PHYSICAL REVIEW A **88**, 033801 (2013)

Geometric phase accompanying SU(2) coherent states for quantum polarized light

Hiroshi Kuratsuji*
Faculty of Science and Engineering, Ritsumeikan University-BKC, Kusatsu City 525-8577, Japan
(Received 3 January 2013; published 3 September 2013)

The geometric phase is studied for quantum polarized light described by an SU(2) coherent state. This is revealed by using quantum interference. The SU(2) coherent state is constructed from the quantized Stokes parameters. The quantum interferometry is arranged by the evolution operator implemented by the Hamiltonian inspired from the birefringence of the Faraday and Kerr effect. The geometric phase is given by the coherent state path integral. In particular, by considering a resonant Hamiltonian, the geometric phase is extracted from the total factor that consists of both geometric and dynamical phases. This extraction forms the basis of an experimental way to detect the geometric phase.

DOI: 10.1103/PhysRevA.88.033801 PACS number(s): 42.50.Ct, 42.25.Ja, 42.25.Lc

I. INTRODUCTION

Polarization is one of the most important key words in modern physics. In quantum mechanics, this is represented by two states. Historically, the most appealing object expressed in terms of two states is undoubtedly the "spin" of electron; that is described by the Pauli spin matrix. Shortly after the advent of spin, Jordan pointed out a possible analogy between electron spin and light polarization [1]. Jordan's recognition may be, in some sense, a real breakthrough in the history of modern quantum optics, though it was almost forgotten in the celebrated history of modern quantum physics.

The concept of polarization has been exhaustively studied within the framework of classical optics or radiation theory [2,3], in which the two-component nature of *waves* (not states) corresponding to left- and right-handed polarization plays a key role. Indeed, it is crucial that two waves have a *spinor form*, as had been early recognized by Jordan [1]. In an anisotropic optical substance, there occurs mutual change between these two states, which gives rise to the so-called "optical activity." In classical optics, the polarization is described by the Stokes parameters, which is very analogous to the spin vector that is obtained by sandwiching the Pauli spin in terms of the spinor form of light waves (see [4–7]). This may be regarded as a variant of the well-known procedure of describing the vector in terms of spinor, the so-called Fierz decomposition [8].

On the other hand, the essence of modern quantum optics [9], which is based on Dirac's quantum theory of radiation, is concisely stated by just one term, *quantized field operator*, which represents the creation and annihilation operator of photons. Within the framework of the quantum theory of radiation, quantized light polarization can be described by the field operator that consists of two independent elements corresponding to the left and right polarization. If we substitute two field operators into Jordan's spinor, one can naturally obtain the quantized Stokes parameter. This has been treated in previous papers (see, e.g., Refs. [10] and [11]) in a different context from the present attempt that is given below.

An algebraic structure behind the quantum Stokes parameters is SU(2) algebra; this was suggested early on in connection with the unpolarized light [12] and further elaborated in [13]. The SU(2) aspect of quantum polarized light has been recently developed in several ramifications, e.g., the quantum tomography [14,15] and entanglement of photons [16].

The purpose of this article is to explore a characteristic aspect of the quantum polarized light that is based on a quite foreign view from the studies investigated so far [14–16]. The object we are concerned with is the geometric phase [17] accompanying the quantum polarization state, which is realized as the SU(2) coherent state [18–20]. The geometric phase provides a simple means to extract the geometric structure inherent in a wide class of quantum systems [17]. Hence it is highly expected that possible hidden aspects of the quantum polarized light may be uncovered by investigating the geometric phase.

The starting point is to construct the SU(2) coherent state from the quantized Stokes parameters, which is naturally connected with the classical Stokes parameters and is obtained as an expectation value with respect to the coherent state. The central point is to arrange the quantum interferometry implemented by an evolution of the quantum polarized light transmitted in anisotropic media, for which the evolution is described by the Hamiltonian inspired from the birefringence caused by the Faraday and Kerr effects. The interference in this process is expressed by a transition amplitude along a cyclic path in the polarization space, which is naturally evaluated by a coherent state path integral [21,22]. The path integral results in the phase factor that consists of the geometric as well as dynamical phases [17]. Here it is crucial to discriminate the geometric phase from the dynamical one. This can be achieved by considering a special case of the resonant Hamiltonian. The way to extract the geometric phase may be used to design an experiment to detect the geometric phase.

The content of the paper is organized as follows. In the next section we present a preliminary account of the the quantized Stokes parameters together with the construction of the polarization coherent state. In Sec. III, the geometric phase is formulated in a general form. In Sec. IV, we give the details of the extraction of the geometric phase. The last section is a summary.

II. SU(2) COHERENT STATE FOR QUANTUM POLARIZED LIGHT

In this section a brief account is given of the SU(2) coherent state by constructing the quantized Stokes parameters in terms of the Schwinger boson.

*kra@se.ritsumei.ac.jp

A. Quantized Stokes parameters

We start with the quantized electric and magnetic. Specifically, we are concerned with the case of a single mode. For the present purpose it is sufficient to consider this case and an extension to many modes is straightforward. Taking account of the polarization degree, the electric and magnetic field read in a complex form [23],

$$\mathbf{E} = i\sqrt{\frac{\hbar\omega}{2V}}\{(\mathbf{e}_+ a^\dagger + \mathbf{e}_- b^\dagger) f(\mathbf{x},t) + \text{H.c.}\},$$
$$\mathbf{B} = i\sqrt{\frac{\hbar\omega}{2V}}\hat{\mathbf{k}} \times \{(\mathbf{e}_+ a^\dagger + \mathbf{e}_- b^\dagger) f(\mathbf{x},t) + \text{H.c.}\}. \tag{1}$$

Here we have mind of the quantized wave contained in the box of volume V, which consists of the plane wave propagating in the z direction ($\hat{\mathbf{k}}$ denotes the unit vector along z direction); $f(\mathbf{x},t) = \exp\{i[k(z-ct)]\}$, with the polarization in the plane perpendicular to the z axis, namely, the (x,y) plane. The pair (a,b) stands for the photon operator corresponding to the left and right polarization that are denoted by $\mathbf{e}_\pm$ [24]. Here the circular polarization basis is written in terms of the linear basis $(\mathbf{e}_x,\mathbf{e}_y)$ as follows:

$$\mathbf{e}_\pm = \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y), \tag{2}$$

which represents the basis vector in the plane that is perpendicular to the propagation direction of photon.

Here we note a connection between the circular basis thus defined and the linear polarization basis treated by previous literature (see, e.g., [10,11]). Namely, let (a_x,a_y) be the photon operator in terms of linear polarization basis, then this is related to the operator expressed in terms of the circular basis:

$$(a,b) = (a_x,a_y)T^\dagger, \tag{3}$$

where T stands for the unitary matrix:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}. \tag{4}$$

In classical optics the polarization state is described by the Stokes parameters, the quantum counterpart of which plays a role of the pseudospin $\mathbf{S}$ that is constructed from a two-photon state. The classical Stokes parameters are defined in terms of the light wave of two components, $(\psi_l,\psi_r) = \psi$ [7],

$$\mathbf{S}^{cl} = \psi^\dagger \sigma \psi,$$

with σ being the Pauli matrix. By replacing ψ by the pair of photon operators, $\psi = (a,b)$, one can define the quantized version of the Stokes parameter, which becomes

$$S_z = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \quad S_+ = a^\dagger b, \quad S_- = b^\dagger a, \tag{5}$$

with $S_\pm = S_x \pm iS_y$. The commutation relation holds for $\mathbf{S} \equiv (S_x,S_y,S_z)$, that is, $\mathbf{S}\times\mathbf{S} = i\mathbf{S}$. The physical meaning of pseudospin $S_\pm$, which is simply the quantum counterpart of the Stokes parameters in classical optics, is that it plays a role of mutual change of photon state from the left polarization to the right polarization and vice versa.

Here a remark is in order concerning a peculiar feature of the circular basis to describe the quantized Stokes parameters. This comes from the fact that the circular basis is closely connected with the *optical activity* leading to the concept of *chirality*. In this sense, the circular basis is suitable only for the present purpose, because it utilizes the characteristics of the Faraday effect that is essentially an induced optical anisotropy of chiral nature.

B. Polarization coherent state

a. General remarks. In order to construct the quantum state for an assembly of polarized photons, we start with a preparatory account for the fact about the classical and quantum correspondence.

The first point to be noted is that the polarization state can be written as a superposition,

$$|\psi\rangle_p = \sum_{N_l+N_r=N} C_{N_l,N_r}|N_l,N_r\rangle, \tag{6}$$

with appropriate coefficients and the fixed number of photons N. Here $|N_l,N_r\rangle$ means the substrate, with (N_l,N_r) being the number of photons of the left- and right-handed polarization. The point here is how to prepare the fixed photon number. According to the quantized field theory, it has been a consensus that it is not feasible to create the state with fixed photon number; however, this is not the case. Recent progress in *cavity quantum electrodynamics* has enabled us to realize the fixed photon number [25]. Thus we have a basis to consider the state of a fixed number of photons in principle. In particular, once having the state with a fixed number of photons, the completely polarized state is obtained by filtering the state (6) through appropriate ellipsometry.

The second point to be noted is that the quantum polarized state of the form (6) should be in conformity with the classical Stokes parameters. That is, the quantized Stokes parameters lead to the classical Stokes parameters by choosing the coefficients C_{N_l,N_r} appropriately. Ideally such a state could be realized by an analogous state with the *spin coherent state*, if we note the parallelism between the spin operator and the quantized Stokes parameters. For the spin coherent state, the expectation value of the quantum spin leads to the classical spin, though this cannot result in the eigenstate of the spin operators. In this way, a similar procedure can be applied to obtain the quantum polarized state as the SU(2) coherent state.

b. SU(2) coherent state. Now by taking account of the general procedure given in the above, we shall construct the polarization state in terms of the SU(2) coherent state. Let us express the state $|N_l,N_r\rangle$ by using two kinds of photons [26]:

$$|N_l,N_r\rangle = \frac{1}{\sqrt{N_l!N_l!}}(a^\dagger)^{N_l}(b^\dagger)^{N_r}|0,0\rangle, \tag{7}$$

which is nothing but the Fock state with $|0,0\rangle \equiv |0\rangle_l|0\rangle_r$. $|0\rangle_{l,r}$ means the vacuum for the left (right) polarized photon. By arranging the quantum number such that $N_l = S+M, N_r = S-M$ (note that $N = 2S$), the Fock state is rewritten as

$$|S,M\rangle = \frac{1}{\sqrt{(S+M)!(S-M)!}}(a^\dagger)^{S+M}(b^\dagger)^{S-M}|0\rangle, \tag{8}$$

with $|0\rangle \equiv |0,0\rangle$. In particular, we have for $M=-S$,

$$|S,-S\rangle = \frac{1}{\sqrt{(2S)!}}(b^\dagger)^{2S}|0\rangle, \tag{9}$$

which stands for the completely right-handed circular polarization state, with the photon number $N_r = N$, satisfying $\hat{S}_z|S,-S\rangle = -S|S,-S\rangle$. By using this basis state, the general polarization state is written as

$$|S,M\rangle = \frac{1}{\sqrt{(S+M)!(S-M)!}}(a^\dagger b)^{S+M}|S,-S\rangle. \quad (10)$$

Now according to the SU(2) group structure inherent in the quantum polarization state, the polarization change takes place transitively from one polarized state to the others, which can be realized by applying the rotational operator [unitary group SU(2)] to a specific starting state. Noting this fact, by choosing the state $|S,-S\rangle$ (completely right-handed circular polarization state) as the *fiducial* state [18–20], we obtain the spin [SU(2)] coherent state (alias Bloch state). The procedure is carried out in the following manner: Let $R(\theta,\phi)$ be the rotational operator that gives the rotation of an angle θ about the fixed axis $\mathbf{n} = (\sin\phi, -\cos\phi)$, namely, $R = \exp[-i\theta S_{\mathbf{n}}] = \exp[-i\theta(S_x\sin\phi - S_y\cos\phi)]$, which is expressed in the complex form

$$R(\theta,\phi) = \exp[\xi S_+ - \xi^* S_-], \quad (11)$$

with $\xi = \frac{1}{2}\theta\exp[-i\phi]$. By applying this to the starting, completely right-handed polarized state $|S,-S\rangle$, we obtain

$$|\theta,\phi\rangle = R(\theta,\phi)|S,-S\rangle. \quad (12)$$

Here a remark is given regarding the convention for the angle variables θ,ϕ, which is measured from the *south pole* following Ref. [19]. In what follows, we adopt the alternative, $\theta \to \pi - \theta$, which means the angle measured from the *north pole*, in order to be fitted to the coordinate the Poincarè sphere (see below).

Now according to the entanglement formula [18–20], we have the form

$$|z\rangle = (1+|z|^2)^{-S}e^{zS_+}|0\rangle, \quad (13)$$

and z is the (complex valued) stereographic coordinate,

$$z = \tan\left(\frac{\pi-\theta}{2}\right)\mathrm{e}^{-i\phi} = \cot\frac{\theta}{2}\mathrm{e}^{-i\phi}. \quad (14)$$

Here if introducing a non-normalized coherent state denoted as $|\tilde{z}\rangle$ with the norm

$$(1+z^*z)^{2S} \equiv F(z,z^*), \quad (15)$$

then the normalized coherent state $|z\rangle \equiv \frac{1}{\sqrt{F(z,z^*)}}|\tilde{z}\rangle$. The coherent state constructed in the above is regarded as an ideal state to describe the polarization state.

The relation of paramount importance is the relation of partition of unity (or completeness relation), namely,

$$\int |z\rangle d\mu(z)\langle z| = 1, \quad (16)$$

with the invariant measure on the sphere,

$$d\mu(z) = \frac{2S+1}{4\pi}\frac{dz^*\wedge dz}{(1+z^*z)^2}.$$

The relation of partition of unity (16) plays a key role in constructing the path integral that will be used later.

In order to verify the correspondence between the quantum and classical Stokes parameters, we calculate the expectation value of the quantized Stokes operators [27],

$$\begin{aligned}\langle z|S_x|z\rangle &= \frac{2Sz^*}{1+z^*z} = S\sin\theta\cos\phi,\\ \langle z|S_y|z\rangle &= \frac{2Sz}{1+z^*z} = S\sin\theta\sin\phi,\\ \langle z|S_z|z\rangle &= -S\frac{1-z^*z}{1+z^*z} = S\cos\theta.\end{aligned} \quad (17)$$

Thus the classical Stokes parameters can be recovered. From a pictorial point of view, these represent the point on the Poincarè sphere, the point of which is parametrized by the polar angle (θ,ϕ).

III. GEOMETRIC PHASE ACCOMPANYING THE POLARIZATION COHERENT STATE

In this section, as a forerunner to the next section, we present a general setting of the geometric phase by using the quantum interference arranged by the coherent state path integral.

A. Interferometry arranged by birefringent Hamiltonian

We first consider a way to detect the phase using the "light beam" carrying the polarization coherent state. We have in mind the interferometer that consists of the photon source and the transmitter of the photon. The transmitter plays the role of changing the incident photon state to a variety of states through the *unitary transformation* that is controlled by the mechanism built in the interferometer.

Let us suppose an apparatus consisting of two "paths," which is a schematic experimental setup similar to the Aharonov-Bohm (AB) type device [28,29] (see Fig. 1). That is, an incident beam is injected and it is split into two beams traveling along two paths: the one is AMB and the other is ATB. Point B is a region of interference. Here the carrier wave (plane-wave component) is discarded and only the polarization state is considered, which is sufficient for the present purpose. At the initial junction point (point A in Fig. 1), the polarization state is set to be $|z\rangle$. Just after A, this is split into two states,

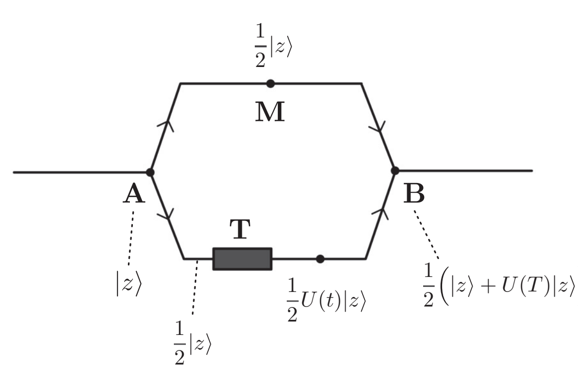

FIG. 1. Schematic setup of AB-type interferometry. At point A $|z\rangle$ is split into two states $1/2|z\rangle$ and $1/2|z\rangle$. Going through the transmitter and arriving at B after the time T, the two states are recombined to give $|\psi\rangle = 1/2(|z\rangle + U(T)|z\rangle)$, which results in the interference.

$\frac{1}{2}|z\rangle$ and $\frac{1}{2}|z\rangle$. For the state of the one beam in which there is no transmitter, the state $\frac{1}{2}|z\rangle$ arrives at junction B, keeping the state in the same state as $\frac{1}{2}|z\rangle$. On the other hand, the state in the other beam, which goes through the transmitter devised as "time dependent," a *pseudomagnetic* field is built, and it is converted to $\frac{1}{2}U(t)|z\rangle$ after the time interval t. Here $U(t)$ means the unitary operator which governs the evolutional behavior of the polarization state. Thus, after time T, the two beams are combined at the final point (point B in Fig. 1). Here we have the interference given by the superposition following the procedure of the AB type [29]:

$$|\psi\rangle = \tfrac{1}{2}[|z\rangle + U(T)|z\rangle]. \tag{18}$$

Then the interference effect can be observed by the overlap $\langle\psi|\psi\rangle$, which turns out to be

$$\langle\psi|\psi\rangle = \tfrac{1}{4}(2 + \langle z|U(T)|z\rangle + \langle z|U^\dagger(T)|z\rangle). \tag{19}$$

Putting $\langle z|U^\dagger(T)|z\rangle) = \exp[i\Phi(C)]$, we have the relation

$$\langle\psi|\psi\rangle = \frac{1}{2}[1 + \cos\Phi(C)] = \cos^2\frac{\Phi(C)}{2}, \tag{20}$$

which results in the interference pattern. The details of the meaning of Eqs. (19) and (20) are given here. First, the interference at junction B in Fig. 1 may be regarded as selecting the state $|z\rangle$ from the transmitted state $U(T)|z\rangle$, that is, the reduction of $U(T)|z\rangle$ to the state $|z\rangle$. Second, the phase $\Phi(C)$ incorporates the *history* of the polarization during passage through the transmitter, which is given by a closed path C starting from z and ending at z in the parameter space for designating the coherent state (this may have a similarity with constructing a "polygon" on the Poincarè sphere, see Ref. [29]). In the next subsection we give a general expression of $\Phi(C)$ in terms of path integral. In Sec. IV the concrete analysis will be discussed by adopting the special form of the Hamiltonian.

Hamiltonian connected with linear birefringence. The unitary transformation is governed by the Hamiltonian generator, which is constructed by using characteristics of the optical device controlled by an externally driven electric field (as for similar sorts of optical devices, see, e.g., [30,31]). The medium we are here concerned with is the dielectrics, which reflects an anisotropic structure giving rise to birefringence. The classical counterpart of the birefringent effect has been given in Ref. [32].

The Hamiltonian is given by the standard form of $H = \int(\mathbf{E}^\dagger\hat{\epsilon}\mathbf{E} + \mu\mathbf{B}^\dagger\mathbf{B})dx$($\hat{\epsilon}$ and μ_0 mean the dielectric tensor and magnetic susceptibility, respectively, and μ is assumed to be scalar and equal to the value in the vacuum state). Then the Hamiltonian is written in terms of the field operators: $H = H_0 + \hat{H}$. The first term is the Hamiltonian for the photon in the vacuum state, that is, $H_0 = \hbar\omega(a^\dagger a + b^\dagger b)$, so this is not a concern here. The remaining term represents the effect of the linear birefringence, which consists of the two main terms of the Faraday and the Kerr effects. The details will be given in Appendix A, and the final result is written in the form of a pseudomagnetic field in the form

$$G = \begin{pmatrix} \kappa & \alpha + i\beta \\ \alpha - i\beta & -\kappa \end{pmatrix}, \tag{21}$$

where the diagonal term represents the Faraday effect and the off-diagonal term comes from the external Kerr effect. The corresponding Hamiltonian becomes

$$\begin{aligned}\hat{H} &= \frac{\hbar\omega}{2}(a^\dagger,b^\dagger)G\begin{pmatrix} a \\ b\end{pmatrix} \\ &= \gamma(a^\dagger a - b^\dagger b) + Ka^\dagger b + K^* b^\dagger a,\end{aligned} \tag{22}$$

with $\gamma = \frac{\hbar\omega\kappa}{2}$ and $K = \frac{\hbar\omega}{2}(\alpha + i\beta)$. (The explicit form of the parameters is given in Appendix A). We note that these parameters have time dependence in general, and the special feature of the time dependence will play a crucial role in investigating the geometric phase, as will be shown in the next section.

B. Geometric phase devised by the path integral

The overlap function $\langle z|U(T)|z\rangle \equiv K(zT;z0)$ in Eq. (19) represents the *quantum cyclic change* along a closed path from z to z, as was discussed in the previous subsection. Having this in mind, let us write the propagator:

$$K(zT;z0) = \langle z|P\left\{\exp\left[-\frac{i}{\hbar}\int_0^T \hat{H}(t)\,dt\right]\right\}|z\rangle, \tag{23}$$

where P means the time-ordered product. (The time-ordered product is denoted by using the symbol P, instead of T, in order to be discriminated from the time interval T.) Equation (23) represents the probability amplitude of *coincidence*, the amplitude for a cyclic change that the system starts with the state $|z_0\rangle$ and returns to the same state after a time interval T. This implies that the polarization state proceeds along closed paths in the Hilbert space spanned by the set of coherent state. The polarization state changes successively from state to state, and by inserting the relation of partition of unity (16) at each infinitesimal time interval the transition amplitude is written as a product integral [27]:

$$K = \int\prod_{k=1}^{\infty}\langle z_{k+1}|z_k\rangle\exp\left[-\frac{i}{\hbar}\int_C\langle z|\hat{H}|z\rangle dt\right]D\mu(z), \tag{24}$$

with $\mathcal{D}\mu(z) \equiv \prod_{t=0}^{T} d\mu[z(t)]$. The overlap of polarization coherent states between an infinitesimal time interval is given by

$$\langle z_{k+1}|z_k\rangle = \frac{(1 + z_{k+1}^* z_k)^{2S}}{\{(1 + z_{k+1}^* z_{k+1})(1 + z_k^* z_k)\}^S}, \tag{25}$$

and the infinite product in (24) represents the finite connection along the closed loop in the complex parameter space, in which each infinitesimal factor represents the connection between two infinitesimally separated points. If use is made of the approximation $\langle z_{k-1}|z_k\rangle \simeq \exp[i\langle z|\frac{\partial}{\partial t}|z\rangle dt]$, (24) is written as the functional integral over all closed paths,

$$K = \int\exp\left[\frac{i}{\hbar}\Phi(C)\right]D\mu(z), \tag{26}$$

where Φ is the action functional:

$$\begin{aligned}\Phi(C) &= \int_0^T\langle z|i\hbar\frac{\partial}{\partial t} - \hat{H}(t)|z\rangle dt \\ &\equiv \Gamma(C) - \Delta(C).\end{aligned} \tag{27}$$

The first term Γ in (27) represents the geometric phase, whereas the second term is the Hamiltonian (dynamical) term, $\langle z|\hat{H}|z\rangle \equiv H(z,z^*,t)$. If one uses the norm $F(z,z^*) = \langle\tilde{z}|\tilde{z}\rangle$, (note that $|\tilde{z}\rangle$ is the un-normalized coherent state), Γ is cast into the form

$$\Gamma(C) = \oint \left(\frac{\partial \ln F}{\partial z} dz - \frac{\partial \ln F}{\partial z^*} dz^* \right). \tag{28}$$

In order to calculate the explicit form of Γ, it is crucial to single out a specific cyclic path $C[z(t)]$ from the path integral. This may be realized by considering the semiclassical limit of (27), the stationary phase condition $\delta\Phi = 0$ yielding the equations of motion for z [33]. The semiclassical limit corresponds to the adiabatic approximation for the adiabatic (Berry's) phase [21]. In other words, the Planck constant plays a role of substitute of the parameter that measures the *adiabaticity*. The propagator is thus reduced to a simple form,

$$K_{sc} = \exp\left[\frac{i\Gamma(C)}{\hbar}\right] \exp\left[-\frac{i\Delta}{\hbar}\right]. \tag{29}$$

Namely, if there exist closed paths, the propagator can be expressed as the overlap between two polarization states: $K_{sc} = \langle z(T)|z(0)\rangle$, the end point of which coincides with z_0 at the time T. In this way the final state may accumulate the history which the system evolves.

As the above formula suggests, the total phase (action integral) consists of two terms: $\Phi = \Gamma - \Delta$. In particular, our concern is how to to detect the geometric phase Γ; in other words, the geometric phase should be discriminated from the total phase. This may be realized by using a characteristic property of a *cyclic motion*. However, it is not easy to find out the cyclic path that is suitable for evaluating the phase $\Gamma(C)$. So we have to restrict the argument to a special case that enables us to extract the cyclic path in a simple way, which will be given in the next section.

IV. EXTRACTION OF THE GEOMETRIC PHASE

We now come to the central part of the present paper, namely, we consider how to manage the geometric phase using the specific feature of the interferometry.

A. The geometric phase associated with the resonant Hamiltonian

We are particularly concerned with the Hamiltonian in such a form that it enables us to extract the information of geometric phase in a concise way. This feature can be achieved by considering the Kerr effect that is arranged by the external electric field exhibiting sinusoidal oscillation:

$$E_x(e) = E_0 \cos\omega_c t, \quad E_y(e) = E_0 \sin\omega_c t. \tag{30}$$

Hence, noting that $\alpha = \frac{k}{2}E_0^2 \cos 2\omega_c t, \beta = \frac{k}{2}E_0^2 \sin 2\omega_c t$ we have the Hamiltonian in a form which is called the "resonant Hamiltonian,"

$$\hat{H} = \gamma S_z + \eta(S_+ \exp[2i\omega_c t] + S_- \exp[-2i\omega_c t]) \equiv \mathbf{G}(t)\cdot\mathbf{S}, \tag{31}$$

with the "sinusoidal oscillating field"

$$\mathbf{G}(t) = (\eta\cos 2\omega_c t, \eta \sin 2\omega_c t, \gamma). \tag{32}$$

Here we use the notation

$$\eta = \frac{\hbar\omega}{2} d, \tag{33}$$

with $d \equiv \frac{k}{2}E_0^2$. It is notable that the Kerr effect is usually much smaller than the Faraday effect, namely, $\eta << \gamma$ holds.

Special orbit leading to the geometric phase. We now examine the geometric phase that is derived from the resonant Hamiltonian for the pseudospin. Using the formula (17), the expectation value of the Hamiltonian $H(t) = H(z,z^*,t)$ is given by

$$H(\theta,\phi) = S[\eta \sin\theta \cos(\phi - ft) - \gamma\cos\theta], \tag{34}$$

where we put $f = 2\omega_c$. The equation of motion is thus written in terms of the angular variable [33]:

$$\begin{aligned} \dot{\phi} &= \frac{1}{S\hbar \sin\theta} \frac{\partial H}{\partial \theta} = \frac{1}{\hbar}[\eta \cot\theta \cos(\phi - ft) + \gamma], \\ \dot{\theta} &= -\frac{1}{S\hbar\sin\theta}\frac{\partial H}{\partial\phi} = -\frac{\eta}{\hbar}\sin(\phi - ft), \end{aligned} \tag{35}$$

where use is made of the kernel function $F(z,z^*)$ [Eq. (15)], together with the stereographic projection $z = \cot\frac{\theta}{2}\exp[-i\phi]$. We see that this form of equations of motion allows a special solution

$$\phi = ft, \quad \theta = \theta_0 (=\text{const}), \tag{36}$$

which just corresponds to the solution in the "rotating frame" in analogy with the NMR (see Appendix B), where the following relation should hold among the parameters θ_0,γ,η:

$$\cot\theta_0 = \frac{\hbar f - \gamma}{\eta} \equiv \frac{2f - \kappa\omega}{d\omega}. \tag{37}$$

The set of parameters (γ,η,f) satisfying (37) for a fixed value $\theta = \theta_0$ belongs to a family of solutions. Indeed, this set of parameters forms a surface in the parameter space (γ,η,f), which we call the "invariant surface" hereafter and characterizes the resonance condition. Equation (36) gives a definite cyclic trajectory (θ,ϕ) with the period $2\pi/f$ on the parameter space (Poincaré sphere).

Following the general formula given in the previous section, (20) and (27), Γ and Δ are evaluated as follows, namely, for a closed loop with the period, $T = \frac{2\pi}{f}$:

$$\Gamma(C) = \oint S\hbar(1-\cos\theta)d\phi = 2S\pi\hbar(1-\cos\theta_0), \tag{38}$$

which is nothing but the solid angle subtended by curve C at the origin of S^2 as the space (Fig. 2). On the other hand, the dynamical (Hamiltonian) phase Δ is given by

$$\Delta(C) = \frac{2S}{f}\pi(\eta\sin\theta_0 - \gamma\cos\theta_0). \tag{39}$$

The crucial point is that the phase Γ depends only on θ_0. Therefore any point lying on the "invariant surface" gives the same Γ. This fact plays a crucial role for extracting the geometric part Γ from the total phase, as will be explained below. On the other hand, the dynamical phase is not determined solely by θ_0.

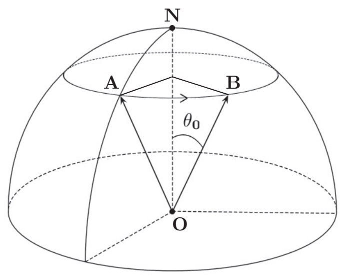

FIG. 2. The orbit on the Poincarè sphere, a circle traced by an arrow with the latitude θ_0. The endpoints, A and B, of two arrows represent the coordinates z and z'. When it holds that $z = z'$, the orbit is a closed orbit, resulting in the geometric phase that gives rise to interference.

B. Arrangement to detect the geometric phase

We now examine the way to extract the geometric phase in a pure manner, namely, we arrange the way such that the dynamical phase Δ vanishes ($\Delta = 0$). Then we have the relation $\cot\theta_0 = \frac{\eta}{\gamma}$, which turns out to be the relation between f and ω: $\cot\theta_0 = \frac{d}{\kappa}$. By combining this with the relation of "invariant surface" (37), we get

$$\hbar f = \frac{\eta^2 + \gamma^2}{\gamma}. \tag{40}$$

This relation is a constraint to be imposed for the parameters which are controlled from the external conditions. The geometric phase Γ thus becomes

$$\Gamma(C) = 2S\pi\left(1 - \frac{\eta}{\sqrt{\eta^2+\gamma^2}}\right). \tag{41}$$

The phase factor obtained by exponentiation turns out to be $\exp[i\Gamma(C)] = \pm\exp[\tilde{\Gamma}(C)]$, with

$$\tilde{\Gamma} \equiv -\frac{2S\pi\eta}{\sqrt{\eta^2+\gamma^2}}. \tag{42}$$

Then, going back to the starting formula (20), the interference pattern turns out to be

$$\langle\psi|\psi\rangle = \frac{1}{2}\left[1 \pm \cos\tilde{\Gamma}(C)\right] = \begin{cases}\cos^2\frac{\tilde{\Gamma}(C)}{2}\\ \sin^2\frac{\tilde{\Gamma}(C)}{2},\end{cases} \tag{43}$$

according to the $\pm$ sign. The sign depends on whether $2S$ is even or odd, respectively. This feature may be called the "even-odd" effect, which could be discriminated experimentally in principle.

The form of this reduced phase (42) is significant, because it indicates that the photon number $N = 2S$ appears as a multiplicative factor which plays the role of an enhancement factor for the tiny value of the parameter η. Thus if we arrange the number of photons such that it is extremely large, we expect that the magnitude of Γ is enhanced.

Now, following the schematic AB-type device discussed in Sec. III A, we can propose a way to reveal the interference that is performed at junction B (Fig. 1). The point to be noted here is that the device is arranged such that the interference occurs only for the case that the pseudospin figures a closed circle on the Poincarè sphere starting from point z and ending at the same point z, leading to the phase $\Gamma(C)$ (see Fig. 2). If there does not form a closed loop, we do not have interference. Furthermore, the basic period of the closed path is $T(=\frac{2\pi}{f})$, which is the period of pseudomagnetic field. Actually, besides the basic period, we have a multiple period $nT(n = 1,\ldots)$, resulting in the geometric phase $n\tilde{\Gamma}(C)$, which leads to a sequence of the interference pattern $\frac{1}{2}[1 \pm \cos n\tilde{\Gamma}(C)](n = 1,2,\ldots)$.

The interference pattern (43) may be rewritten in a refined form by noting the constraint (40) that holds among (η,γ,f). Let us consider a specific case that the frequency f is fixed. Then we have only one free parameter instead of (η,γ), and if we choose the angular parametrization such that

$$\eta = \frac{\hbar f}{2}\sin x, \quad \gamma = \frac{\hbar f}{2}(1 + \cos x),$$

then we have an alternative form of the interference pattern,

$$\langle\psi|\psi\rangle = \frac{1}{2}\left\{1 \pm \cos\left(2S\pi\cos\frac{x}{2}\right)\right\}. \tag{44}$$

With this equation, the interference pattern is calibrated by the parameter x. In this way, the problem is reduced to the modulation of the parameters that are built in the Faraday and (external) Kerr effects. The former is induced by a uniform magnetic field, whereas the latter comes from an externally driven sinusoidal electric field. This feature may form the basis of designing an experimental means to detect the geometric phase.

Remarks (i). Here we examine the case that the resonance condition is satisfied, namely, $\gamma = \hbar f$, which leads to $\cos\theta_0 = 0$; hence it follows that $\exp[\frac{i}{\hbar}\Gamma] = \exp[2S\pi i] = \pm 1$. From this we see that the interference pattern is given as

$$\frac{1}{2}(1 \pm \cos\Delta) = \begin{cases}\cos^2\frac{\Delta(C)}{2}\\ \sin^2\frac{\Delta(C)}{2},\end{cases}$$

with $\Delta = \frac{2S\pi\eta}{f}$. In this way, the resonance condition does not bring about any explicit effect of the geometric phase on the interference pattern.

Remarks (ii). The whole scheme of extracting the geometric phase developed in the above is based on the SU(2) coherent state starting from the state with fixed photon number.

Actually, we have to check the effect of fluctuation (uncertainty) on the geometric phase that is inherent in the deviation from the fixed number constraint. Let $|\psi'\rangle$ be the small deviation of order ρ from the (un-normalized) coherent state with the fixed number of photon; $|\tilde{z}\rangle$; $|\tilde{\psi}\rangle = |\tilde{z}\rangle + |\psi'\rangle$. $|\psi'\rangle$ may cause the *renormalized* values of the parameters appearing in the geometric phase. In particular, we try to find a renormalization for the spin magnitude S. A *first aid* procedure of the renormalization may be given by the following *ansatz* for the norm:

$$\langle\tilde{\psi}|\tilde{\psi}\rangle = (1 + zz^*)^{2S'}, \tag{45}$$

where $S' = S + \epsilon$, with ϵ being small of the same order as ρ. In what follows, it is sufficient to restrict our argument to the state accompanying the classical orbit (36) for which $z = z_0 \equiv \tan(\theta_0/2)\exp[-i\phi]$. Thus ϵ is estimated to be [34]

$$\epsilon = \frac{\langle z_0|\psi'\rangle + \text{c.c.}}{|z_0|^2(1+|z_0|^2)^{2S}}.$$

The factor $f(z_0) = \langle z_0|\psi'\rangle$ can be expanded in power series of z_0:

$$f(z_0) = \rho(C_0 + C_1 z_0 + \cdots + C_2 z_0^2 + \cdots).$$

Hence ρ has the dependence on ϕ. This dependence is eliminated by integrating with respect to ϕ to average out, so we get

$$\bar{\epsilon} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\langle z_0|\psi'\rangle + \text{c.c.}}{|z_0|^2(1+|z_0|^2)^{2S}} d\phi = \frac{\rho C_0}{|z_0|^2(1+|z_0|^2)^{2S}}.$$

Thus we have obtained a crude estimation for the shift of S, which leads to the fluctuation in the geometric phase.

V. SUMMARY

The geometric phase has been investigated for quantum polarized light described by the SU(2) coherent state. This study may enable us to explore the *geometric structure* inherent in the quantum polarized light [35].

The main consequence is that the geometric phase can be discriminated from the dynamical phase by using the specific feature of the resonant Hamiltonian. The most appealing point of the present theory is that the geometric phase is controlled by the external conditions [36]. This is due to the structure of the resonant Hamiltonian, as has been suggested in the previous section. We have also examined the fluctuation effect caused by the uncertainty of photon numbers, which is indeed required actually, because one has to check the accuracy in case of experimental tests of geometric phase.

The geometric phase is regarded as a manifestation of the "holonomy" in quantum mechanics [37]. This implies that the geometric phase is relevant only for the case of a nonstationary problem, namely, the time-dependent Hamiltonian, a sharp contrast to the time-independent Hamiltonian. For the latter case, we note a conventional aspect of the eigenvalue problem for a pseudospin system [38]. For example, the eigenvalue problem, which is relevant to the present subject, might relate to a problem of quasistationary state in birefringent materials, specifically a case that reveals the nonlinear birefringence. The details of this will be investigated elsewhere.

ACKNOWLEDGMENT

The author thanks Dr. Masao Matsumoto for useful discussions and help in preparing the initial draft of this manuscript. The present work was carried out under the auspices of a Grant-in-Aid for Scientific Research from the Ministry of Science and Education (No. 24656057).

APPENDIX A: THE FORM OF THE LINEAR BIREFRINGENCE

In what follows we give a sketch for the birefringence that leads to the medium in the presence of the external electromagnetic field, which is used in the Sec. III A.

a. Birefringence caused by the Faraday effect. It is known that an external magnetic field gives rise to optical activity in materials. We start with the relation between the quantum electric and dielectric field in the linear polarization basis [2]:

$$\mathbf{D} = \hat{\varepsilon}_0\mathbf{E} + i\mathbf{E}\times\kappa = \hat{\epsilon}\mathbf{E}. \tag{A1}$$

Here the dielectric tensor is expressed as

$$\hat{\epsilon} = \begin{pmatrix} \epsilon_0 & i\kappa \\ -i\kappa & \epsilon_0 \end{pmatrix} = \epsilon_0\mathbf{1} + i\sigma_y. \tag{A2}$$

This form is based on the symmetry law of kinetic coefficients [2]; that is, the dielectric tensor under a magnetic field should be Hemitian. The first term in (A1) is a contribution from the vacuum that is absorbed in the H_0, and so is omitted here. The second term is dominant, where the vector $\kappa = (\kappa_1,\kappa_2,\kappa_3)$ is known as the gyration vector, which indicates an axial-vector nature of the Faraday effect. κ is expressed as $\kappa = v\mathbf{H}$ for a transparent medium in an external magnetic field $\mathbf{H}$. Here v is known as the Verde constant. The second term of (A2) is written in terms of the circular polarization basis $\mathbf{e}_\pm$:

$$\hat{\epsilon} = \kappa\sigma_z = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}. \tag{A3}$$

The corresponding *pseudomagnetic field*, i.e., $\mathbf{G}$, is written as $\mathbf{G}_{\text{ax}} = (0,0,\kappa)$.

b. Birefringence caused by the external Kerr effect. This case is considered to be of *nonaxial* nature. The dielectric tensor is expressed in terms of the external electric field $\epsilon_{ij} = kE(e)_i E(e)_j$ (see, e.g., Ref. [2]), where i and j take the x and y, and k is proportional to the Kerr constant. Hence this can be written in the linear polarization basis as $\hat{\epsilon} = (\text{scalarmatrix}) + \bar{\epsilon}$, with

$$\begin{cases} \bar{\epsilon}_{11} = \frac{1}{2}k\big(E(e)_x^2 - E(e)_y^2\big) = \alpha, \\ \bar{\epsilon}_{22} = -\frac{1}{2}k\big(E(e)_x^2 - E(e)_y^2\big) = -\alpha, \\ \bar{\epsilon}_{12} = kE(e)_x E(e)_y = \beta. \end{cases} \tag{A4}$$

The components $E(e)_x$, $E(e)_y$ represent the two components perpendicular to the propagation direction. [$E(e)$ stands for the externally applied field.] Note that the scalar (isotropic) term is also discarded as in the case of the Faraday effect, because this term does not contribute to the birefringence. If writing the above tensor in terms of the circular basis, we have the pseudomagnetic field which is written as $G_{na} = (\alpha,\beta,0)$.

APPENDIX B: ANALOGY WITH NMR

We here derive the the rotating solutions in analogy with the procedure adopted in NMR [39]. Let us consider the unitary transformation to the frame rotating around the z axis with the constant angular velocity f, namely, $R(t) = \exp[-\frac{i}{\hbar}ftS_z]$. With this transformation we have the relation $\mathbf{S} = R^\dagger\mathbf{S}'R$, where $\mathbf{S}$ and $\mathbf{S}'$ are the polarization spin operators for the rest and rotating frame. Substituting the Heisenberg equation of motion in the rest frame, it follows that

$$i\hbar\frac{d\mathbf{S}}{dt} = i\hbar R^\dagger f(S_z\mathbf{S}' - \mathbf{S}'S_z)R + R^\dagger i\hbar\frac{d\mathbf{S}'}{dt}R, \tag{B1}$$

and noting that $[\mathbf{S},\hat{H}] = R^\dagger[\mathbf{S}',\hat{H}]R$, we obtain the equation of motion in the rotating frame,

$$i\hbar\frac{d\mathbf{S}'}{dt} = [\mathbf{S}',\tilde{H}], \tag{B2}$$

with $\tilde{H} = H' - fS_z = \eta S_x + (\gamma - \hbar f)S_z$, where use is made of the relation $R^\dagger\hat{H}R = H' = \eta S_x + \gamma S_z$. The above equation of motion turns out to be

$$\frac{d\mathbf{S}'}{dt} = \mathbf{S}' \times \mathbf{B}' \tag{B3}$$

with $\mathbf{B}' = (\eta,0,\hbar f - \gamma)$. This is written in the components,

$$\frac{dS_x}{dt} = \tilde{\gamma}S_y, \quad \frac{dS_y}{dt} = \eta\gamma S_z - \tilde{\gamma}S_x$$

and $\frac{dS_z}{dt} = -\eta S_y$, with $\tilde{\gamma} = \gamma - \hbar f$. This can be treated in the semiclassical manner. We immediately have the constant of motion $S_z'' = \eta S_x + \tilde{\gamma}S_z = C$. Using this, the equation of motion for S_y is obtained, $d^2S_y/dt^2 = -(\tilde{\gamma}^2 + \eta^2)S_y$, leading to $S_y = S\sin(\Omega t)$ with $\Omega = \sqrt{\tilde{\gamma}^2 + \eta^2}$ and hence $S_z = -\frac{\eta S}{\Omega}\cos(\Omega t), S_x = \frac{\tilde{\gamma}S}{\Omega}\cos(\Omega t)$, which represents the precession around the S_z'' axis introduced in the above. As a special case, we consider the stationary solution, $\frac{d\mathbf{S}'}{dt} = 0$, which leads to $\mathbf{S}' = \mathbf{S}(0) = (S\sin\theta_0,0,-S\cos\theta_0)$. Noting that $\mathbf{S}'//\mathbf{B}'$, we have $\cot\theta_0 = (\frac{\hbar f - \gamma}{\eta})$, which is nothing but the relation (37).

[1] P. Jordan, Eur. Phys. J. A **44**, 292 (1927); see also, J. Oppenheimer, Phys. Rev. **38**, 725 (1931).

[2] L. D. Landau and I. Lifschitz, *Electrodynamics in Continuous Media, Course of Theoretical Physics*, 2nd ed. (Butterworth-Heinemann, Oxford, 1983), Vol. 8.

[3] M. Born and E. Wolf, *Principles of Optics* (Pergamon, Oxford, 1975).

[4] K. L. Sala, Phys. Rev. A **29**, 1944 (1984).

[5] M. V. Tratnik and J. E. Sipe, Phys. Rev. A **35**, 2965 (1987).

[6] C. Brosseau, *Statistical Theory of Light Polarization* (Academic Press, New York, 1998).

[7] H. Kuratsuji and S. Kakigi, Phys. Rev. Lett. **80**, 1888 (1998).

[8] L. D. Landau and I. M. Lifschitz, *Quantum Mechanics: Non-Relativistic Theory*,3rd ed. (Pergamon, Oxford, 1977), Vol. 3.

[9] See, e.g., R. Glauber, Phys. Rev. **130**, 2529 (1963).

[10] V. P. Karasev, J. Sov. Laser Res. **12**, 431 (1991).

[11] V. P. Karassiov and A. V. Masalov, Laser Physics **12**, 948 (2002).

[12] H. Prakash and N. Chandra, Phys. Rev. A **4**, 796 (1971).

[13] G. S. Agarwal, Lett. Nuovo Cimento **1**, 53 (1971); G. S. Agarwal, J. Lehler, and H. Paul, Opt. Commun. **129**, 369 (1996); J. Lehler, H. Paul, and G. S. Agarwal, *ibid.* **139**, 262 (1997).

[14] A. Sehat, J. Soderholm, G. Bjork, P. Espinoza, A. B. Klimov, and L. L. Sanchez-Soto, Phys. Rev. A **71**, 033818 (2005).

[15] A. B. Klimov *et al.*, Phys. Rev. Lett. **105**, 153602 (2010).

[16] T. Tsegaye, J. Soderholm, M. Atature, A. Trifonov, G. Bjork, A. V. Sergienko, B. E. A. Saleh, and M. C. Teich, Phys. Rev. Lett. **85**, 5013 (2000).

[17] *Geometric Phases in Physics*, edited by A. Shapere and F. Wilzcek (World Scientific, Singapore, 1989).

[18] J. M. Radcliffe, J. Phys. A **4**, 313 (1971).

[19] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972).

[20] A. M. Perelomov, *Generalized Coherent States* (Springer Verlag, Berlin, 1986).

[21] H. Kuratsuji, Phys. Rev. Lett. **61**, 1687 (1988) , and references cited therein.

[22] A primitive idea was suggested in our previous paper without reference to actual physical situations; H. Kuratsuji and M. Matsumoto, Phys. Lett. A **155**, 99 (1991).

[23] K. Gottfried, *Quantum Mechanics, Fundamentals* (Benjamin, New York, 1966), Vol. 1.

[24] J. J. Sakurai, *Advanced Quantum Mechanics* (Wiley, New York, 1967).

[25] See, e.g., S. Brattke, B. T. H. Varcoe, and H. Walther, Phys. Rev. Lett. **86**, 3534 (2001), and references cited therein. Actually, the Fock state realized in this paper is restricted to the single photon. However, the Fock state consisting of arbitrary number of photons may be achieved in principle. Our argument is based on such a Fock state of multiple number of photons.

[26] J. J. Sakurai, *Modern Quantum Mechanics* (Addison Wesley, New York, 1994).

[27] H. Kuratsuji and T. Suzuki, J. Math. Phys. **21**, 472 (1980).

[28] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[29] M. V. Berry, J. Mod. Opt. **34**, 1401 (1987).

[30] P. D. Drummond and D. F. Walls, J. Phys. A **13**, 725 (1980).

[31] R. Loudon, *Quantum Theory of Light* (Pergamon, Oxford, 1973).

[32] H. Kuratsuji, R. Botet, and R. Seto, Prog. Theor. Phys. **117**, 195 (2007).

[33] The equation of motion is explicitly written as $i\hbar g_{z\bar{z}}\frac{dz}{dt} = \frac{\partial H}{\partial z^*}, i\hbar g_{z\bar{z}}\frac{dz^*}{dt} = \frac{\partial H}{\partial z}$, where $g_{z\bar{z}} = \frac{\partial^2 \ln F}{\partial z \partial z^*}$, which denotes the metric of the generalized phase space—the so-called Kaehher metric.

[34] Use is made of an approximation, $\langle\psi|\psi\rangle \simeq (1 + zz^*)^{2S} + \langle z|\psi'\rangle$, together with the expansion, $(1 + zz^*)^{2S'} \simeq (1 + zz^*)^{2S} + \epsilon zz^*(1 + zz^*)^{2S}$, up to the order of ϵ^2.

[35] Here we mention the other kind of geometric phase accompanying the polarized light, which is known as Pancharatnam's phase: S. Pancharatnam, Proc. Ind. Acad. Sci. Ser. A **44**, 247 (1956), (see the reprint volume [17]). This object, however, is understood by purely classical wave theory, whereas the present one is genuinely quantum mechanical.

[36] The idea of separation of the geometric phase from the dynamical phase has been suggested in another context, e.g., a design to detect very tiny gravitational effects; E. M. Martinez, I. Fuentes, and R. B. Mann, Phys. Rev. Lett. **107**, 131301 (2011) , and references cited therein.

[37] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983).

[38] H. Kuratsuji and Y. Mizobuchi, Phys. Lett. A **82**, 279 (1981).

[39] I. Rabi, N. F. Ramsey, and J. Schwinger, Rev. Mod. Phys. **26**, 167 (1954).