Collective modes in the anisotropic unitary Fermi gas and the inclusion of a backflow term

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We study the collective modes of the confined unitary Fermi gas under anisotropic harmonic confinement as a function of the number of atoms. We use the equations of extended superfluid hydrodynamics, which take into account a dispersive von Weizsäcker-like term [C. F. von Weizsäcker, Z. Phys. **96**, 431 (1935)] in the equation of state. We also discuss the inclusion of a backflow term in the extended superfluid Lagrangian and the effects of this anomalous term on sound waves and the Beliaev damping of phonons [S. T. Beliaev, Sov. Phys. JETP **7**, 299 (1958)].

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I. INTRODUCTION

In this paper we calculate the collective monopole and quadrupole modes of the unitary Fermi gas (characterized by an infinite s-wave scattering length) under axially symmetric anisotropic harmonic confinement by using the extended Lagrangian density of superfluids, which we proposed a few years ago [1], to study the unitary Fermi gas [1–8]. The internal energy density of our extended Lagrangian density contains a term proportional to the kinetic energy of a uniform noninteracting gas of fermions, plus a gradient correction of the von-Weizsacker form $\lambda\hbar^2/(8m)(\nabla n/n)^2$ [9]. The inclusion of a gradient term has been adopted for studying the quantum hydrodynamics of electrons by March and Tosi [10] and by Zaremba and Tso [11]. In the context of the BCS-BEC crossover, the gradient term is quite standard [12–20]. In the last part of this paper we consider the inclusion of backflow terms [21,22] in the extended superfluid Lagrangian. By using our equations of extended superfluid hydrodynamics with backflow we calculate sound waves, the static response function, and the structure factor of a generic uniform superfluid and also the effect of the backflow on the Beliaev damping of phonons [23].

II. EXTENDED SUPERFLUID LAGRANGIAN AND HYDRODYNAMIC EQUATIONS

The extended Lagrangian density of dilute and ultracold superfluids is given by [1–8]

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_W, \tag{1}$$

where

$$\mathscr{L}_0 = -\hbar\,\dot{\theta}\,n - \frac{\hbar^2}{2m}(\nabla\theta)^2\,n - U(\mathbf{r})\,n - \mathcal{E}_0(n) \tag{2}$$

is the familiar Popov's Lagrangian density [24] of superfluid hydrodynamics, with $n(\mathbf{r},t)$ being the local density and $\theta(\mathbf{r},t)$ half [25] of the phase of the condensate order parameter of Cooper pairs for superfluid fermions (or the phase of the condensate order parameter for superfluid bosons). Here $U(\mathbf{r})$ is the external potential acting on particles and $\mathcal{E}_0(n)$ is the bulk internal energy of the system. The additional term

$$\mathscr{L}_W = -\lambda\frac{\hbar^2}{8m}\frac{(\nabla n)^2}{n} \tag{3}$$

generalizes superfluid hydrodynamics by explicitly taking into account local density gradients' contributions to the local internal energy density, which becomes

$$\mathcal{E}_0(n(\mathbf{r},t),\nabla n) = \mathcal{E}_0(n(\mathbf{r},t)) + \lambda\frac{\hbar^2}{8m}\frac{[\nabla n(\mathbf{r},t)]^2}{n(\mathbf{r},t)}, \tag{4}$$

where, as previously mentioned, $\mathcal{E}_0(n)$ is the internal energy of a uniform unitary Fermi gas with density n. The parameter λ giving the gradient correction must be determined from microscopic calculations or from comparison with experimental data.

By using the Lagrangian density (1) the Euler-Lagrange equation for θ gives

$$\frac{\partial n}{\partial t} + \frac{\hbar}{m}\nabla\cdot(n\;\nabla\theta) = 0, \tag{5}$$

while the Euler-Lagrange equation for n leads to

$$\hbar\,\dot{\theta} + \frac{\hbar^2}{2m}(\nabla\theta)^2 + U(\mathbf{r}) + X(n,\nabla n) = 0, \tag{6}$$

where

$$X(n,\nabla n) = \frac{\partial\mathcal{E}}{\partial n} - \nabla\cdot\frac{\partial\mathcal{E}}{\partial(\nabla n)}, \tag{7}$$

which describes how the internal energy varies as the local density and its gradient vary, may be considered a local chemical potential. The local velocity field $\mathbf{v}(\mathbf{r},t)$ of the superfluid is related to $\theta(\mathbf{r},t)$ by

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t). \tag{8}$$

This definition ensures that the velocity is irrotational, i.e., $\nabla\times\mathbf{v} = \mathbf{0}$. By using definition (8) in both Eqs. (5) and (6) and applying the gradient operator ∇ to Eq. (6) one finds the following extended hydrodynamic equations of superfluids:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\;\mathbf{v}) = 0. \tag{9}$$

$$m\frac{\partial\mathbf{v}}{\partial t} + \nabla\left[\frac{1}{2}m\mathbf{v}^2 + U(\mathbf{r}) + X(n,\nabla n)\right] = \mathbf{0}. \tag{10}$$

Since in equilibrium both $\mathbf{v}(\mathbf{r},t) = \mathbf{0}$ and $\frac{\partial\mathbf{v}(\mathbf{r},t)}{\partial t} = 0$ must hold, from Eq. (6) it follows that in the presence of an external confinement, $U(\mathbf{r})$, the equilibrium (ground state) density $n_0(\mathbf{r})$ obeys:

$$U(\mathbf{r}) + X(n_0,\nabla n_0) = \mu, \tag{11}$$

and the space- and time-independent constant μ may be identified with the chemical potential (in fact it may thought of as the Lagrangian parameter which fixes the total number of fermions). The equilibrium conditions' position-independent phase $\theta_0(\mathbf{r},t) = -\mu t/\hbar$ obviously satisfies the equilibrium conditions on $\mathbf{v}$ and its time derivative.

III. COLLECTIVE MODES OF THE ANISOTROPIC UNITARY FERMI GAS

In the case of the unitary Fermi gas the bulk internal energy can be written as

$$\mathcal{E}_0(n) = \xi \, \frac{3}{5} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \, n^{5/3}, \tag{12}$$

where $\xi \simeq 0.4$ is a universal parameter [1,2,16,26,27] and various approaches [1,2,16,19,26,27] suggest that $\lambda \simeq 0.25$. The local chemical potential is then

$$X(n,\nabla n) = \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \xi \, n^{2/3} - \lambda \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n}}{\sqrt{n}}, \tag{13}$$

with the abovementioned values of ξ and λ.

In this section we consider the unitary Fermi gas under the anisotropic axially symmetric harmonic confinement

$$U(\mathbf{r}) = \frac{m}{2} \omega_\rho^2 (x^2 + y^2) + \frac{m}{2} \omega_z^2 z^2, \tag{14}$$

where ω_ρ is the cylindric radial frequency while ω_z is the axial frequency. In this case, Eq. (11) for the ground-state density profile $n_0(\mathbf{r})$ becomes

$$\frac{m}{2} \omega_\rho^2 (x^2 + y^2) + \frac{m}{2} \omega_z^2 z^2 + \frac{\hbar^2}{2m} (3\pi^2)^{2/3} \xi \, n_0(x,y,z)^{2/3} - \lambda \frac{\hbar^2}{2m} \frac{\nabla^2 \sqrt{n_0(x,y,z)}}{\sqrt{n_0(x,y,z)}} = \mu. \tag{15}$$

We have solved numerically this three-dimensional partial differential equation by using a finite-difference predictor-corrector Crank-Nicholson method [28] with imaginary time after chosing $\xi = 0.42$ and $\lambda = 0.25$. In the case of the isotropic trap ($\omega_\rho/\omega_z = 1$) the fermionic cloud is spherically symmetric and consequently axial and radial density profiles coincide. Instead, as expected, by increasing the trap anisotropy also the fermionic cloud becomes more anisotropic.

We are interested in calculating the frequencies of low-lying collective oscillations of the anisotropic unitary Fermi gas. Exact scaling solutions for the unitary Fermi gas have been considered by Castin [29] and, for the linearized hydrodynamic equations with no gradient quantum pressure term, by Hou, Pitaevskii, and Stringari [30]. Unfortunately, in the presence of anisotropic trapping potential these scaling solutions are no longer exact when one considers the gradient term.

For this reason we solve numerically the extended hydrodynamic equations, Eqs. (9) and (10). In particular, by using our finite-difference predictor-corrector Crank-Nicholson code in real time [28], we integrate the time-dependent nonlinear Schrödinger equation, which is fully equivalent (see [1,5]) to Eqs. (9) and (10).

Figure 1 refers to the unitary Fermi gas under isotropic ($\omega_\rho = \omega_z$) harmonic confinement. In the two panels we plot the monopole frequency Ω_0 (upper panel) and the quadrupole frequency Ω_2 (lower panel) as a function of the number N of atoms. As expected [29], the frequency Ω_0 of the monopole mode does not depend on the number N of particles and it is given by

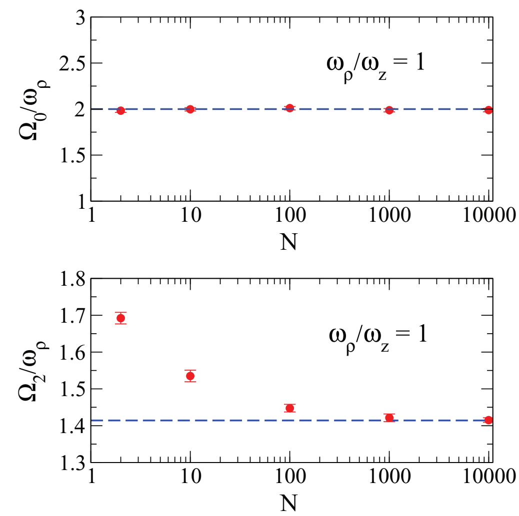

FIG. 1. (Color online) Unitary Fermi gas under isotropic ($\omega_\rho = \omega_z$) harmonic confinement. In the two panels there are the monopole frequency Ω_0 (upper panel) and the quadrupole frequency Ω_2 (lower panel) as a function of the number N of atoms. Solid circles with error bars: numerical results obtained solving Eqs. (9) and (10) with Eq. (13) and $\lambda = 0.25$. Dashed lines: analytical results, i.e., exact Eq. (16) and Thomas-Fermi Eq. (17). Universal parameter of the unitary Fermi gas: $\xi = 0.42$.

$$\Omega_0 = 2\omega_\rho. \tag{16}$$

On the contrary, the figure shows that the frequency Ω_2 of the quadrupole mode depends on N and for large values of N it approaches asymptotically the value $\sqrt{2}\omega_\rho$, appropriate to the case of neglecting the gradient term (see [31]). The solid circles are the results with $\lambda = 0.25$ while the dashed lines show the analytical results [29,31]. Remarkably, for small values of N the gradient term enhances the quadrupole frequency Ω_2. In the isotropic case ($\omega_\rho = \omega_z$) the quadrupole frequency Ω_2 in the limit $N \to \infty$ reduces to the Thomas-Fermi result (i.e., without the gradient term) [31]

$$\Omega = \sqrt{2}\omega_\rho, \tag{17}$$

as expected. On the contrary, in the small N limit it approaches $\Omega = 2\omega_\rho$, which is the quadrupole oscillation frequency of non-interacting atoms (the same result holds for ideal fermions and ideal bosons) [32].

In Figure 2 we consider the unitary Fermi gas under anisotropic but axially symmetric ($\omega_\rho = 2\omega_z$) harmonic confinement. In this case monopole and quadrupole modes are coupled and we have determined numerically the two associated frequencies, $\Omega_{0,2}^{(a)}$ and $\Omega_{0,2}^{(b)}$. Also in this case the gradient term increases the frequencies for small values of N. Moreover, for large values of N these frequencies reduce to

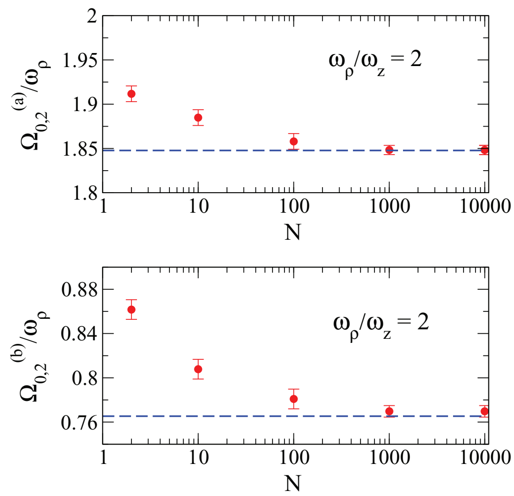

FIG. 2. (Color online) Unitary Fermi gas under anisotropic but axially symmetric ($\omega_\rho = 2\omega_z$) harmonic confinement. In the two panels there are the two frequencies, $\Omega^{(a)}_{0,2}$ and $\Omega^{(b)}_{0,2}$, of the coupled monopole and quadrupole modes as a function of the number N of atoms. Solid circles with error bars: numerical results obtained solving Eqs. (9) and (10) with Eq. (13) and $\lambda = 0.25$. Dashed lines: analytical results, i.e., Thomas-Fermi Eq. (18). Universal parameter of the unitary Fermi gas: $\xi = 0.42$.

the results without the gradient term [31],

$$\Omega^{(a),(b)}_{0,2} = \sqrt{\tfrac{5}{3}\omega_\rho^2 + \tfrac{4}{3}\omega_z^2 \pm \tfrac{1}{3}\sqrt{25\omega_\rho^4 + 16\omega_z^4 - 32\omega_\rho^2\omega_z^2}}, \tag{18}$$

which correspond to the dashed lines. Our calculations show that the frequency Ω_2 of Fig. 1 and the frequencies $\Omega^{(a)}_{0,2}$ and $\Omega^{(b)}_{0,2}$ of Fig. 2 give a clear signature of the presence of the von-Weizsacker gradient term.

We point out that current experiments with ultracold atoms at unitarity can detect deviations from the Thomas-Fermi approximation, as observed some years ago for Bose-Einstein condensates [33].

IV. INCLUSION OF A BACKFLOW TERM

Inspired by the papers of Son and Wingate [26] and Manes and Valle [27] in this section we consider the inclusion of a backflow term in the extended superfluid Lagrangian. This backflow term depends on the velocity strain, as suggested for superfluid ^{4}He many years ago by Thouless [21] and explicitly included in the density functional Lagrangian for superfluid ^{4}He by Dalfovo and collaborators [22]. In particular, we consider the Lagrangian density

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_W + \mathscr{L}_B, \tag{19}$$

where $\mathscr{L}_0$ and $\mathscr{L}_W$ are given by Eqs. (2) and (3), respectively, and the backflow term $\mathscr{L}_B$ reads

$$\mathscr{L}_B = -\frac{\hbar^2}{m} n^{1/3}[\gamma_1(\nabla^2\theta)^2 + \gamma_2(\partial_i\partial_j\theta)^2]. \tag{20}$$

Notice that $i,j = x,y,z$ and summations over repeated indices are implied. Again, for a generic superfluid the parameters γ_1 and γ_2 of the backflow term must be determined from microscopic calculations or from comparison with experimental data.

The Lagrangian density (19) depends on the dynamical variables $\theta(\mathbf{r},t)$ and $n(\mathbf{r},t)$. The conjugate momenta of these dynamical variables are then given by

$$\pi_\theta = \frac{\partial\mathscr{L}}{\partial\dot\theta} = -\hbar n, \tag{21}$$

$$\pi_n = \frac{\partial\mathscr{L}}{\partial\dot n} = 0, \tag{22}$$

and the corresponding Hamiltonian density reads

$$\mathscr{H} = \pi_\theta\dot\theta + \pi_n\dot n - \mathscr{L} = -\hbar n\dot\theta - \mathscr{L}, \tag{23}$$

namely,

$$\mathscr{H} = \frac{\hbar^2}{2m}(\nabla\theta)^2 n + U(\mathbf{r})n + \mathcal{E}_0(n) + \lambda\frac{\hbar^2}{8m}\frac{(\nabla n)^2}{n} + \frac{\hbar^2}{m}n^{1/3}[\gamma_1(\nabla^2\theta)^2 + \gamma_2(\partial_i\partial_j\theta)^2], \tag{24}$$

where the last term $(\hbar^2/m)n^{1/3}[\gamma_1(\nabla^2\theta)^2 + \gamma_2(\partial_i\partial_j\theta)^2] = mn^{1/3}[\gamma_1(\nabla\cdot\mathbf{v})^2 + \gamma_2(\partial_i v_j)^2]$ is the energy density associated with the backflow.

The Hamiltonian density (24) is nothing else than the energy density recently found by Manes and Valle [27] with a derivative expansion from their effective field theory of the the Goldstone field [26,27]. The effective field theory of Manes and Valle [27] traces back to the old hydrodynamic results of Popov [24] and generalizes the one derived by Son and Wingate [26] for the unitary Fermi gas from general coordinate invariance and conformal invariance. Actually, at next-to-leading order Son and Wingate [26] found an additional term proportional to $\nabla^2 U(\mathbf{r})$, which has been questioned by Manes and Valle [27] and which is absent in our approach. In addition, Manes and Valle [27] have stressed that the conformal invariance displayed by the unitary Fermi gas implies

$$\gamma_2 = -3\gamma_1. \tag{25}$$

Note that a paper of Schakel [34] confirms the results of Manes and Valle.

We are interested in the propagation of sound waves in superfluids. For simplicity we set $U(\mathbf{r}) = 0$ and consider a small fluctuation $\phi(\mathbf{r},t)$ of the phase $\theta(\mathbf{r},t)$ around the stationary phase $\theta_0(t) = -(\mu/\hbar)t$, namely,

$$\phi(\mathbf{r},t) = \theta(\mathbf{r},t) - \theta_0(t), \tag{26}$$

and a small fluctuation $\rho(\mathbf{r},t)$ of the density $n(\mathbf{r},t)$ around the constant and uniform density n_0, namely,

$$\rho(\mathbf{r},t) = n(\mathbf{r},t) - n_0. \tag{27}$$

After noticing that $(\nabla^2\theta)^2$ and $(\partial_i\partial_j\theta)^2$ differ by a total derivative [27] and that their coefficients in the full Lagrangian density (19) are constants, one easily derives the quadratic Lagrangian density $\mathscr{L}^{(2)}$ of the fluctuating fields $\phi(\mathbf{r},t)$ and $\rho(\mathbf{r},t)$ in the following form:

$$\mathscr{L}^{(2)} = -\hbar\dot\phi\rho - \frac{\hbar^2 n_0}{2m}(\nabla\phi)^2 - \frac{mc_s^2}{2n_0}\rho^2 - \lambda\frac{\hbar^2}{8mn_0}(\nabla\rho)^2 - \gamma\frac{\hbar^2 n_0^{1/3}}{m}(\nabla^2\phi)^2, \tag{28}$$

where c_s is the sound velocity of the generic superfluid, given by

$$c_s^2 = \frac{n_0}{m}\frac{\partial^2 \mathcal{E}_0(n)}{\partial n^2}\bigg|_{n_0}, \tag{29}$$

and $\gamma = \gamma_1 + \gamma_2$.

The linearized equations of motion associated with $\mathscr{L}_2$ read

$$\frac{\partial}{\partial t}\rho + n_0 \nabla \cdot \mathbf{v} - 2n_0^{1/3}\gamma\, \nabla^2(\nabla \cdot \mathbf{v}) = 0, \tag{30}$$

$$\frac{\partial}{\partial t}\mathbf{v} + \frac{c_s^2}{n_0}\nabla\rho - \frac{\lambda\hbar^2}{4m^2 n_0}\nabla(\nabla^2\rho) = 0, \tag{31}$$

with $\mathbf{v} = (\hbar/m)\nabla\phi$. They can be arranged in the form of a wave equation:

$$\left[\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2 + \left(\lambda\frac{\hbar^2}{4m^2} + \gamma\frac{2c_s^2}{n_0^{2/3}}\right)\nabla^4 - \lambda\gamma\frac{\hbar^2}{2m^2 n_0^{2/3}}\nabla^6\right] \times \rho(\mathbf{r},t) = 0. \tag{32}$$

which admits monochromatic plane-wave solutions, whose frequency ω and wave vector $\mathbf{q}$ are related by the dispersion relation:

$$\hbar\,\omega(q) = \sqrt{\left(\frac{\hbar^2 q^2}{2m} + \gamma\frac{\hbar^2 q^4}{m n_0^{2/3}}\right)\left(\lambda\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)}. \tag{33}$$

Notice that a negative value of γ implies that the frequency $\omega(q)$ becomes imaginary for $q > n_0^{1/3}/\sqrt{2|\gamma|}$. However, since γ is expected to be very small, the wave vectors where this happens are outside the range of validity of hydrodynamics.

It is instead useful to expand $\omega(q)$ for small values of q (long-wavelength hydrodynamic regime), finding

$$\hbar\omega(q) = c_s\hbar q + \frac{\hbar}{2}\left(\lambda\frac{\hbar^2}{4m^2 c_s} + \gamma\frac{2c_s}{n_0^{2/3}}\right)q^3 + \cdots. \tag{34}$$

The dispersion relation is linear in q only for small values of the wave number q and the coefficient of cubic correction depends on a combination of the gradient and backflow parameters λ and γ. For $\gamma = 0$ one recovers the dispersion relation we proposed some years ago [1], while setting also $\lambda = 0$ one gets the familiar linear dispersion relation $\omega = c_s q$ of phonons. For the unitary Fermi gas one gets

$$c_s^2 = \frac{\hbar^2}{m^2}\frac{\xi}{3}(3\pi^2)^{2/3} n_0^{2/3}. \tag{35}$$

Moreover, we have seen that the backflow parameters are related by the formula (25), which means

$$\gamma = \gamma_1 + \gamma_2 = -2\gamma_1. \tag{36}$$

Consequently, at the cubic order in q Eq. (33) gives

$$\frac{\omega(q)}{c_s k_F} = \frac{q}{k_F} + \Gamma\frac{q^3}{k_F^3}, \tag{37}$$

where $k_F = (3\pi^2 n_0)^{2/3}$ is the Fermi wave number and

$$\Gamma = \frac{3\lambda}{8\xi} - 2(3\pi^2)^{2/3}\gamma_1. \tag{38}$$

Within a mean-field approximation Manes and Valle [27] have found $\gamma_1 \simeq 0.006$, which implies $\gamma \simeq -0.012$ and $\Gamma \simeq 0.12$, using $\xi = 0.4$ and $\lambda = 0.25$. As recently discussed by Mannarelli, Manuel, and Tolos [35], the sign of Γ has a dramatic effect on the possible phonon interaction channels: the three-phonon Beliaev process, i.e., the decay of a phonon into two phonons [23], is allowed only for positive values of Γ. Under this condition ($\Gamma \geqslant 0$) the phonon has a finite lifetime and the frequency $\omega(q)$ possesses an imaginary part $\mathrm{Im}[\omega(q)]$ due to this three-phonon decay [23,36]. In particular, we find

$$\mathrm{Im}[\omega(q)] = -\frac{\hbar q^5}{270\pi m n_0}. \tag{39}$$

This formula of Beliaev damping is easily derived from Beliaev theory [23,36] taking into account Eq. (35).

It is important to point out that the sign of Γ in Eq. (37) was debated also without the backflow term. In 1998 Marini, Pistolesi, and Strinati [37] found $\Gamma > 0$ at unitarity by including Gaussian fluctuations to the mean-field BCS-BEC crossover. In 2005 Combescot, Kagan, and Stringari [38] derived Eq. (37) with a negative Γ at unitarity on the basis of a dynamical BCS model. In 2011 Schakel [34] obtained a positive Γ at unitarity by using a derivative expansion technique, finding exactly the values of Γ predicted by Ref. [37] in the full BCS-BEC crossover.

To conclude this section, we observe that, for a generic many-body system, the dispersion relation can be written as [32]

$$\hbar\omega(q) = \sqrt{\frac{m_1(q)}{m_{-1}(q)}}, \tag{40}$$

where $m_n(q)$ is the n moment of the dynamic structure function $S(q,\omega)$ of the many-body system under investigation, namely [32],

$$m_n(q) = \int_0^\infty d\omega S(q,\omega)(\hbar\omega)^n. \tag{41}$$

In our problem, Eq. (32), it is straightforward to recognize (see also [22]) that

$$m_1(q) = \frac{\hbar^2 q^2}{2m} + \gamma\frac{\hbar^2 q^4}{m n_0^{2/3}} \tag{42}$$

and

$$m_{-1}(q) = \frac{1}{\lambda\frac{\hbar^2 q^2}{2m} + 2mc_s^2}. \tag{43}$$

In general, the static response function $\chi(q)$ is defined as [32]

$$\chi(q) = -2\, m_{-1}(q); \tag{44}$$

in our problem it reads

$$\chi(q) = -\frac{2}{\lambda\frac{\hbar^2 q^2}{2m} + 2mc_s^2}, \tag{45}$$

which satisfies the exact sum rule $\chi(0) = -1/mc_s^2$ [32]. The static structure factor $S(q)$, defined as [32]

$$S(q) = m_0(q) = \int_0^\infty d\omega S(q,\omega), \tag{46}$$

can be approximated by the expression

$$S(q) = \sqrt{m_1(q)\, m_{-1}(q)} = \sqrt{\frac{\frac{\hbar^2 q^2}{2m} + \gamma \frac{\hbar^2 q^4}{m n_0^{2/3}}}{\lambda \frac{\hbar^2 q^2}{2m} + 2 m c_s^2}}, \qquad (47)$$

which gives an upper bound of $S(q)$ [32] and reduces to $S(q) = \hbar q/(2 m c_s)$ for small q.

Finally, we remark that one can also calculate the frequencies Ω of collective oscillations of the unitary Fermi gas under the action of the trapping potential given by Eq. (14) taking into account the backflow. We have verified that in the case of spherically symmetric harmonic confinement ($\omega_\rho = \omega_z$) the monopole mode Ω_0 is not affected by the backflow term, i.e., $\Omega_0 = 2\omega_\rho$. Moreover, for large values of N the contribution due to the backflow becomes negligible, similarly to the von Weizsäcker one.

V. CONCLUSIONS

We have calculated collective modes of the anisotropic unitary Fermi gas by using the equations of extended superfluid hydrodynamics. In particular, we have shown that a gradient correction of the von-Weizsacker form in the hydrodynamic equations strongly affects the frequencies of collective modes of the system under axially symmetric anisotropic harmonic confinement. We have found that, for both monopole and quadrupole modes, this effect becomes negligible only in the regime of a large number of fermions, where one recovers the predictions of superfluid hydrodynamics [31]. In the last part of the paper we have considered the inclusion of a backflow term in the extended hydrodynamics of superfluids.

We believe our results can trigger the interest of experimentalists. Some years ago beyond-Thomas-Fermi effects due to the dispersive gradient term have been observed by measuring the frequencies of collective modes in trapped Bose-Einstein condensates [33]. Moreover, the spectrum of phonon excitations and Beliaev decay have been observed in a quasiuniform Bose-Einstein condensate with Bragg pulses [39]. Performing similar measurements in the unitary Fermi gas can shed light on the role played by gradient and backflow corrections in superfluid hydrodynamics.

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