Møller scattering and Einstein-Podolsky-Rosen spin correlations

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In this paper we present and discuss the relativistic correlation function in a bipartite system of two electrons, originating from the $e^-e^- \longrightarrow e^-e^-$ scattering of a polarized electron beam on an unpolarized target. We also calculate and investigate the probabilities of the definite outcomes of spin-projection measurements performed by two observers. The presented results might help in experimentally verifying whether relativistic quantum theory is able to reproduce the behavior of real quantum systems.

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I. INTRODUCTION

Starting from the pioneering paper by Czachor [1] one can notice a rise of interest in relativistic aspects of the Einstein-Podolsky-Rosen–type (EPR-type) correlations in systems of massive fermions (see, e.g., Refs. [2–13] and references therein). The behavior of the relativistic correlations is in general different than in the nonrelativistic case. Theoretical analysis showed that relativistic correlations for massive particles may be described by a nonmonotonic function of particle momenta. This unexpected behavior was first found in bipartite vector boson systems and spin-$1/2$ fermion systems [14,15] and was reflected in the degree of violation of the Bell-type inequalities, which in some configurations was a nonmonotonic function of momentum, too. Moreover, it has been shown that there exist configurations for which the degree of the inequality violation increases with particle momenta and reaches its maximal value in the ultrarelativistic limit. Let us also stress that local extrema do not appear for bipartite photon systems.

All the results mentioned previously strongly suggest that the existence of local extrema is a characteristic feature of relativistic correlations for massive particles. For these reasons it is important to measure this correlation function experimentally. Such experiments might be treated as a test of nonlocal aspects of relativistic quantum theory. Thus, the question arises of whether the relativistic corrections can be measured. Our purpose is to show that it is possible to verify the unexpected predictions of relativistic quantum theory mentioned above in the nonlocal correlation experiment by using Møller electrons as the EPR pair.

As far as we know, there have been only three correlation experiments to date performed by means of massive relativistic fermions (protons). Their aim was to test Bell-type inequalities. These experiments were the Lamehi-Rachti-Mittig (LRM) experiment [16] performed about thirty years ago at CEN-Saclay and two recent experiments: the first one at the Kernfysisch Versneller Instituut (KVI, Holland) by Hamieh *et al.* [17] and the second one by Sakai *et al.* [18] at the RIKEN Accelerator Research Facility (Japan). In all three experiments the proton-proton spin correlations were measured. The LRM team tested Bell-type inequalities with the use of the low-energy (13.5 MeV) proton beam, which corresponds to a proton velocity $v \sim 0.17c$. On the other hand, in the KVI experiment, the spin correlations of proton pairs in a ${}^1S_0$ intermediate state, obtained from the ${}^{12}\mathrm{C}(d,{}^2\mathrm{He}){}^{12}\mathrm{B}$ nuclear charge-exchange reaction, were measured for protons with a kinetic energy $\sim$86 MeV ($v \sim 0.4c$). Finally, in the RIKEN experiment the proton pair was created in the ${}^1\mathrm{H}(d,{}^2\mathrm{He})n$ charge-exchange reaction with a proton energy $\sim$135 MeV ($v \sim 0.5c$). In all these experiments correlation functions were measured only for some special configurations and the results were in agreement with the nonrelativistic quantum mechanics predictions. From our estimate it follows that in order to observe a difference between predictions of relativistic and nonrelativistic quantum mechanics, the kinetic energy of the EPR particles should be at least of the order of the particles rest mass. The experiments [16–18] did not meet this condition.

A realistic experiment can be performed using polarized electrons undergoing Møller scattering, resulting in a pair of final-state electrons as EPR particles [19]. Such a state is easily prepared in a laboratory. Moreover, the state of electrons after the scattering can be determined experimentally with sufficient precision. Presently, the corresponding experiment is under preparation by the QUEST Collaboration [20]. It will use a polarized electron beam incident on a stationary, unpolarized target and the Mott polarimetry technique for determining spin projections of outgoing electrons.

The aim of the present paper is to calculate the correlation function and the joint probabilities in a bipartite system of two electrons originating from $e^-e^- \longrightarrow e^-e^-$ scattering. To this end we calculate the outgoing density matrix and the correlation function of a pair of Møller electrons for arbitrary polarization and momenta of the scattered electrons. We focus on the case corresponding to the above experimental method and conditions. In particular, we analyze the degree of entanglement in this case.

The paper is organized as follows: In Sec. II we discuss the initial-state preparation procedure and the corresponding density matrix. In Sec. III we recall the spin operator, for which the correlation functions and the probabilities will be calculated. In Sec. IV we give the general formulas for the correlation functions and the corresponding joint probabilities for two electrons produced in Møller scattering, and in Sec. V we discuss the special case of two electrons originating from the Møller scattering of the electron beam from a stationary

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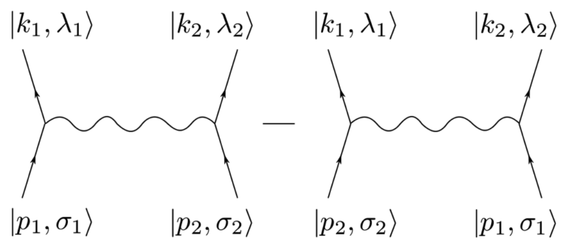

FIG. 1. First-order Feynman graphs illustrating the process of Møller scattering $e^-e^- \longrightarrow e^-e^-$.

target. In this section we discuss also entanglement of the state of two electrons in this case. In Sec. VI we study the behavior of the correlation function and the probabilities for scattering from a stationary target when spin of the electrons can be projected onto an arbitrary direction. In particular, we study the case when the directions on which the spin is projected are perpendicular to the respective electrons momenta (such a measurement can be realized by means of Mott polarimetry).

We use natural units with $\hbar = c = 1$, the Minkowski metric tensor $\eta^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, and adopt the convention $\varepsilon^{0123} = 1$.

II. PREPARATION OF INITIAL STATE

Møller scattering to first order in radiative corrections is illustrated in the Fig. 1. In high-energy physics the states of colliding electrons are prepared separately, therefore the initial state of two electrons (the state before the scattering) has the product form

$$\rho^{\mathrm{in}}_{(\tau_1,\tau_2),(\tau_1',\tau_2')}(q_1,q_2,q_1',q_2') = \rho^{1\mathrm{in}}_{\tau_1\tau_1'}(q_1,q_1')\rho^{2\mathrm{in}}_{\tau_2,\tau_2'}(q_2,q_2'), \quad (1)$$

where τ_i,τ_i', $i = 1,2$ are spin indices which take values $\pm 1/2$ while q_i, q_i' denote four-momenta, which are well determined for the initial electrons. Thus, the matrices $\rho^{i\,\mathrm{in}}$ can be written in the form describing particles with sharp four-momenta, i.e.,

$$\rho^{i\,\mathrm{in}}_{\tau_i\tau_i'}(q_i,q_i') = \frac{2p_i^0}{\delta^3(\mathbf{0})}\delta^3(\mathbf{q}_i - \mathbf{p}_i)\delta^3(\mathbf{q}_i' - \mathbf{p}_i)\frac{1}{2}(\mathbb{1} + \boldsymbol{\xi}_i\cdot\boldsymbol{\sigma})_{\tau_i\tau_i'}, \quad (2)$$

where $\boldsymbol{\xi}_i$ and p_i denotes polarization vector and the four-momentum of the ith electron, respectively; $\boldsymbol{\sigma} = (\sigma_1,\sigma_2,\sigma_3)$ and σ_k are standard Pauli matrices.

The final state after the scattering can be written as

$$\hat\rho^{\mathrm{out}} = \frac{\hat M\hat\rho^{\mathrm{in}}\hat M^\dagger}{\mathrm{Tr}\{\hat M\hat\rho^{\mathrm{in}}\hat M^\dagger\}}, \quad (3)$$

where M is the scattering amplitude. The denominator $\mathrm{Tr}\{\hat M\hat\rho^{\mathrm{in}}\hat M^\dagger\}$ is proportional to the cross section of the $e^-e^- \longrightarrow e^-e^-$ process. Let us stress that transition from the separable state (2) to the outgoing state (3) is not a unitary operation (in spite of unitarity of the S matrix). Therefore, we can expect that for some configurations the outgoing state can be entangled.

The scattering amplitude matrix element (in the first-order approximation) is given by [21]

$$\begin{aligned} M_{(\lambda_1,\lambda_2),(\tau_1,\tau_2)}(r_1,r_2,q_1,q_2) &= \frac{i(2\pi)^4e^2m^2}{V^2\sqrt{r_1^0r_2^0q_1^0q_2^0}}\left\{\frac{[\bar u^{\lambda_1}(r_1)\gamma_\mu u^{\tau_1}(q_1)][\bar u^{\lambda_2}(r_2)\gamma^\mu u^{\tau_2}(q_2)]}{(q_1-r_1)^2}\right.\\ &\quad\left.-\frac{[\bar u^{\lambda_1}(r_1)\gamma_\mu u^{\tau_2}(q_2)][\bar u^{\lambda_2}(r_2)\gamma^\mu u^{\tau_1}(q_1)]}{(q_2-r_1)^2}\right\}\\ &\quad\times\delta^4(r_1+r_2-q_1-q_2), \end{aligned} \quad (4)$$

where V is the volume element, m is the electron mass, e is the elementary charge, γ_μ are the Dirac matrices, $u^\tau(q)$ designates the Dirac field amplitude, and bar denotes Dirac conjugation.

The state defined in Eq. (3) is a general state of two electrons produced in Møller scattering. However, we are interested in the spin-density matrix describing the final state of two electrons with well-determined four-momenta; say k_1 and k_2. This matrix can be obtained by projecting the general state given in Eq. (3) onto the subspace spanned by two-particle states with four-momenta k_1 and k_2.

Thus, let $\rho^f(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2)$ be such a density matrix describing the spin state of two electrons with four-momenta k_1, k_2 originating from Møller scattering of initial electrons with four momenta p_1, p_2 and polarizations $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. The conservation of four-momentum implies

$$p_1 + p_2 = k_1 + k_2. \quad (5)$$

Taking into account Eqs. (1), (2), and (3), we get the matrix elements of $\rho^f(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2)$:

$$\begin{aligned} \rho^f_{\lambda_1\lambda_2,\lambda_1'\lambda_2'}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2) = \frac{1}{|\mathcal{F}|^2}\Bigg\{&\frac{1}{(p_1-k_1)^4}\mathrm{Tr}[u^{\lambda_1'}(k_1)\bar u^{\lambda_1}(k_1)\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu]\,\mathrm{Tr}[u^{\lambda_2'}(k_2)\bar u^{\lambda_2}(k_2)\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &+\frac{1}{(p_1-k_2)^4}\mathrm{Tr}[u^{\lambda_1'}(k_1)\bar u^{\lambda_1}(k_1)\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu]\,\mathrm{Tr}[u^{\lambda_2'}(k_2)\bar u^{\lambda_2}(k_2)\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\mathrm{Tr}[u^{\lambda_1'}(k_1)\bar u^{\lambda_1}(k_1)\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu u^{\lambda_2'}(k_2)\bar u^{\lambda_2}(k_2)\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\mathrm{Tr}[u^{\lambda_1'}(k_1)\bar u^{\lambda_1}(k_1)\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu u^{\lambda_2'}(k_2)\bar u^{\lambda_2}(k_2)\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\Bigg\}. \end{aligned} \quad (6)$$

In the above equation we have used the notation

$$\Omega^i(\boldsymbol{\xi}_i,p_i) = u(p_i)\tfrac{1}{2}(\mathbb{1} + \boldsymbol{\xi}_i\cdot\boldsymbol{\sigma})\bar u(p_i), \quad (7)$$

where $u(p_i)$ stands for a 4×2 matrix: $u(p_i) = [u^{1/2}(p_i), u^{-1/2}(p_i)]$. The normalization factor $|\mathcal{F}|^2$ can be calculated by tracing out the spin indices in the numerator of Eq. (7) and is given by the following formula:

$$
\begin{aligned}
|\mathcal{F}|^2 = \frac{1}{4m^2}\Bigg\{ & \frac{1}{(p_1-k_1)^4}\,\mathrm{Tr}[(k_1\gamma + m\,\mathbb{1})\gamma_\mu \Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu]\,\mathrm{Tr}[(k_2\gamma + m\,\mathbb{1})\gamma^\mu \Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu] \\
& + \frac{1}{(p_1-k_2)^4}\,\mathrm{Tr}[(k_1\gamma + m\,\mathbb{1})\gamma_\mu \Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu]\,\mathrm{Tr}[(k_2\gamma + m\,\mathbb{1})\gamma^\mu \Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu] \\
& - \frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}[(k_1\gamma + m\,\mathbb{1})\gamma_\mu \Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu (k_2\gamma + m\,\mathbb{1})\gamma^\mu \Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu] \\
& - \frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}[(k_1\gamma + m\,\mathbb{1})\gamma_\mu \Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu (k_2\gamma + m\,\mathbb{1})\gamma^\mu \Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\Bigg\},
\end{aligned} \tag{8}
$$

where $k_i\gamma = k_i^\mu \gamma_\mu$, $p_i\gamma = p_i^\mu \gamma_\mu$, $i = 1,2$.

The matrix $\Omega^i(\boldsymbol{\xi}_i, p_i)$ defined in Eq. (7) can be written as [7]

$$
\Omega^i(\boldsymbol{\xi}_i,p_i) = \frac{1}{4}\left(\frac{p_i\gamma}{m} + \mathbb{1}\right)\left(\mathbb{1} + 2\gamma^5 \frac{w_i\gamma}{m}\right), \tag{9}
$$

where the components of the four-vectors w_i are given by

$$
w_i^0 = \frac{\mathbf{p}_i \cdot \boldsymbol{\xi}_i}{2}, \quad \mathbf{w}_i = \frac{1}{2}\left[m\boldsymbol{\xi}_i + \frac{\mathbf{p}_i(\mathbf{p}_i \cdot \boldsymbol{\xi}_i)}{m + p_i^0}\right]. \tag{10}
$$

As before, the quantities with index i correspond to the ith particle.

If we write $|\mathcal{F}|^2$ in the form

$$
|\mathcal{F}|^2 = \frac{1}{4m^2}\left[\frac{K_1}{(p_1-k_1)^4} + \frac{K_2}{(p_1-k_2)^4} - \frac{K_3}{(p_1-k_1)^2(p_1-k_2)^2}\right], \tag{11}
$$

then after simple but tedious calculations we get

$$
K_1 = \frac{2}{m^2}\{2[m^2 - (k_1p_1)][m^2 - 4(w_1w_2)] + (p_1p_2)^2 + (k_1p_2)^2 - 4(k_1w_1)(k_2w_2)\}, \tag{12a}
$$

$$
K_2 = \frac{2}{m^2}\{2[m^2 - (k_2p_1)][m^2 - 4(w_1w_2)] + (p_1p_2)^2 + (k_1p_1)^2 - 4(k_2w_1)(k_1w_2)\}, \tag{12b}
$$

$$
\begin{aligned}
K_3 = {} & 8[(p_1p_2) + (w_1w_2)] - \frac{4}{m^2}[4(p_1p_2)(w_1w_2) + (p_1p_2)^2] + \frac{8}{m^4}\{(p_1w_2)[(k_2p_1)(k_2w_1) + (k_1p_1)(k_1w_1)] \\
& + (p_2w_1)[(k_2p_1)(k_1w_2) + (k_1p_1)(k_2w_2)] - (p_1p_2)[(k_1w_1)(k_2w_2) + (k_1w_2)(k_2w_1) + (p_1w_2)(p_2w_1)] \\
& - (w_1w_2)[(k_1p_1)^2 + (k_2p_1)^2 - (p_1p_2)^2]\},
\end{aligned} \tag{12c}
$$

where p_1p_2, p_1k_2, ... designate Minkowski scalar products.

Hence, inserting Eqs. (12) into Eq. (9) we get finally

$$
\begin{aligned}
|\mathcal{F}|^2 = {} & \frac{1}{8m^4(m^2 - k_1p_1)^2(m^2 - k_2p_1)^2}\big([(p_1p_2)^2 + (k_1p_2)^2 - 4(k_1w_1)(k_2w_2)](m^2 - k_2p_1)^2 \\
& + [(p_1p_2)^2 + (k_1p_1)^2 - 4(k_2w_1)(k_1w_2)](m^2 - k_1p_1)^2\big) \\
& + \frac{1}{4m^2(m^2 - k_1p_1)(m^2 - k_2p_1)}\Bigg(m^2 + 2(w_1w_2) - 3(p_1p_2) + \frac{1}{m^2}(p_1p_2)^2 \\
& + \frac{2}{m^4}\{-(p_1w_2)[(k_2w_1)(k_2p_1) + (k_1w_1)(k_1p_1)] - (p_2w_1)[(k_1w_2)(k_2p_1) + (k_2w_2)(k_1p_1)] \\
& + (p_1p_2)[(k_1w_2)(k_2w_1) + (k_2w_2)(k_1w_1) + (p_1w_2)(p_2w_1)] + (w_1w_2)[(k_1p_1)^2 + (k_2p_1)^2 - (p_1p_2)^2]\}\Bigg).
\end{aligned} \tag{13}
$$

The term $|\mathcal{F}|^2$ is related to the differential cross section for $e^-e^- \longrightarrow e^-e^-$ scattering as follows:

$$
d\sigma = \frac{4\pi e^4 m^4}{s(s - 4m^2)}|\mathcal{F}|^2 dt \frac{d\varphi}{2\pi}, \tag{14}
$$

where s and t are two of three Mandelstam variables:

$$
s = (p_1 + p_2)^2, \quad t = (p_1 - k_1)^2, \quad u = (p_1 - k_2)^2. \tag{15}
$$

III. RELATIVISTIC SPIN OPERATOR

Consider the spin square operator which can be uniquely defined in terms of the generators of the Poincaré group as

$$
\hat{\mathbf{S}}^2 = -\frac{1}{m^2}\hat{W}^\mu \hat{W}_\mu, \tag{16}
$$

where $\hat{W}^\mu$ is the Pauli-Lubanski four-vector and $\hat{W}^\mu = \frac{1}{2}\epsilon^{\nu\alpha\beta\mu}\hat{P}_\nu \hat{J}_{\alpha\beta}$ and $\hat{J}_{\alpha\beta}$ are the generators of the Lorentz group.

Assuming linearity in the components of $\hat{W}^\mu$ one can easily derive the formula for the spin operator in the enveloping algebra of the Lie algebra of the Poincaré group. In fact, demanding that the spin operator transform like a vector under rotations and like a pseudovector under reflections, commute with spacetime observables, and fulfill the standard canonical commutation relations, we obtain the spin operator

$$\hat{\mathbf{S}} = \frac{1}{m}\left(\hat{\mathbf{W}} - \hat{W}^0 \frac{\hat{\mathbf{P}}}{\hat{P}^0 + m}\right). \tag{17}$$

Its action S on the spin indices is given by the Pauli matrices, i.e.,

$$\mathbf{S} = \frac{\boldsymbol{\sigma}}{2}, \tag{18a}$$

whereas in the bispinor basis the action takes the form

$$\mathbf{S} = -\frac{1}{2m}\left[-p^0\gamma^5\gamma^0\boldsymbol{\gamma} + \frac{\mathbf{p}}{p^0+m}\gamma^5\gamma^0(\mathbf{p}\cdot\boldsymbol{\gamma}) + i\gamma^0(\mathbf{p}\times\boldsymbol{\gamma})\right]. \tag{18b}$$

The operator (17) coincides with the spin observable defined in the quantum-field-theory framework [8]. Moreover, it has been shown [22] that the spin operator (17) is equivalent to the action of the mean-spin operator introduced by Foldy and Wouthuysen [23]. In the Dirac theory, the spin operator is not uniquely defined. For an exhaustive discussion of this problem see Refs. [22] and [24] and references therein.

IV. PROBABILITIES AND CORRELATION FUNCTION

Now let two observers, Alice and Bob, perform measurements on the scattered electrons. We assume that Alice measures the (normalized to 1) spin projection of the electron with four-momentum k_1 on direction $\mathbf{a}$ while Bob measures the (normalized to 1) spin projection of the electron with four-momentum k_2 on direction $\mathbf{b}$. The probability of receiving outcome a by Alice and b by Bob ($a,b = \pm1$) is given by the following formula:

$$\begin{aligned}\mathcal{P}^{ab}&(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2;\mathbf{a},\mathbf{b})\\ &= \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \rho^f_{\lambda_1\lambda_2,\lambda_1'\lambda_2'}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2)\\ &\quad\times \pi^a_{\lambda_1'\lambda_1}(\mathbf{a})\pi^b_{\lambda_2'\lambda_2}(\mathbf{b}),\end{aligned} \tag{19}$$

where $\pi^{\pm1}(\mathbf{n})$, $\mathbf{n} = \mathbf{a},\mathbf{b}$, are the projectors from the spectral decomposition of the operator $2\mathbf{n}\cdot\mathbf{S}$ (in the one-particle spin basis) corresponding to the eigenvalues ±1, respectively. Since the matrix of the spin operator $\mathbf{S}$ in the spin basis is $\frac{1}{2}\boldsymbol{\sigma}$ (see, e.g., Ref. [8]), we have

$$\pi^{\pm1}(\mathbf{n}) = \tfrac{1}{2}(\mathbb{1} \pm \mathbf{n}\cdot\boldsymbol{\sigma}). \tag{20}$$

Inserting Eqs. (7) and (20) into Eq. (19) we get

$$\begin{aligned}\mathcal{P}^{ab}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2;\mathbf{a},\mathbf{b}) = \frac{1}{|\mathcal{F}|^2}\Bigg\{&\frac{1}{(p_1-k_1)^4}\,\mathrm{Tr}[\Pi^a(k_1,\mathbf{a})\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu]\,\mathrm{Tr}[\Pi^b(k_2,\mathbf{b})\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &+\frac{1}{(p_1-k_2)^4}\,\mathrm{Tr}[\Pi^a(k_1,\mathbf{a})\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu]\,\mathrm{Tr}[\Pi^b(k_2,\mathbf{b})\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}[(\Pi^a(k_1,\mathbf{a})\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu\Pi^b(k_2,\mathbf{b})\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}\left[\Pi^a(k_1,\mathbf{a})\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu\Pi^b(k_2,\mathbf{b})\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu\right]\Bigg\},\end{aligned} \tag{21}$$

with

$$\Pi^{\pm1}(k_i,\mathbf{n}) = u(k_i)\pi^{\pm1}(\mathbf{n})\bar{u}(k_i) \tag{22}$$

being the counterparts of $\pi^{\pm1}(\mathbf{n})$ in the bispinor basis and $|\mathcal{F}|^2$ given in Eq. (14).

With the help of the probabilities given in Eq. (19) we can define the correlation function

$$\begin{aligned}\mathcal{C}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2;\mathbf{a},\mathbf{b}) &= \sum_{a,b=\pm1} ab\mathcal{P}^{ab}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2;\mathbf{a},\mathbf{b}) = \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} \rho^f_{\lambda_1\lambda_2,\lambda_1'\lambda_2'}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,k_1,k_2)\\ &\quad\times(\mathbf{a}\cdot\boldsymbol{\sigma})_{\lambda_1'\lambda_1}(\mathbf{b}\cdot\boldsymbol{\sigma})_{\lambda_2'\lambda_2},\end{aligned} \tag{23}$$

which together with Eq. (7) yields the general form

$$\begin{aligned}\mathcal{C}(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2;k_1,k_2;\mathbf{a},\mathbf{b}) = \frac{1}{|\mathcal{F}|^2}\Bigg\{&\frac{1}{(p_1-k_1)^4}\,\mathrm{Tr}[S(k_1,\mathbf{a})\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu]\,\mathrm{Tr}[S(k_2,\mathbf{b})\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &+\frac{1}{(p_1-k_2)^4}\,\mathrm{Tr}[S(k_1,\mathbf{a})\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu]\,\mathrm{Tr}[S(k_2,\mathbf{b})\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}[(S(k_1,\mathbf{a})\gamma_\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma_\nu S(k_2,\mathbf{b})\gamma^\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma^\nu]\\ &-\frac{1}{(p_1-k_1)^2(p_1-k_2)^2}\,\mathrm{Tr}[S(k_1,\mathbf{a})\gamma_\mu\Omega^2(\boldsymbol{\xi}_2,p_2)\gamma_\nu S(k_2,\mathbf{b})\gamma^\mu\Omega^1(\boldsymbol{\xi}_1,p_1)\gamma^\nu]\Bigg\}.\end{aligned} \tag{24}$$

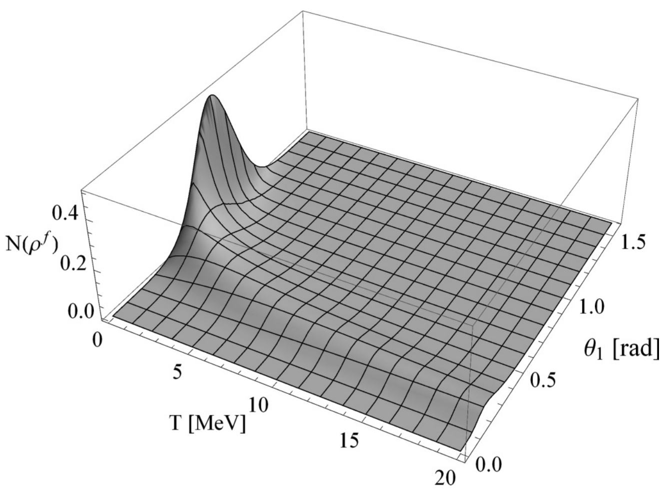

FIG. 2. Negativity of the state ρ^f of two electrons originating from scattering from an unpolarized target as a function of beam kinetic energy T and an angle at which the first electron is scattered, θ_1. Beam polarized along Z axis, $|\boldsymbol{\xi}_1| = 1$.

V. SCATTERING FROM A STATIONARY TARGET

Now let us discuss a special case of an electron beam scattering from an unpolarized stationary target, $w_2 = 0$, $\mathbf{p}_2 = \mathbf{0}$, which could be considered as the simplest experimental setup for generating electron pairs for the correlation experiment. Under these conditions Eq. (14) reduces to

$$|\mathcal{F}|^2 = \frac{1}{4m^4\left(k_1^0 - m\right)^2\left(k_1^0 - p_1^0\right)^2} \times \left\{k_1^0\left(m + p_1^0\right)\left[m^2 - 4mp_1^0 - 2\left(p_1^0\right)^2\right] - 2\left(k_1^0\right)^3\left(m + p_1^0\right) + 3\left(k_1^0\right)^2 p_1^0\left(2m + p_1^0\right) + \left(k_1^0\right)^4 + p_1^0\left[-m^3 + 4m^2 p_1^0 + (p_1^0)^3\right]\right\}. \quad (25)$$

In terms of the Mandelstam variables (15) the above equation takes the form

$$|\mathcal{F}|^2 = \frac{1}{4m^4}\left\{\frac{1}{t^2}\left[\frac{s^2+u^2}{2} + 4m^2(t-m^2)\right] + \frac{1}{u^2}\left[\frac{s^2+t^2}{2} + 4m^2(u-m^2)\right] + \frac{4}{tu}\left(\frac{s}{2} - m^2\right)\left(\frac{s}{2} - 3m^2\right)\right\}. \quad (26)$$

The above formula coincides with the results obtained in Ref. [25]. We should note that it does not depend on the beam polarization.

A. Entanglement of initial state

The scattered Møller state of two electrons is not an irreducible singlet or triplet state but rather a mixture of these states (more precisely, the density matrix is in this case a reducible Wigner-Eckart tensor operator). We have assumed that the state of two electrons before scattering, Eq. (1), is separable. But it does not mean that the state after scattering, $\rho^f(p_1,p_2,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,k_1,k_2)$ given in Eq. (7), is separable, too. We analyze the degree of entanglement of the state ρ^f originating from the scattering from an unpolarized target. To this end we calculate the value of entanglement measure for this state. We use the negativity introduced in Ref. [26]. The explicit form of the matrix $\rho^f(p_1,p_2 = (m,\mathbf{0}),\boldsymbol{\xi}_1,\boldsymbol{\xi}_2 = \mathbf{0},k_1,k_2)$ is complicated, therefore in Figs. 2, 3, and 4 we present only the plots of numerically calculated negativity under some choice of parameters. The parametrization utilized for the plots is given in Appendix A. The behavior of negativity for other choices of beam polarization is qualitatively similar.

VI. CORRELATION FUNCTION FOR SCATTERING FROM A TARGET

After some calculation we can also derive a formula for the correlation function for scattering from a target. Namely, we

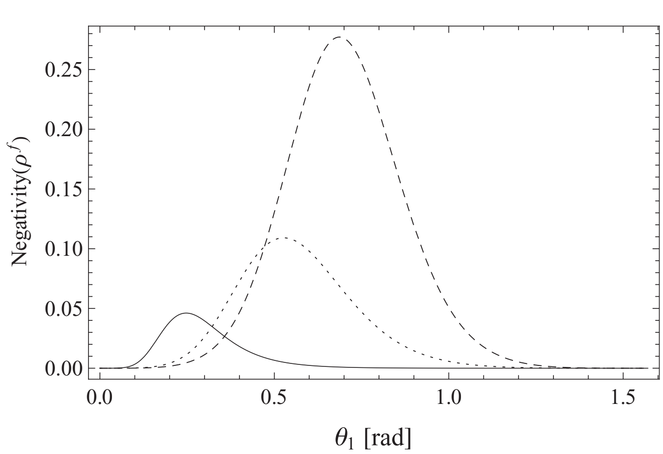

FIG. 3. Sections of the surface presented in Fig. 2 for fixed values of beam kinetic energy: $T = 0.5$ MeV (dashed line), $T = 2$ MeV (dotted line), $T = 15$ MeV (solid line).

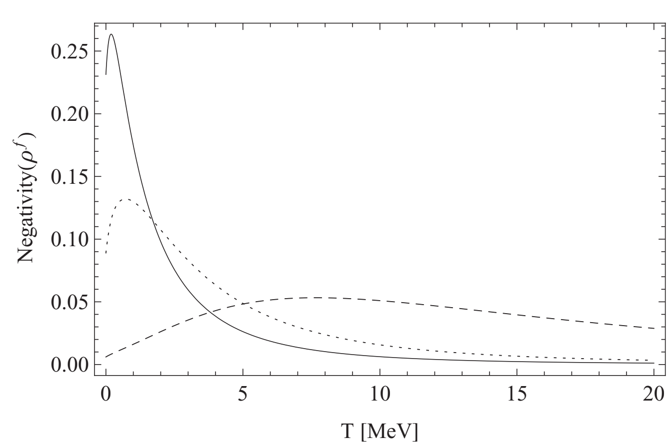

FIG. 4. Sections of the surface presented in Fig. 2 for fixed values of scattering angle of the first electron: $\theta_1 = 0.3$ rad (dashed line), $\theta_1 = 0.5$ rad (dotted line), $\theta_1 = 0.6$ rad (solid line).

have

$$
\begin{aligned}
\mathcal{C}(p_1;k_1,k_2;\mathbf{a},\mathbf{b}) = \Big(&\big(k_1^{0^2}-m^2\big)\big\{2(\mathbf{a}\cdot\mathbf{b})\big(k_1^0-p_1^0\big)\big[k_1^{0^3}-k_1^{0^2}\big(3m+2p_1^0\big)+k_1^0\big(m^2+6p_1^0m+p_1^{0^2}\big)+m\big(2m^2-5p_1^0m-3p_1^{0^2}\big)\big]\\
&+(\mathbf{a}\cdot\mathbf{p}_1)\big[(\mathbf{b}\cdot\mathbf{p}_1)\big(k_1^0-p_1^0\big)\big(k_1^0-2m-p_1^0\big)-(\mathbf{b}\cdot\mathbf{k}_2)\big(2k_1^{0^2}-3mk_1^0+p_1^0k_1^0-2m^2-3p_1^{0^2}+5mp_1^0\big)\big]\big\}\\
&+(\mathbf{a}\cdot\mathbf{k}_1)\big\{(\mathbf{b}\cdot\mathbf{k}_2)\big[4k_1^{0^4}-8k_1^{0^3}\big(m+p_1^0\big)-k_1^{0^2}\big(5m^2-28p_1^0m-17p_1^{0^2}\big)\\
&+k_1^0\big(9m^3-11p_1^0m^2-33p_1^{0^2}m-13p_1^{0^3}\big)+m\big(2m^3-11p_1^0m^2+14p_1^{0^2}m+15p_1^{0^3}\big)\big]\\
&-(\mathbf{b}\cdot\mathbf{p}_1)\big(k_1^0-p_1^0\big)\big[2k_1^{0^3}-k_1^{0^2}\big(5m+7p_1^0\big)-k_1^0\big(m^2-18p_1^0m-5p_1^{0^2}\big)\\
&+m\big(6m^2-11p_1^0m-7p_1^{0^2}\big)\big]\big\}\Big)\big\{2\big(k_1^0+m\big)\big(k_1^0-2m-p_1^0\big)\big[k_1^{0^4}-2k_1^{0^3}\big(m+p_1^0\big)+3p_1^0k_1^{0^2}\big(2m+p_1^0\big)\\
&+k_1^0\big(m+p_1^0\big)\big(m^2-4p_1^0m-2p_1^{0^2}\big)-p_1^0\big(m^3-4p_1^0m^2-p_1^{0^3}\big)\big]\big\}^{-1}.
\end{aligned}
\tag{27}
$$

A. Mott polarimetry

A measurement of the spin projection can be realized by means of Mott polarimetry. The method is sensitive only to the spin projection on a direction perpendicular to the Mott scattering plane (i.e., $\mathbf{a}\perp\mathbf{k}_1$, $\mathbf{b}\perp\mathbf{k}_2$). In such a special case the formula (27) reduces to a simpler form:

$$
\begin{aligned}
\mathcal{C}(p_1;k_1,k_2;\mathbf{a},\mathbf{b}) = \Big(&\big(k_1^0-m\big)\big(k_1^0-p_1^0\big)\big\{2(\mathbf{a}\cdot\mathbf{b})\big[-k_1^0\big(m+p_1^0\big)+k_1^{0^2}-m\big(m-3p_1^0\big)\big]+(\mathbf{a}\cdot\mathbf{p}_1)(\mathbf{b}\cdot\mathbf{p}_1)\big\}\Big)\\
&\times\big\{2\big[k_1^0\big(m^3-3m^2p_1^0-6mp_1^{0^2}-2p_1^{0^3}\big)-2k_1^{0^3}\big(m+p_1^0\big)\\
&+3k_1^{0^2}p_1^0\big(2m+p_1^0\big)+k_1^{0^4}-p_1^0\big(m^3-4m^2p_1^0-p_1^{0^3}\big)\big]\big\}^{-1}.
\end{aligned}
\tag{28}
$$

Notice that the correlation function (28) does not depend on the beam polarization.

In Fig. 5 one can see a set of plots showing the dependence of the correlation function on the angle θ_1 at which the first

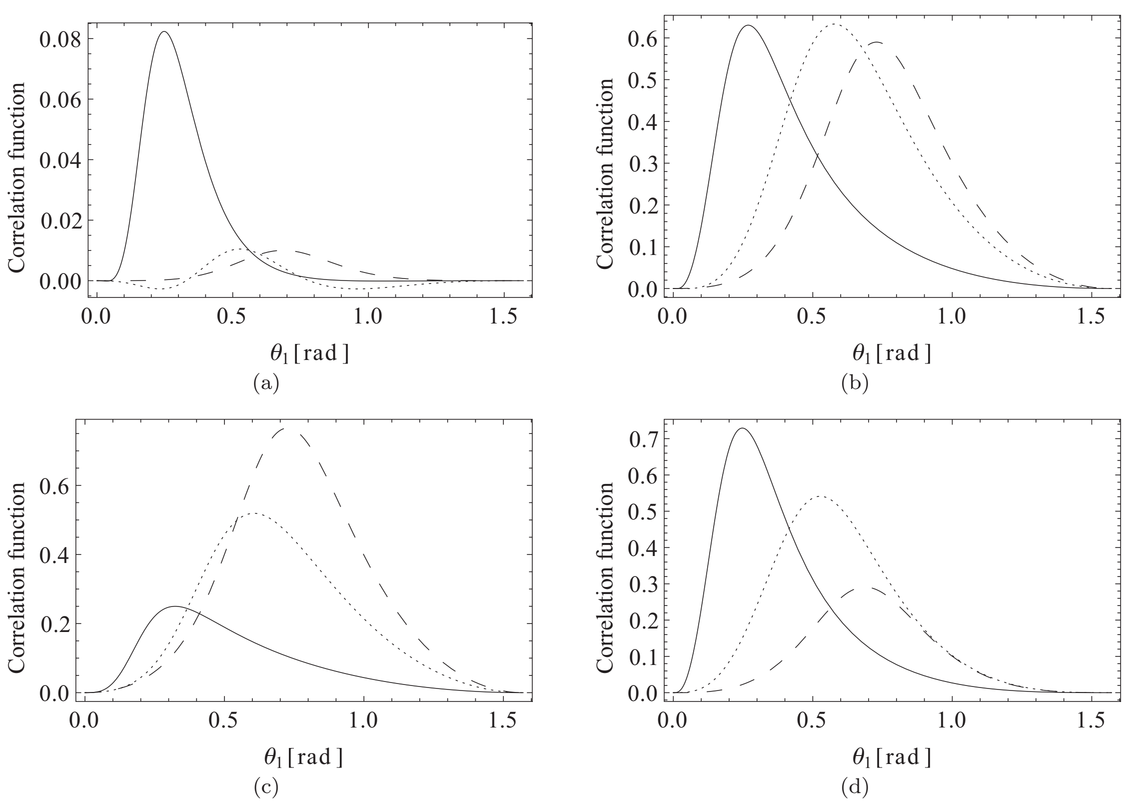

FIG. 5. Correlation function vs the angle at which one of the electrons is scattered for beam kinetic energy $T = 0.5$ MeV (dashed line), $T = 2$ MeV (dotted line), and $T = 15$ MeV (solid line); $\mathbf{a}$ and $\mathbf{b}$ in the scattering plane and (a) $\mathbf{a}\perp\mathbf{k}_1$ and $\mathbf{b}\perp\mathbf{k}_2$ (Mott polarimetry); (b) $\angle(\mathbf{a},\mathbf{k}_1)=\pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2)=\pi/12$; (c) $\mathbf{a}\perp\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$; (d) $\mathbf{a}||\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$.

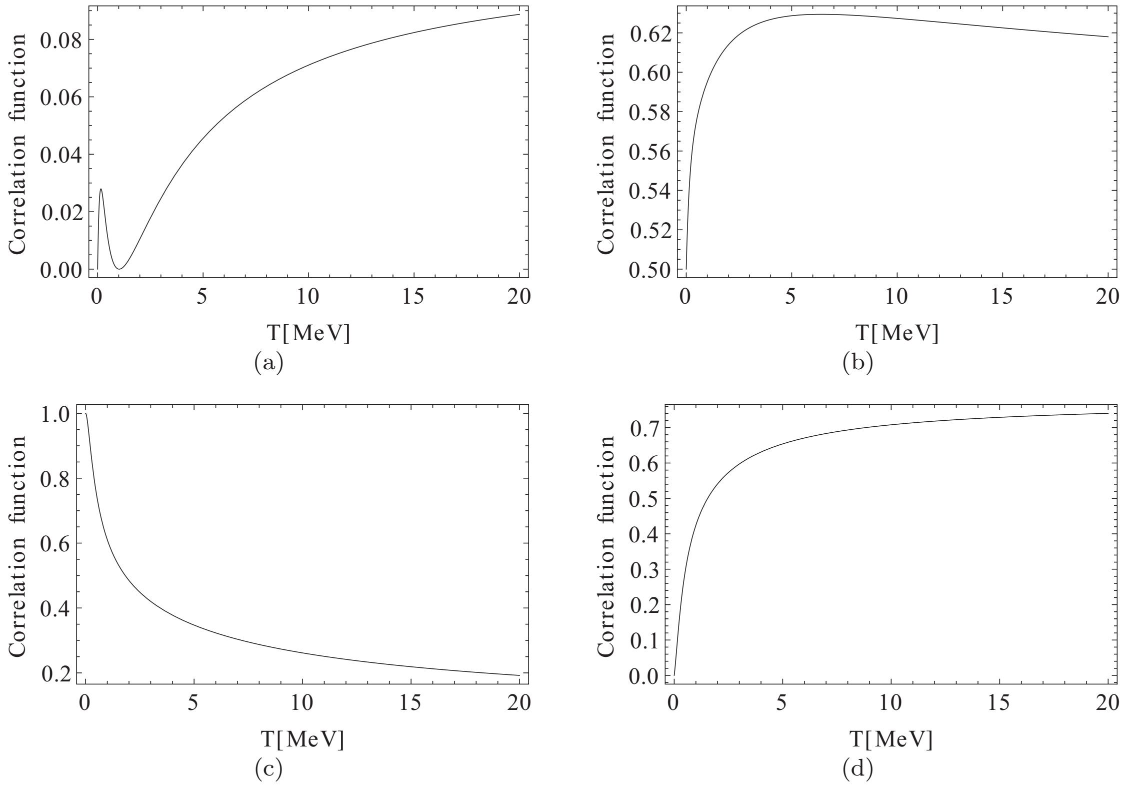

FIG. 6. Correlation function vs beam kinetic energy for **a** and **b** in the scattering plane; equal final-state momenta (symmetric scattering—see Appendix B) (a) $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b} \perp \mathbf{k}_2$ (Mott polarimetry); (b) $\angle(\mathbf{a},\mathbf{k}_1) = \pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2) = \pi/12$; (c) $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$; (d) $\mathbf{a}||\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$. Note that for panels (a) and (b) the correlation function is nonmonotonic and in panel (d) it is increasing with energy.

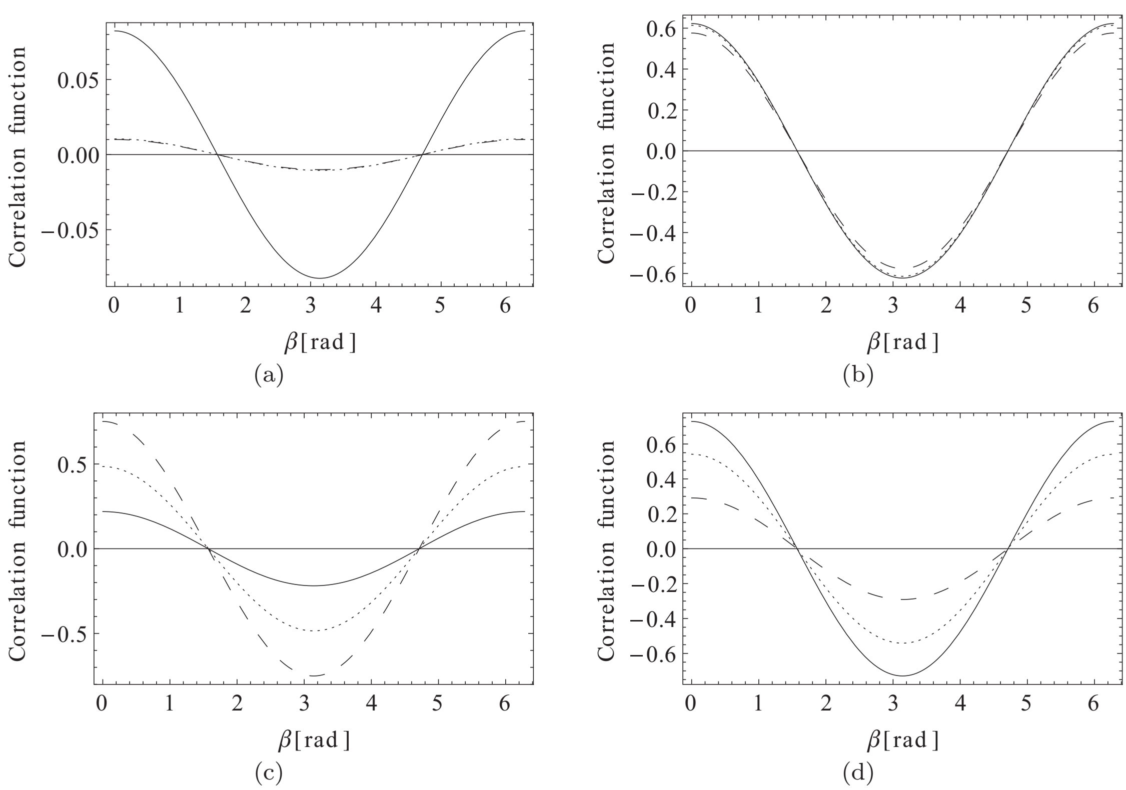

FIG. 7. Correlation function vs the angle between the scattering plane and the **b** direction for beam kinetic energy $T = 0.5$ MeV (dashed line), $T = 2$ MeV (dotted line), and $T = 15$ MeV (solid line); **a** in the scattering plane, equal final-state momenta (symmetric scattering—see Appendix B) (a) $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b} \perp \mathbf{k}_2$ (Mott polarimetry); (b) $\angle(\mathbf{a},\mathbf{k}_1) = \pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2) = \pi/12$; (c) $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$; (d) $\mathbf{a}||\mathbf{k}_1$ and $\boldsymbol{b}||\boldsymbol{k}_2$.

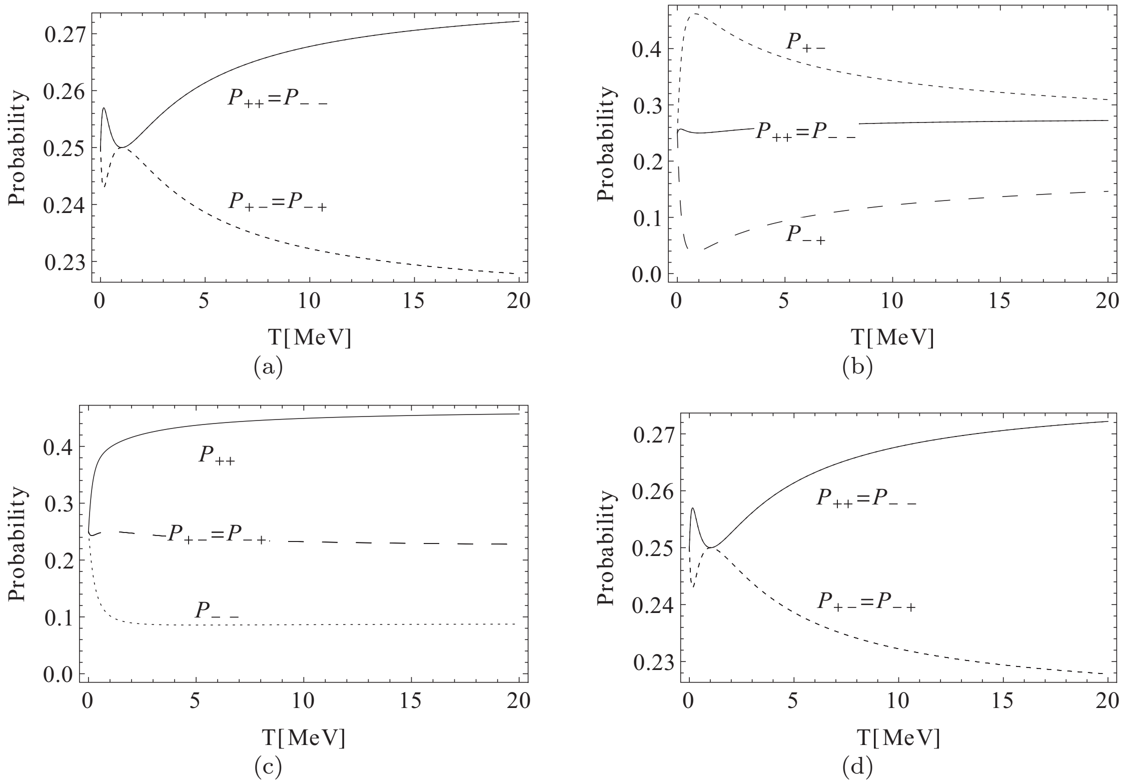

FIG. 8. Probabilities P_{++}, P_{--}, P_{+-}, P_{-+} vs the beam kinetic energy; **a** and **b** in the scattering plane, equal final-state momenta (symmetric scattering—see Appendix B). Individual plots show the probabilities for (a) unpolarized beam (implies $P_{++} = P_{--}$ and $P_{+-} = P_{-+}$); (b) beam polarized 85% along X axis (implies $P_{++} = P_{--}$); (c) beam polarized 85% along Y axis (implies $P_{+-} = P_{-+}$); (d) beam polarized 85% along Z axis (implies $P_{++} = P_{--}$ and $P_{+-} = P_{-+}$).

electron is scattered [compare with parametrization given in Appendix A, especially Eq. (A3)] for the beam kinetic energy $T = 15$ MeV. Figure 5(a) shows the case of $\boldsymbol{a} \perp \boldsymbol{k}_1$ and $\boldsymbol{b} \perp \boldsymbol{k}_2$ (such a measurement can be achieved by means of Mott polarimetry). Figure 5(b) corresponds to $\angle(\mathbf{a},\mathbf{k}_1) = \pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2) = \pi/12$, Fig. 5(c) to $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$, and Fig. 5(d) to $\mathbf{a}||\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$. All the above figures have local maxima for the case of $\theta_1 = \theta_2$ (symmetric scattering).

Figure 6 shows the dependence of the correlation function on the beam kinetic energy for vectors **a** and **b** in the scattering plane and for the case of symmetric scattering (see Appendix B). Figure 6(a) corresponds to $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b} \perp \mathbf{k}_2$

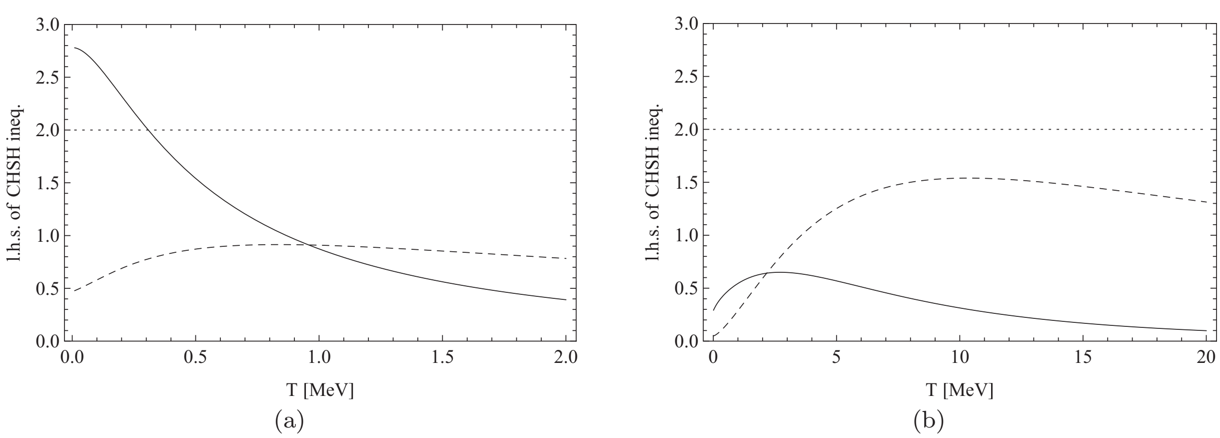

FIG. 9. Left-hand side of the CHSH inequality [Eq. (30)] as a function of energy in the case of Møller scattering from a stationary target. We consider two configurations in which vectors **a**, **b**, **c**, **d** are determined according to Eqs. (A9) and (A10) with the following values of angles: $\alpha = 2.513$ rad, $\alpha' = 0.241$ rad, $\beta = 4.366$ rad, $\beta' = -1.130$ rad, $\gamma = 0.879$ rad, $\gamma' = -0.109$ rad, $\delta = 0$ rad, $\delta' = -1.570$ rad (solid line), and $\alpha = 0.911$ rad, $\alpha' = 0.706$ rad, $\beta = 0$ rad, $\beta' = -0.141$ rad, $\gamma = 0.062$ rad, $\gamma' = 0.251$ rad, $\delta = 5.811$ rad, $\delta' = -0.204$ rad (dashed line). In panel (a) the scattering angle $\theta_1 = 0.782$ rad, in panel (b) the scattering angle $\theta_1 = 0.298$ rad.

(Mott polarimetry)—we point out that for low beam energies the value of the correlation function dynamically varies, which is reflected in two local extrema. Figure 6(b) corresponds to $\angle(\mathbf{a},\mathbf{k}_1) = \pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2) = \pi/12$ and has maximum for $T = 6.41$ MeV. Figure 6(c) corresponds to $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$ and decreases monotonically from the maximal value 1 for $T = 0$ and Fig. 6(d) corresponding to $\mathbf{a}||\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$ monotonically rises.

In Fig. 7 one can see the set of figures showing the dependence of the correlation function on the angle β at which the vector $\mathbf{b}$ lies with respect to the scattering plane for the symmetric-scattering case (see Appendix B). Again Fig. 7(a) corresponds to $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b} \perp \mathbf{k}_2$ (Mott polarimetry), Fig. 7(b) to $\angle(\mathbf{a},\mathbf{k}_1) = \pi/4$ and $\angle(\mathbf{b},\mathbf{k}_2) = \pi/12$, Fig. 7(c) to $\mathbf{a} \perp \mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$, and Fig. 7(d) to $\mathbf{a}||\mathbf{k}_1$ and $\mathbf{b}||\mathbf{k}_2$.

B. Probabilities

The experimental method assumes counting of coincidences of spin-projection outcomes $++$, $+-$, $-+$, $--$ (denoted by N_{++}, N_{--}, N_{+-}, N_{-+}) which, when divided by the number of events (N), gives the joint probabilities of obtaining specific results by Alice and Bob (P_{++}, P_{--}, P_{+-}, P_{-+}). Thus, the joint probabilities P_{ab} are more primary quantities than the correlation function and are going to be measured directly. The experimental correlation function can be calculated as

$$\mathcal{C}^{\text{expt}}(\mathbf{a},\mathbf{b}) = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N} \tag{29}$$

and contains less information about behavior of the real quantum systems than the probabilities themselves.

Because of the complexity of the explicit formulas for the probabilities we do not include them in the present paper. The behavior of the probabilities is shown in Fig. 8.

In Fig. 8 one can see the dependence of the probabilities P_{++}, P_{--}, P_{+-}, P_{-+} on the beam kinetic energy in the case of symmetric scattering (see Appendix B) for $\mathbf{a}$ and $\mathbf{b}$ lying in the scattering plane and for the beam polarization vector $\boldsymbol{\xi}$ equal to $\boldsymbol{\xi} = (0,0,0)$ [Fig. 8(a)], $\boldsymbol{\xi} = 0.85(1,0,0)$ [Fig. 8(b)], $\boldsymbol{\xi} = 0.85(0,1,0)$ [Fig. 8(c)], and $\boldsymbol{\xi} = 0.85(0,0,1)$ [Fig. 8(d)]. We point out that although the beam polarization does not affect the shape of the correlation function it has influence on the behavior of the probabilities.

C. Bell-type inequalities

Having at our disposal the general correlation function we can also discuss Bell-type inequalities in the case of Møller scattering. Violation of such inequalities can be treated as a signature of the fact that correlations are really quantum and cannot be reproduced in any local realistic theory [27]. We consider here the Clauser-Horne-Shimony-Holt (CHSH) inequality [28] which has the form

$$|\mathcal{C}(p_1;k_1,k_2;\mathbf{a},\mathbf{b}) - \mathcal{C}(p_1;k_1,k_2;\mathbf{a},\mathbf{d}) + \mathcal{C}(p_1;k_1,k_2;\mathbf{c},\mathbf{b}) + \mathcal{C}(p_1;k_1,k_2;\mathbf{c},\mathbf{d})| \leqslant 2. \tag{30}$$

For the correlation function given in Eq. (27) there exist such configurations that the CHSH inequality (30) is violated and such that this inequality holds. In Fig. 9 we have plotted the left-hand side of the CHSH inequality as a function of energy for two chosen configurations and for two energy scales. Let us notice that the region where the CHSH inequality is violated corresponds to the region where the entanglement measure has a relatively big value (cf. Fig. 2).

VII. CONCLUSIONS

In this paper we have derived and analyzed formulas for the correlation function and joint probabilities in a bipartite system of two relativistic electrons produced in scattering experiments. The correlation functions and the probabilities have been calculated in the state originating from the $e^-e^- \longrightarrow e^-e^-$ Møller scattering. We have also analyzed the entanglement of the two-electron final state. Finally, we have discussed briefly the CHSH inequality in this case. We showed that for some configurations the inequality is violated.

Our analysis and results can serve as the theoretical basis for the experimental test of predictions of relativistic quantum theory in EPR-type experiments. It seems that such a test will be possible in the experiment prepared by the QUEST Collaboration mentioned in the introduction.

ACKNOWLEDGMENTS

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APPENDIX A: PARAMETRIZATION USED

Let us analyze the kinematical situation in the case of electron beam scattering from the stationary target, i.e., $p_2 = (m,0,0,0)$ (see Fig. 10). Without loss of generality we can assume that the beam propagates along the X direction and the scattered electrons move in the XY plane. Conservation of four-momentum implies that there are only two independent kinematical variables. As these independent variables we choose the energy of the incoming electron p_1^0 and the angle θ_1 at which the electron with four-momentum k_1 is scattered. Those two variables unambiguously determine other

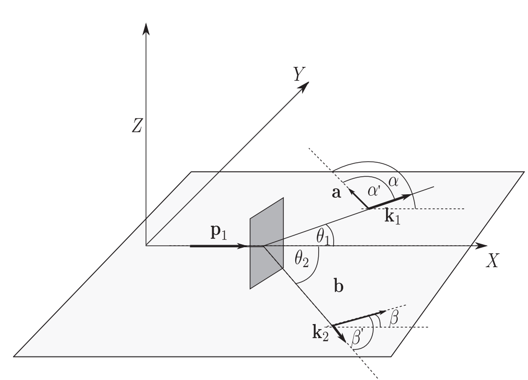

FIG. 10. Møller scattering in the case of electron beam scattering from the stationary target: $\mathbf{p}_1 = \mathbf{k}_1 + \mathbf{k}_2, m + p_1^0 = k_1^0 + k_2^0$; arbitrary $\mathbf{a}$ and $\mathbf{b}$.

kinematical variables, and the corresponding formulas are the following:

$$\mathbf{p}_1 = \sqrt{\left(p_1^0\right)^2 - m^2}\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{A1}$$

$$k_1^0 = m\frac{p_1^0 + m + \left(p_1^0 - m\right)\cos^2\theta_1}{p_1^0 + m - \left(p_1^0 - m\right)\cos^2\theta_1}, \tag{A2}$$

$$\mathbf{k}_1 = \frac{2m\sqrt{\left(p_1^0\right)^2 - m^2}\cos\theta_1}{p_1^0 + m - \left(p_1^0 - m\right)\cos^2\theta_1}\begin{pmatrix}\cos\theta_1\\ \sin\theta_1\\ 0\end{pmatrix}, \tag{A3}$$

$$k_2^0 = \frac{p_1^0\left(p_1^0 + m\right)\sin^2\theta_1 + 2m^2\cos^2\theta_1}{p_1^0 + m - \left(p_1^0 - m\right)\cos^2\theta_1}, \tag{A4}$$

$$\mathbf{k}_2 = |\mathbf{k}_2|\begin{pmatrix}\cos\theta_2\\ -\sin\theta_2\\ 0\end{pmatrix}, \tag{A5}$$

where

$$|\mathbf{k}_2| = \frac{\sqrt{\left(p_1^0 + m\right)^2\sin^2\theta_1 + 4m^2\cos^2\theta_1}}{p_1^0 + m - \left(p_1^0 - m\right)\cos^2\theta_1}\sqrt{\left(p_1^0\right)^2 - m^2}\sin\theta_1, \tag{A6}$$

and

$$\cos\theta_2 = \frac{\left(p_1^0 + m\right)\sin\theta_1}{\sqrt{\left(p_1^0 + m\right)^2\sin^2\theta_1 + 4m^2\cos^2\theta_1}}, \tag{A7}$$

$$\sin\theta_2 = \frac{2m\cos\theta_1}{\sqrt{\left(p_1^0 + m\right)^2\sin^2\theta_1 + 4m^2\cos^2\theta_1}}. \tag{A8}$$

The (unit) directions **a** and **b** on which the spin projections are measured are arbitrary (Fig. 10). We use the following parametrization of the vectors **a** and **b**:

$$\mathbf{a} = \begin{pmatrix}\cos\alpha\cos\left(\alpha' + \theta_1\right)\\ \cos\alpha\sin\left(\alpha' + \theta_1\right)\\ \sin\alpha\end{pmatrix}, \tag{A9}$$

$$\mathbf{b} = \begin{pmatrix}\cos\beta\cos\left(\beta' - \theta_2\right)\\ \cos\beta\sin\left(\beta' - \theta_2\right)\\ \sin\beta\end{pmatrix}, \tag{A10}$$

where $\alpha',\beta' \in \langle 0,\pi\rangle$, $\alpha,\beta \in \langle 0,2\pi\rangle$. If the measurement is based on Mott scattering, **a** and **b** must be perpendicular to $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. This condition corresponds to the choice $\alpha' = \beta' = \pi/2$ in Eqs. (A9) and (A10). The polarization vector of the beam is parametrized as follows:

$$\boldsymbol{\xi} = |\boldsymbol{\xi}|\begin{pmatrix}\cos\chi\\ \sin\chi\cos\psi\\ \sin\chi\sin\psi\end{pmatrix}, \tag{A11}$$

where $|\boldsymbol{\xi}| \in \langle 0,1\rangle$, $\chi \in \langle 0,\pi\rangle$, $\psi \in \langle 0,2\pi\rangle$.

APPENDIX B: SYMMETRIC PARAMETRIZATION

Let us now analyze a case of the symmetric scattering, i.e., $\theta_1 = \theta_2 = \theta$ (see Fig. 10). Again we assume that the beam (in direction X) impinges on a stationary target [$\boldsymbol{p}_2 = m(0,0,0)$] and the scattering takes place in the XY plane. In such a case $k_1 = k_2$ and four-momentum conservation implies

$$k_1^0 = k_2^0 = \frac{m + p^0}{2}, \tag{B1}$$

$$\mathbf{k}_1 = \frac{1}{2}\begin{pmatrix}\sqrt{p^{0^2} - m^2}\\ \sqrt{2m(p^0 - m)}\\ 0\end{pmatrix}, \tag{B2}$$

$$\mathbf{k}_2 = \frac{1}{2}\begin{pmatrix}\sqrt{p^{0^2} - m^2}\\ -\sqrt{2m(p^0 - m)}\\ 0\end{pmatrix}. \tag{B3}$$

Notice that the parametrization of k_i depends on p_1^0 which also means the dependence on the beam kinetic energy T. This implies that for every T the angle θ takes different value.

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[20] The Polish-German QUEST Collaboration includes teams from the following institutions: Institute of Nuclear Physics (Kracow, Poland); Jagiellonian University (Kracow, Poland); Technische Universität Darmstadt (Darmstadt, Germany); University of Lodz (Lodz, Poland); University of Warsaw (Warsaw, Poland). Contact person is Jacek Ciborowski (cib@fuw.edu.pl). The main goal of the QUEST Collaboration is to perform correlation experiments with Møller electrons [19]. The experiment will be done with the help of the superconducting electron linear accelerator S-DALINAC at Darmstadt.

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