma^\mu\partial_\mu) + i\Sigma^{(2)\text{anom}}]\psi(x) = 0, \tag{6.8}$$

with $\Sigma^{(2)\text{anom}} = \frac{g^2}{24\pi^2}i\gamma^\mu n_\mu n^\nu\partial_\nu$. From the identity

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{6.9}$$

we know that the change of the light velocity from c to $\bar{c} = e^{-\alpha_e}c$ can be achieved via a change of the γ matrix,

$$\gamma^\mu \to \bar{\gamma}^\mu = \gamma^\mu - (1 - e^{\alpha_e})\gamma^\nu n_\nu n^\mu. \tag{6.10}$$

Thus, letting $\alpha_e = \ln(1 - \frac{g^2}{24\pi^2})$ and noting that $\Sigma^{(2)R}(i\bar{\gamma}^\mu\partial_\mu) = \Sigma^{(2)R}(i\gamma^\mu\partial_\mu) + O(g^4)$, Eq. (6.8) can be rewritten as

$$[\bar{\gamma}^\mu\partial_\mu + im + i\Sigma^{(2)R}(i\bar{\gamma}^\mu\partial_\mu)]\psi(x) = 0, \tag{6.11}$$

which is covariant under the Lorentz transformation with the light velocity $\bar{c} = e^{-\alpha_e}c$. Thus, for a Λ-QED with $N_f = 3$, all the order-g^2 Lorentz anomalies will be eliminated by renormalizing the light velocity with $\bar{c} = (1 + \frac{g^2}{24\pi^2})c$.

Now we turn to discuss the Lorentz symmetry breaking in the low-energy regime predicted by the Λ-QED in the continuum limit when the Lorentz anomalous terms cannot be completely eliminated by a renormalization of the light velocity. We shall (i) ignore all radiative corrections except Lorentz anomalies, (ii) neglect the $O(k^3)$ terms in the Lorentz anomalies, and (iii) eliminate the Lorentz anomaly in the photon propagator via a renormalization of the light velocity. Then we obtain the following Maxwell-Dirac equations:

$$\Box A^\mu = g\bar{\psi}[\gamma^\mu - \lambda_e\gamma^\nu n_\nu n^\mu]\psi, \tag{6.12}$$

$$[i(\gamma^\mu - \lambda_e\gamma^\nu n_\nu n^\mu)(\partial_\mu + igA_\mu) - m]\psi = 0, \tag{6.13}$$

where $\lambda_e(\gamma n)(pn)$ is the Lorentz anomaly in the free-electron propagator, $n^\mu = (1 - v^2)^{-\frac{1}{2}}\{1,\mathbf{v}\}$ specifies the motion of the reference frame with respect to the vacuum, and the renormalized light velocity $c = 1$ remains valid in all inertial frames.

A. The free electron

From Eq. (6.13), we obtain the free-electron propagator,

$$S_F(p) = \frac{i}{(p_\mu - \lambda_e p_\nu n^\nu n_\mu)\gamma^\mu - m}. \tag{6.14}$$

Let $\mathbf{p} = E\mathbf{u}$; the dispersion relation for a free electron can be written in the form

$$E^2 = \frac{m^2}{1 - u^2 - \frac{\lambda_e(2-\lambda_e)(1-\mathbf{u}\cdot\mathbf{v})^2}{1-v^2}}. \tag{6.15}$$

Then, the rest mass and the group velocity of the electron are given by

$$m_e = m\sqrt{\frac{1 - v^2}{(1 - \lambda_e)^2 - v^2}} \tag{6.16}$$

and

$$\mathbf{u}_g = \frac{(1 - v^2)\mathbf{u} - \lambda_e(2 - \lambda_e)(1 - \mathbf{u}\cdot\mathbf{v})\mathbf{v}}{(1 - \lambda_e)^2 - v^2 + \lambda_e(2 - \lambda_e)\mathbf{u}\cdot\mathbf{v}}, \tag{6.17}$$

respectively. We note that the rest mass m_e varies under the boost of the reference frame and, in the preferred frame with $v = 0$, we have

$$E^2 = \frac{m^2}{(1 - \lambda_e)^2 - u^2} = \frac{m_e^2}{1 - (1 - \lambda_e)^2 u_g^2}. \tag{6.18}$$

This implies that, for a free electron, the maximal attainable velocity relative to the vacuum is $(1 - \lambda_e)^{-1}c$.

Furthermore, the conserved current density of the free electron can be defined as

$$j^\mu = \bar{\psi}(\gamma^\mu - \lambda_e\gamma^\nu n_\nu n^\mu)\psi, \tag{6.19}$$

which, like the rest mass m_e, also depends explicitly on the motion of the reference frame relative to the vacuum.

B. The electron in a Coulomb field

According to the Maxwell-Dirac equations (6.12) and (6.13), the potential energy of an electron in the Coulomb field of a static point charge $-Zg$ in the given reference frame is

$$V(r) = -\frac{Zg^2}{r}, \tag{6.20}$$

and the motion of the electron in the Coulomb field is described by

$$\begin{aligned}&\left[1 - \frac{\lambda_e(1 - \mathbf{v}\cdot\alpha)}{1 - v^2}\right][H - V(r)]\psi \\ &= \left\{\left[\alpha - \frac{\lambda_e(1 - \mathbf{v}\cdot\alpha)\mathbf{v}}{1 - v^2}\right]\cdot\mathbf{p} + \beta m\right\}\psi.\end{aligned} \tag{6.21}$$

Equation (6.21) can be rewritten in the form

$$H\psi = \left\{\frac{1 - \lambda_e - v^2}{(1 - \lambda_e)^2 - v^2}\left[\alpha\cdot\mathbf{p} + \beta m - \frac{Z\alpha_v}{r}\right] + K^{\text{anom}}\right\}\psi, \tag{6.22}$$

where

$$K^{\text{anom}} = -\lambda_e \frac{(\mathbf{v}\cdot\alpha)(\alpha\cdot\mathbf{p}+\beta m)+(1-\lambda_e-\alpha\cdot\mathbf{v})(\mathbf{p}\cdot\mathbf{v})}{(1-\lambda_e)^2-v^2} \quad (6.23)$$

is the anomalous part of the kinetic energy and

$$\alpha_v = \frac{(1-\lambda_e)^2-v^2}{1-\lambda_e-v^2}g^2 \quad (6.24)$$

denotes the fine-structure constant in the given reference frame. In fact, if the Coulomb system is at rest in the preferred frame with $v=0$, the energy eigenvalues of the system will be given by the well-known Sommerfeld's formula

$$E_{n,\kappa} = m_e\left\{1+\frac{Z^2\alpha_0^2}{\left(n-|\kappa|+\sqrt{\kappa^2-Z^2\alpha_0^2}\right)^2}\right\}^{-\frac{1}{2}}, \quad (6.25)$$

with $\alpha_0 = (1-\lambda_e)g^2$.

In these two examples, we can see two types of manifestation of the Lorentz anomalies in Λ-QED: (i) the fundamental constants m_e and α_v become varying under the boost of the reference frame, and (ii) the Hamiltonian of the free electron contains an anomalous kinetic term K^{anom} which may lead to spatially anisotropic and/or parity violation effects observable in the reference frames other than the preferred one.

During the past decades, mainly motivated by ideas about quantum gravity, there has been tremendous interest in searching for the Lorentz-invariance violation, and a number of theoretical frameworks have been developed to discuss the possible ways of violating Lorentz invariance [5,12]. As the mechanism of violating Lorentz invariance in the momentum-cutoff model of quantum field is quite different, it is anticipated that the research on this model will be useful to search for and explain the evidences of Lorentz violation.

In conclusion, the research conducted so far on momentum-cutoff vacuums has indicated that it is worthwhile to do further research on this subject, both in theory and experiments, not only for providing another version of workable quantum field theory but also for discovering unknown features of the local geometry of the vacuum which, in the conventional QED, we had assumed to be well known.

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APPENDIX

We give a proof of Eq. (3.19), which is valid also for non-Abelian gauge fields. Let $F[\varphi]$ be a functional of the field $\varphi_r(x)$ and $f(r,x|\varphi) = \delta F[\varphi]/\delta\varphi_r(x)$. Then we have

$$F[\varphi+\Delta\varphi]-F[\varphi] = \sum_r \int d^4x \int_0^1 d\lambda f(r,x|\varphi+\lambda\Delta\varphi)\Delta\varphi_r(x). \quad (A1)$$

Thus, for a current density given by

$$J^{\mu a}(x|A) = -\frac{\delta W[A]}{\delta A_\mu^a(x)}, \quad (A2)$$

we obtain

$$W[A+\phi]-W[A] = -\text{Tr}\left\{\int_0^1 d\lambda J_\mu[A+\lambda\phi]\phi^\mu\right\}, \quad (A3)$$

where $J_\mu[A]$ is a current density operator which satisfies

$$\langle x|\text{tr}\{J_\mu[A]T^a\}|x\rangle = J_\mu^a(x|A). \quad (A4)$$

It is easy to derive Eq. (3.19) from (A3) by defining the current density operator as

$$J_\mu[A] = -gG[A]\gamma_\mu. \quad (A5)$$

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