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Quantum parameter estimation using general single-mode Gaussian states

O. Pinel,[1] P. Jian,[2] N. Treps,[2] C. Fabre,[2] and D. Braun[3]

[1]*Centre for Quantum Computation and Communication Technology, Department of Quantum Science, The Australian National University, Canberra, ACT 0200, Australia*

[2]*Laboratoire Kastler Brossel, Université Pierre et Marie Curie-Paris 6, ENS, CNRS, 4 place Jussieu, 75252 Paris, France*

[3]*Laboratoire de Physique Théorique, Université Paul Sabatier, Toulouse III and CNRS, 118 route de Narbonne, 31062 Toulouse, France*

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We calculate the quantum Cramér-Rao bound for the sensitivity with which one or several parameters, encoded in a general single-mode Gaussian state, can be estimated. This includes in particular the interesting case of mixed Gaussian states. We apply the formula to the problems of estimating phase, purity, loss, amplitude, and squeezing. In the case of the simultaneous measurement of several parameters, we provide the full quantum Fisher information matrix. Our results unify previously known partial results and constitute a complete solution to the problem of knowing the best possible sensitivity of measurements based on a single-mode Gaussian state.

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Metrology using electromagnetic fields as a probe is of fundamental importance in many areas of science and technology. Applications include, among many others, distance measurements with laser range finders or radar, measurement of the shape and composition of objects in microscopy and spectroscopy, angular velocities with laser gyroscopes, and attempts of gravitational wave detection using large interferometers such as VIRGO and Laser Interferometer Gravitational-Wave Observatory (LIGO). In all these schemes, one or several parameters of the system under investigation are encoded in the state of light, and one subsequently tries to recover that value by detecting the light in a suitable way. It is important to know with what precision such a parameter can be measured in principle; that is, once all technical noise sources are eliminated, measurement instruments are ideally precise, and the system can be prepared in the same identical state as often as desired [1].

Quantum parameter estimation theory provides an answer to this question in the form of the quantum Cramér-Rao bound, which constitutes a lower bound to the fluctuations of an estimator of a parameter θ, given the knowledge of how the quantum-mechanical state ρ depends on the parameter. The bound is essentially due to quantum uncertainty and is given by the inverse quantum Fisher information I_{Fisher} associated with the state ρ_θ, where I_{Fisher} measures the distinguishability (or, in a complementary way, the fidelity) of two close-by quantum states that differ infinitesimally in θ. The result can be intuitively understood in quantum information terms. For neighboring states that differ slightly in the value of a parameter θ, the more distinguishable the states are, the more precisely θ can be measured. The quantum Cramér-Rao bound is applicable to any quantum-mechanical system and provides often a generalized uncertainty relation, even if no Hermitian operator can be simply associated with a given observable, as is the case, for example, for phase estimation [2–4].

In quantum optics, a particularly useful class of states is the class of Gaussian states, which are defined generally as states with a Gaussian Wigner function. This class includes coherent states (e.g., the light emitted by a laser operating far above threshold), thermal light, squeezed light, and, in the case of several modes, some entangled states such as Einstein-Podolsky-Rosen (EPR) states. These states are readily available in the laboratory with large photon numbers [5] and play an important role in quantum metrology and information processing [6]. In [7] quantum Fisher information was calculated for pure Gaussian states with arbitrarily many modes, and a measurement scheme was proposed that saturates the quantum Cramér-Rao bound. However, the need to calculate the square root of two different operators renders the calculation, in general, very difficult for mixed states of infinite dimensional systems. Partial early results include those by Twamley, who calculated the Bures distance between squeezed thermal states [8], and Paraoanu and Scutaru, who did so for displaced thermal states [9]. Scutaru found the fidelity for thermal states that are both displaced and squeezed [10]. Monras and Paris found the quantum Fisher information for the particular problem of loss estimation with displaced squeezed thermal states [11], and Aspachs *et al.* considered phase estimation with thermal states [12]. These results all refer to single-mode states. Very recently, Marian and Marian produced a result for the fidelity between arbitrary one- or two-mode Gaussian states [13]. Other recent works on the quantum Cramér-Rao bound for two-mode Gaussian states include [14–16].

Here we provide a comprehensive analysis for general single-mode Gaussian states. They can be parameterized by five real parameters that we will describe below. Our analysis is based on a general expression for the Bures distance between two Gaussian one-mode states in [10], which we expand up to second order in the infinitesimal difference $d\theta$ in the parameters between the two neighboring states ρ_θ and $\rho_{\theta+d\theta}$. This yields the quantum Fisher information. For the case of simultaneous estimation of several parameters, we calculate the complete quantum Fisher matrix, which sets a lower bound to the covariance matrix of the parameters in the sense of a matrix inequality [17].

Gaussian states. The quadratures of an electromagnetic field mode (in units with $\hbar = 2$) are defined in terms of the annihilation and creation operators a and $a^\dagger$ of the mode as [18]

$$\hat{x} = a^\dagger + a, \quad \hat{p} = i(a^\dagger - a). \tag{1}$$

In the Wigner function description of the state, the quadratures correspond to two phase-space coordinates x and p, which

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we group into a two-dimensional vector $\mathbf{X}$, $\mathbf{X}^\top = (x,p)$. The Wigner function for an arbitrary quantum state given in terms of its density matrix ρ is then defined as

$$W(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi\, e^{-ip\xi}\langle x-\xi|\rho|x+\xi\rangle\,. \tag{2}$$

For a single-mode Gaussian state that depends on the parameter θ, the Wigner function takes the general form

$$W_\theta(\mathbf{X}) = \frac{1}{2\pi|\Sigma_\theta|^{1/2}} e^{-\frac{1}{2}(\mathbf{X}-\overline{\mathbf{X}}_\theta)^\top\Sigma_\theta^{-1}(\mathbf{X}-\overline{\mathbf{X}}_\theta)}\,, \tag{3}$$

where $\overline{\mathbf{X}}_\theta$ are the parameter-dependent expectation values of the quadratures in the state ρ_θ, Σ_θ is the covariance matrix [19], and $|\Sigma_\theta|$ is its determinant. Σ_θ is a real symmetric matrix with matrix elements

$$\Sigma_{\theta,ij} = \tfrac{1}{2}\langle X_iX_j + X_jX_i\rangle - \langle X_i\rangle\langle X_j\rangle\,, \tag{4}$$

and $\langle\cdots\rangle \equiv \mathrm{tr}(\rho\cdots)$. We see that the Wigner function is parameterized with five real parameters. The purity of the state is given by $P_\theta = \mathrm{tr}\rho_\theta^2 = |\Sigma_\theta|^{-1/2}$.

Quantum Cramér-Rao bound. The (squared) sensitivity $(\delta\theta)^2$ with which a parameter θ can be estimated from Q measurement results a_i of some observable A is defined as the variance of the deviation from the true value of θ of an estimator of θ, $\theta_{\mathrm{est}}(a_1,\ldots,a_Q)$, which depends solely on the measurements results: $\delta\theta^2 = \langle[\theta_{\mathrm{est}}(a_1,\ldots,a_Q)-\theta]^2\rangle_s$, where $\langle\cdots\rangle_s$ corresponds to the statistical mean. It is bounded from below by the inverse of the quantum Fisher information,

$$(\delta\theta)^2 \geqslant \frac{1}{Q\,I_{\mathrm{Fisher}}(\rho_\theta)}\,, \tag{5}$$

where I_{Fisher} is defined here as the quantum Fisher information for a single measurement. The bound is optimized over all possible positive operator-valued measurements and classical postprocessing of data (i.e., all estimator functions). For an unbiased estimator it can be saturated in the limit of a large number of measurements and thus represents the ultimate reachable bound of sensitivity. The quantum Fisher information is given in terms of the Bures distance between two close-by states $\rho_\theta,\rho_{\theta+\epsilon}$ as

$$I_{\mathrm{Fisher}}(\rho_\theta) = 4\left(\left.\frac{\partial d_{\mathrm{Bures}}(\rho_\theta,\rho_{\theta+\epsilon})}{\partial\epsilon}\right|_{\epsilon=0}\right)^2. \tag{6}$$

The Bures distance between two quantum states ρ_1,ρ_2 is defined as

$$d_{\mathrm{Bures}}(\rho_1,\rho_2) = \sqrt{2}\sqrt{1-\sqrt{F(\rho_1,\rho_2)}}, \tag{7}$$

where $F(\rho_1,\rho_2) = [\mathrm{tr}(\sqrt{\rho_1}\rho_2\sqrt{\rho_1})^{1/2}]^2$ denotes the fidelity between the two states. In [10] it was found that for two arbitrary single-mode Gaussian states ρ_1,ρ_2 of the form (3)

$$F(\rho_1,\rho_2) = \frac{2\exp\left[-\frac{1}{2}\boldsymbol{\Delta}\mathbf{X}^\top(\boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2)^{-1}\boldsymbol{\Delta}\mathbf{X}\right]}{\sqrt{|\boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2|+(1-|\boldsymbol{\Sigma}_1|)(1-|\boldsymbol{\Sigma}_2|)}-\sqrt{(1-|\boldsymbol{\Sigma}_1|)(1-|\boldsymbol{\Sigma}_2|)}}, \tag{8}$$

where $\boldsymbol{\Delta}\mathbf{X} = \langle\mathbf{X}_1-\mathbf{X}_2\rangle$ is the mean relative displacement. Under a smoothness hypothesis, necessary for any Cramér-Rao bound, we have $\frac{\partial F(\rho_\theta,\rho_{\theta+\epsilon})}{\partial\epsilon}|_{\epsilon=0} = 0$ and

$$I_{\mathrm{Fisher}}(\rho_\theta) = -2\left.\frac{\partial^2F(\rho_\theta,\rho_{\theta+\epsilon})}{\partial\epsilon^2}\right|_{\epsilon=0}. \tag{9}$$

The first- and second-order derivatives of the determinant of a differentiable, invertible matrix A_θ with respect to θ can be written conveniently as

$$|A_\theta|' = |A_\theta|\mathrm{tr}\big(A_\theta^{-1}A_\theta'\big) \tag{10}$$

and

$$|A_\theta|'' = |A_\theta|\big\{\mathrm{tr}\big[A_\theta^{-1}A_\theta'' - \big(A_\theta^{-1}A_\theta'\big)^2\big] + \big[\mathrm{tr}\big(A_\theta^{-1}A_\theta'\big)\big]^2\big\}, \tag{11}$$

where A_θ' is the term by term derivative of A_θ with respect to θ [20].

After a straightforward but long and tedious expansion of the fidelity to second order we find

$$I_{\mathrm{Fisher}}(\rho_\theta) = \frac{1}{2}\frac{\mathrm{tr}\big[\big(\boldsymbol{\Sigma}_\theta^{-1}\boldsymbol{\Sigma}_\theta'\big)^2\big]}{1+P_\theta^2} + 2\frac{P_\theta'^2}{1-P_\theta^4} + \boldsymbol{\Delta}\mathbf{X}_\theta'^\top\boldsymbol{\Sigma}_\theta^{-1}\boldsymbol{\Delta}\mathbf{X}_\theta'. \tag{12}$$

Equation (12) shows that the quantum Fisher information depends on three terms representing the information carried by (1) the evolution of the noise properties of the state encoded in $\boldsymbol{\Sigma}_\theta$, (2) the evolution of the purity P_θ with θ, and (3) the "speed" of displacement $\boldsymbol{\Delta}\mathbf{X}_\theta' = d\langle\mathbf{X}_{\theta+\epsilon}-\mathbf{X}_\theta\rangle/d\epsilon|_{\epsilon=0}$ of the state in phase space. Equation (12) provides a generalization of the result for pure Gaussian single-mode states [7] and constitutes the main result of this Rapid Communication. The second term vanishes if, for the value of θ under consideration, the state is pure, $P_\theta = 1$, under the condition that the eigenvalues of ρ_θ are differentiable at that value of θ.

Unification of previous partial results. We now show that one obtains from (12) previous partial results for particular measurements. We recall that a general single-mode Gaussian state can always be represented as a squeezed displaced thermal state ν [19],

$$\rho = R(\psi)D(\alpha)S(\xi)\nu S(\xi)^\dagger D(\alpha)^\dagger R(\psi)^\dagger\,, \tag{13}$$

where $R(\psi) = \exp(i\psi a^\dagger a)$ is the rotation operator, $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is the displacement operator, and $S(\xi) = \exp(\frac{1}{2}\xi a^{\dagger 2} - \frac{1}{2}\xi^* a^2)$ is the squeezing operator.

The five real parameters can be interpreted physically as (1) the shift of the state along the x quadrature, parameterized by a $\mathbb{R} \ni \alpha > 0$, and the phase of the rotation $\psi \in \mathbb{R}$, (2) a complex squeezing parameter $\xi = re^{i\chi}$, $r,\chi \in \mathbb{R}$, where

$r > 0$ defines the amount of squeezing and χ is the squeezing direction (we will also use the parameter $\sigma = e^{-r}$), and (3) the purity of the initial thermal state ν, $P_0 = 1/(2N_{\rm th} + 1)$, where $N_{\rm th} = {\rm tr}(\nu a^\dagger a)$ denotes the number of thermal photons. Since squeezing and shifting are unitary operations, we have $P_\theta = P_0$ for all possible parameters θ. This implies that the second term in (12) only contributes if θ is a function of $N_{\rm th}$ for the parametrization (13).

With these parameters, $\mathbf{\Delta X} = 2\alpha(\cos\psi, \sin\psi)$, and the general covariance matrix can then be written as [19]

$$\mathbf{\Sigma} = (2N_{\rm th}+1)\begin{pmatrix} \sigma^2\cos^2(\chi+\psi) + \frac{1}{\sigma^2}\sin^2(\chi+\psi) & \frac{1}{2}\left(\sigma^2 - \frac{1}{\sigma^2}\right)\sin(2\chi+2\psi) \\ \frac{1}{2}\left(\sigma^2 - \frac{1}{\sigma^2}\right)\sin(2\chi+2\psi) & \frac{1}{\sigma^2}\cos^2(\chi+\psi) + \sigma^2\sin^2(\chi+\psi) \end{pmatrix} \tag{14}$$

Applying Eq. (5), we find the following expressions for the quantum Fisher information I_θ for all five parameters $\theta \in \{\alpha,\psi,\sigma,\chi,N_{\rm th}\}$ [from now on we replace the subscript Fisher with the parameter(s) θ to be varied]. The quantum Fisher information for the estimation of α reads

$$I_\alpha = 4P_0\left(\frac{1}{\sigma^2}\cos^2(\chi) + \sigma^2\sin^2(\chi)\right). \tag{15}$$

Note that amplitude estimation is directly related to the measurement of the power of the electromagnetic signal. As expected, I_α is maximal when the state is amplitude squeezed. For an unsqueezed state, $\sigma = 1$, we have $I_\alpha = 4P_0$, which for a pure state, $P_0 = 1$, agrees with the result that one may obtain directly from the overlap of two coherent states.

The quantum Fisher information for phase estimation reads

$$I_\psi = 4P_0\alpha^2\left(\sigma^2\cos^2(\chi) + \frac{1}{\sigma^2}\sin^2(\chi)\right) + \frac{1}{1+P_0^2}\frac{(1-\sigma^4)^2}{\sigma^4}. \tag{16}$$

The first term depends on the mean field. It is largest when the state is phase squeezed, i.e., when $\chi = 0$ and $\sigma > 1$. The second term depends only on the squeezing-dependent noise properties of the state and its purity. Each of these two terms corresponds exactly to the results of [12], where the authors analyze displaced thermal states and thermal squeezed states. Equation (16) generalizes these results to the most general single-mode Gaussian states that can be both squeezed and displaced at the same time. Optimizing (16) for a fixed average number of photons $N = {\rm tr}(\rho a^\dagger a)$ leads directly to the result found in [21]: squeezed vacuum states are optimal for phase measurement, leading asymptotically to a precision $\delta\psi = 1/[8(N+N^2)]$. Generating such states with a large squeezing is challenging. Using bright squeezed states instead, for which $\alpha \gg 1$, one gets the asymptotic limit $\delta\psi \simeq 1/(2\sigma\sqrt{N})$ [22].

For the estimation of squeezing, we find asymptotically the same bound as [23,24] for pure Gaussian states. The quantum Fisher information for the estimation of σ^2 reads

$$I_{\sigma^2} = \frac{1}{1+P_0^2}\frac{1}{\sigma^4}. \tag{17}$$

On the other hand, the quantum Fisher information I_r for the squeezing parameter r is a constant, which generalizes the result in [23].

The quantum Fisher information relevant for estimating the squeezing angle is

$$I_\chi = \frac{1}{1+P_0^2}\frac{(1-\sigma^4)^2}{\sigma^4}. \tag{18}$$

Interestingly, both the squeezing and its angle can be estimated with a sensitivity that reaches, for large $N_{\rm th}$, a constant independent of $N_{\rm th}$. This is in contrast to the estimation for the thermal photon number itself, for which the sensitivity keeps getting worse with larger photon number. The corresponding quantum Fisher information reads

$$I_{N_{\rm th}} = \frac{1}{N_{\rm th} + N_{\rm th}^2}. \tag{19}$$

This can be understood as a consequence of increasing thermal smearing of the state as a function of temperature, which leads to larger and larger (thermal) photon number fluctuations. Alternatively, we have the quantum Fisher information for the estimation for purity $I_P = 1/(P^2 - P^4)$, as follows also from $I_{N_{\rm th}}$ by the laws of error propagation.

Equation (12) can also be applied to the estimation of other relevant physical parameters through different parametrizations of the Gaussian state, such as the estimation of losses. Taking as the initial state an amplitude-squeezed state with real amplitude α_0 and variance σ^2 in the amplitude quadrature ($\psi = \chi = 0$), the amplitude and the covariance matrix of the state read, after an attenuation of η, respectively,

$$\alpha(\eta) = \sqrt{1-\eta}\,\alpha_0, \tag{20}$$

$$\Sigma = \begin{pmatrix} \sigma^2 + \eta(1-\sigma^2) & 0 \\ 0 & \frac{1}{\sigma^2} + \eta\left(1 - \frac{1}{\sigma^2}\right) \end{pmatrix}. \tag{21}$$

The quantum Fisher information for the estimation of η is found to be

$$I_\eta = \frac{1}{1-\eta} \times \left(\frac{\alpha_0^2}{\sigma^2 + \eta(1-\sigma^2)} + \frac{[1-2\eta(1-\eta)](1-\sigma^2)^2}{2\eta[2\sigma^2 + \eta(1-\eta)(1-\sigma^2)^2]}\right). \tag{22}$$

This corresponds exactly to the result of [11] if we translate η to the parameter ϕ in that paper, as $1-\eta=\cos^2(\phi)=e^{-\gamma t}$, where γ denotes the rate in the Lindblad master equation and t is the evolution time in the channel.

Extension to multiple parameters. In the case of the simultaneous measurement of several parameters $\boldsymbol{\theta}=\theta_1,\ldots,\theta_p$, the quantum Cramér-Rao bound generalizes to a matrix inequality bounding the covariance matrix $\boldsymbol{\gamma}$ of the estimators, defined through its matrix elements $\gamma_{ij}=\langle\theta_i\theta_j\rangle-\langle\theta_i\rangle\langle\theta_j\rangle$. A lower bound of this matrix is given by the inverse of the quantum Fisher matrix $\mathbf{I}(\boldsymbol{\theta})$ [17],

$$\boldsymbol{\gamma}\geqslant\frac{1}{Q}\mathbf{I}(\boldsymbol{\theta})^{-1}. \tag{23}$$

The inequality is to be understood in the sense that $\mathbf{A}\geqslant\mathbf{B}$ is equivalent to $\mathbf{A}-\mathbf{B}$ being a positive semidefinite matrix. The quantum Fisher matrix $\mathbf{I}(\boldsymbol{\theta})$ is defined through the symmetric logarithmic derivative L_{θ_i} of the state with respect to a parameter θ_i,

$$\mathbf{I}(\boldsymbol{\theta})_{ij}=\mathrm{tr}\left(\rho_{\boldsymbol{\theta}}\frac{L_{\theta_i}L_{\theta_j}+L_{\theta_j}L_{\theta_i}}{2}\right)=\mathrm{tr}\big(\partial_{\theta_i}\rho_{\boldsymbol{\theta}}L_{\theta_j}\big). \tag{24}$$

The symmetric logarithmic derivative can be expressed in terms of the spectral decomposition of the density matrix, $\rho_{\boldsymbol{\theta}}=\sum_n\rho_n(\boldsymbol{\theta})|\psi_n(\boldsymbol{\theta})\rangle\langle\psi_n(\boldsymbol{\theta})|$, as

$$L_{\theta_i}\equiv2\sum_{nm}\frac{\langle\psi_m|\partial_{\theta_i}\rho_{\boldsymbol{\theta}}|\psi_n\rangle}{\rho_n+\rho_m}|\psi_m\rangle\langle\psi_n|. \tag{25}$$

The sum is over all terms with $\rho_n+\rho_m\neq0$. Contrary to the single-parameter case, bound (23) may not necessarily be achievable. In the case of a diagonal quantum Fisher information matrix one gets back result (5). The quantum Fisher matrix defines a Riemannian metric with metric tensor g_{ij} [17,25],

$$d^2_{\mathrm{Bures}}(\rho_{\boldsymbol{\theta}},\rho_{\boldsymbol{\theta}+d\boldsymbol{\theta}})=g_{ij}d\theta_i d\theta_j=\tfrac{1}{4}I_{ij}(\boldsymbol{\theta}). \tag{26}$$

This implies that we can calculate the quantum Fisher matrix by differentiating the Bures distance $d_{\mathrm{Bures}}(\rho_{\boldsymbol{\theta}},\rho_{\boldsymbol{\theta}+d\boldsymbol{\theta}})$ with respect to parameters $d\theta_i$ and $d\theta_j$.

When applying this procedure to (7) with (8) for the fidelity of a single-mode Gaussian state, we obtain the matrix element $I_{\theta_i\theta_j}$ for measuring parameters θ_i and θ_j,

$$\begin{aligned}I_{\theta_i\theta_j}=&\frac{1}{2}\frac{1}{1+P_{\boldsymbol{\theta}}^2}\mathrm{tr}\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}\frac{\partial\boldsymbol{\Sigma}_{\boldsymbol{\theta}}}{\partial\theta_i}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}\frac{\partial\boldsymbol{\Sigma}_{\boldsymbol{\theta}}}{\partial\theta_j}\right)\\&+\frac{2}{1-P_{\boldsymbol{\theta}}^4}\frac{\partial P_{\boldsymbol{\theta}}}{\partial\theta_i}\frac{\partial P_{\boldsymbol{\theta}}}{\partial\theta_j}\\&+\left(\frac{\partial\boldsymbol{\Delta}\mathbf{X}}{\partial\theta_i}\right)^{\top}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{-1}\left(\frac{\partial\boldsymbol{\Delta}\mathbf{X}}{\partial\theta_j}\right).\end{aligned} \tag{27}$$

Compared to (12) we see that squared derivatives with respect to the same parameter θ are simply replaced by mixed derivatives with respect to θ_i and θ_j, such that the diagonal matrix elements agree with (12), $I_{\theta_i\theta_i}=I_{\theta_i}$. With the general expression (27) one can explicitly calculate the entire quantum Fisher information matrix with dimension up to 5×5.

In terms of the parameters introduced in (13), there are only two independent nonvanishing off-diagonal matrix-elements,

$$I_{\chi\psi}=I_\chi \tag{28}$$

$$I_{\alpha\psi}=2P_0\alpha\left(\frac{1}{\sigma^2}-\sigma^2\right)\sin(2\chi). \tag{29}$$

The first equation can be easily understood from (14), where χ and ψ always appear in linear combination. From (29) we see that for states without displacement ($\alpha=0$), without squeezing ($\sigma=1$), or squeezing in direction $\chi=0$ the off-diagonal matrix-element $I_{\alpha\psi}$ vanishes, implying that in this case the matrix-valued quantum Cramér-Rao bound (23) for the combined estimation of amplitude and phase boils down to two separate bounds, one for each of these two estimations. Note, however, that, in general, the two bounds cannot be saturated simultaneously.

Another useful example of the possibility of statistically independent measurements is the simultaneous measurement of attenuation η and phase ψ when light in an interferometer passes through a phase shifter. A realistic phase shifter, such as a thin piece of glass, will indeed not only shift the phase but typically also lead to some attenuation of the signal and thus to a mixed state if one does not keep track of the photon number, such that (24) applies. This situation was considered recently in [26], albeit for a state with a fixed number of photons, in which case the corresponding 2×2 quantum Fisher information matrix is diagonal. Here we see that the same independence holds for all single-mode Gaussian states.

In summary, we have derived the quantum Cramér-Rao bound for the measurement of the five parameters characterizing a general mixed single-mode Gaussian state of light. Our analysis generalizes and unifies several existing approaches for particular states or particular single-parameter measurements [8–12]. We have also derived the quantum Fisher information matrix that gives a matrix-valued lower bound on the covariance matrix of estimators in the case of the simultaneous measurement of several parameters and found that the only two joint measurements which are generically not independent are those of the phase together with the amplitude or together with the phase of the squeezing. Our results constitute a complete solution of the problem of the best possible sensitivity for the measurement of an arbitrary parameter of the most general (not necessarily pure) single-mode Gaussian state.

Note added. Recently, we became aware of an alternative approach for estimation of a single parameter with general multimode Gaussian light by Monras [27]. While our results agree with the general Eq. (13) in that paper when specialized to the single-mode single-parameter case, our Eq. (12) contains an extra term compared to his Eq. (16) due to the variation of the purity with the parameter.

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