Embedding non-Markovian quantum collisional models into bipartite Markovian dynamics

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A wide class of non-Markovian completely positive master equations can be formulated on the basis of quantum collisional models. In this phenomenological approach the dynamics of an open quantum system is modeled through an ensemble of stochastic realizations that consist in the application at random times of a (collisional) completely positive transformation over the system state. In this paper, we demonstrate that these kinds of models can be embedded in bipartite Markovian-Lindblad dynamics consisting of the system of interest and an auxiliary one. In contrast with phenomenological formulations, here the stochastic ensemble dynamics an the interevent time interval statistics are obtained from a quantum measurement theory after assuming that the auxiliary system is continuously monitored in time. Models where the system intercollisional dynamics is non-Markovian [B. Vacchini, Phys. Rev. A **87**, 030101(R) (2013)] are also obtained from the present approach. The formalism is exemplified through bipartite dynamics that lead to non-Markovian system effects such as an environment-to-system backflow of information.

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I. INTRODUCTION

The description of open quantum systems through local-in-time Markovian evolutions is well understood from both mathematical [1] and physical [2] points of view. As is well known, under a completely positive condition, Lindblad equations provide the more general evolution structure of the system density matrix [1,2]. On the other hand, in recent years an ever-increasing interest has been paid to establishing a non-Markovian generalization of the open quantum system theory formulated in terms of nonlocal-in-time evolutions [3]. There exist diverse formalisms for describing memory effects. One leading program consists in generalizing Lindblad equations by replacing the rates of each dissipative channel by a time-convoluted kernel function. A wide class of both phenomenological [4–8] and theoretical approaches [9–22] were formulated for building and characterizing master equations of that kind, which in turn lead to a completely positive solution map.

In the category of phenomenological approaches, quantum collisional models (QCMs) provided a fundamental tool for establishing a non-Markovian generalization of Lindblad equations [4,5]. In this formalism, the evolution of an open quantum system follows from an average performed over an ensemble of stochastic realizations of the system state. Each realization consists in the application, at random times, of a completely positive transformation. The events can be read as a "collision" or interaction with the environment. Depending on the statistics of the collision times and the system interevent dynamics, different non-Markovian master equations were established [5–8]. Over that basis, the emergence of non-Markovian effects such as a system-to-environment backflow of information [23,24] were also analyzed in the recent literature [25,26].

The collisional superoperator, the interevent system dynamics, and the collision time statistics are the main ingredients of the approach. They must be defined, in an arbitrary way, from the beginning. Therefore, besides its usefulness, the QCM model does not have an associated microscopic description, and neither it is completely understood which kind of underlying mechanism may induce the structure of the stochastic dynamics. The main goal of this paper is to provide a rigorous physical framework to answer these issues.

The basic idea consists in embedding the non-Markovian system evolution in a Markovian bipartite dynamics. It is defined by the system of interest and an auxiliary (ancilla) system. We demonstrate that there exist bipartite Markovian interactions that induce the same system non-Markovian dynamics. In this way, "microscopic interactions" that lead to the master equations associated with the QCM are found. On the other hand, by assuming that the auxiliary system is continuously monitored in time, over the basis of a (Markovian) quantum jump approach [27–29], we find that the realizations of the QCM can be put in one-to-one correspondence with the realizations of the measurement apparatus. In this way, the stochastic dynamics of the QCM is established from a quantum measurement theory. In addition, this modeling allows to describe the interevent statistics from the Markovian-Lindblad description.

In Ref. [8] Vacchini introduced a generalized QCM where, in contrast to previous approaches [5–7], the system interevent dynamics is defined by a non-Markovian propagator. On the basis of an underlying tripartite Markovian dynamics, we show that this generalization can also be described within the present framework. Even when the stochastic realizations consist of successive collisional events with a non-Markovian interevent dynamics [8], they cannot be read as the result of a continuous measurement action performed over the system of interest. In fact, in contrast with the results of Ref. [22], here we demonstrate that QCMs can consistently be recovered when measuring the auxiliary ancilla system. The non-Markovian quantum jump approach developed in [22] relies on more general bipartite interactions. Additionally, the monitoring action is performed over the system of interest.

It is interesting to note that collisional models were also proposed as a phenomenological tool for deriving Markovian irreversible dynamics [1,30]. Furthermore, from a quantum information perspective [31], similar approaches were introduced by considering collisions with a string of auxiliary qubits systems [32–34]. When the system-string interaction is defined by partial swap and controlled-NOT qubits operations, specific Markovian master equations describe the system dynamics [33]. Generalization of these ideas to non-Markovian dynamics were considered recently in Refs. [35–37]. The stretched relation of these results with the present formalism is also investigated.

The paper is outlined as follows. In Sec. II we present the Markovian embedding, where the system density matrix is obtained by using projector techniques [3]. In Sec. III, from a standard quantum measurement theory, we obtain the stochastic ensemble dynamic after assuming that the auxiliary system is subjected to a measurement process. These results rely on the standard quantum jump approach [27–29] applied to bipartite dynamics. In Sec. IV, we analyze some examples that exhibit the main features of the present approach. A backflow of information from the system to the environment is explicitly shown. In Sec. V, some generalizations of the standard collisional approach are provided. The dynamics presented in Ref. [8] is recovered from a tripartite Markovian dynamics. The formalisms of Refs. [35–37] are analyzed in this context. In Sec. VI we present the conclusions.

II. MARKOVIAN EMBEDDING

In this section, it is demonstrated that non-Markovian QCMs can be obtained by tracing out a bipartite Markovian dynamics. We deal with the case of stationary renewal statistics.

A. Phenomenological renewal collisional models

The superoperator $\mathcal{E}_s$ that defines each collisional event is written as

$$\mathcal{E}_s[\rho]=\sum_\alpha V_\alpha\rho V_\alpha^\dagger,\qquad \sum_\alpha V_\alpha^\dagger V_\alpha=\mathrm{I}_s, \tag{1}$$

where the set of operators $\{V_\alpha\}$ act on the system Hilbert space. I_s is the identity matrix. Between collision events the system dynamics is defined by an arbitrary Lindblad generator $\mathcal{L}_s$ (unitary plus dissipative contributions). Thus, given that the last event happened at time t', the interevent evolution follows from the propagator $\exp[(t-t')\mathcal{L}_s]$. By assuming that the collision times define a renewal process, with waiting time distribution $w(t)$ [5], it is possible to demonstrate that the average system density matrix ρ_t^s is governed by the equation [6]:

$$\frac{d}{dt}\rho_t^s=\mathcal{L}_s\big[\rho_t^s\big]+\int_0^t dt'k(t-t')\mathcal{C}_s\big\{\exp[(t-t')\mathcal{L}_s]\rho_{t'}^s\big\}. \tag{2}$$

The superoperator $\mathcal{C}_s$ and the kernel function read

$$\mathcal{C}_s=\mathcal{E}_s-\mathrm{I}_s,\quad k(u)=\frac{uw(u)}{1-w(u)}, \tag{3}$$

where u is a Laplace variable $[f(u)\equiv\int_0^\infty dte^{-ut}f(t)]$. Notice that here, due to the assumed (stationary) renewal property, the kernel does not depend separately on the time variables t and t'. On the other hand, if $[\mathcal{C}_s,\mathcal{L}_s]=0$, in an interaction representation with respect to $\mathcal{L}_s$, Eq. (2) (under the replacement $\mathcal{L}_s\to 0$) recovers the evolution introduced in Ref. [5].

B. Bipartite Markovian dynamics

We introduce a bipartite arrangement defined by the system of interest S and an auxiliary (ancilla) system A. Their joint density matrix is ρ_t^{sa}. Therefore their marginal density matrices follow from a partial trace:

$$\rho_t^s=\mathrm{Tr}_a\big[\rho_t^{sa}\big],\quad \rho_t^a=\mathrm{Tr}_s\big[\rho_t^{sa}\big]. \tag{4}$$

The bipartite dynamics is defined by a Markovian Lindblad equation,

$$\frac{d}{dt}\rho_t^{sa}=\mathcal{L}\rho_t^{sa}=(\mathcal{L}_s+\mathcal{L}_a+\mathcal{C}_{sa})\rho_t^{sa}, \tag{5}$$

where the (arbitrary) Lindblad generators $\mathcal{L}_s$ and $\mathcal{L}_a$ define the system and ancilla dynamics, respectively. The contribution $\mathcal{C}_{sa}$ introduces their mutual interaction.

Now we ask about the possibility of finding specific system-ancilla interactions such that the marginal system density matrix ρ_t^s [Eq. (4)] fulfills the evolution (2). With this goal in mind, the superoperator $\mathcal{C}_{sa}$ is defined as

$$\mathcal{C}_{sa}[\rho]=\frac{1}{2}\sum_{\alpha,l}\gamma_l([V_{\alpha l},\rho V_{\alpha l}^\dagger]+[V_{\alpha l}\rho,V_{\alpha l}^\dagger]), \tag{6}$$

where γ_l are dissipative rates and the operator $V_{\alpha l}$ is

$$V_{\alpha l}=V_\alpha\otimes|a_l\rangle\langle a_0|. \tag{7}$$

The set of operators $\{V_\alpha\}$ are the same as those in Eq. (1). The states $\{|a_0\rangle,|a_l\rangle\}$, $l=1,2,\ldots,\dim\{\mathcal{H}_a\}-1$, form a complete orthogonal normalized basis in the Hilbert space $\mathcal{H}_a$ of the ancilla system. Hence, except for the state $|a_0\rangle$, the index l runs over all available states. Notice that operators (7) introduce irreversible ancilla transitions between the state $|a_0\rangle$ and any of the remaining possible states $|a_l\rangle$, that is, $|a_0\rangle\rightsquigarrow|a_l\rangle$.

1. *Ancilla dynamics*

With the previous choice of operators [Eq. (7)], it is simple to write down a closed Markovian evolution for the ancilla state ρ_t^a. From Eqs. (5) and (6) we get

$$\frac{d}{dt}\rho_t^a=\mathbb{L}_a\rho_t^a=(\mathcal{L}_a+\mathcal{C}_a)\rho_t^a. \tag{8}$$

The extra Lindblad term reads

$$\mathcal{C}_a\rho_t^a=\frac{1}{2}\sum_l\gamma_l\big(\big[A_l,\rho_t^aA_l^\dagger\big]+\big[A_l\rho_t^a,A_l^\dagger\big]\big), \tag{9}$$

where $A_l=|a_l\rangle\langle a_0|$. Straightforwardly, this superoperator can be rewritten as

$$\mathcal{C}_a\rho_t^a=-\tfrac{1}{2}\gamma\big\{|a_0\rangle\langle a_0|,\rho_t^a\big\}_++\gamma\langle a_0|\rho_t^a|a_0\rangle\bar{\rho}_a. \tag{10}$$

Here, $\{\cdots\}_+$ denotes an anticommutation operation, and the ancilla state $\bar{\rho}_a$ is

$$\bar{\rho}_a=\sum_l\frac{\gamma_l}{\gamma}|a_l\rangle\langle a_l|,\quad \gamma=\sum_l\gamma_l, \tag{11}$$

which in fact satisfies $\mathrm{Tr}_a[\bar{\rho}_a]=1$.

2. Non-Markovian system dynamics

In contrast to Eq. (8), the evolution of the system state ρ_t^s is non-Markovian. Its calculation is a little more involved, which here is obtained by using a projector technique [2,3]. I now introduce the projectors $\mathcal{P}$ and $\mathcal{Q}$:

$$\mathcal{P}\rho_t^{sa} = \mathrm{Tr}_a[\rho_t^{sa}] \otimes \bar{\rho}_a, \quad \mathcal{P} + \mathcal{Q} = \mathrm{I}_{sa}, \tag{12}$$

where I_{sa} is the identity matrix in the bipartite system-ancilla Hilbert space, and $\bar{\rho}_a$ is the ancilla state (11). The election of this projector definition will become clear in the next section.

The bipartite evolution (5) can be projected in relevant and irrelevant contributions [3]

$$\frac{d}{dt}\mathcal{P}\rho_t^{sa} = \mathcal{P}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho_t^{sa}, \tag{13}$$

$$\frac{d}{dt}\mathcal{Q}\rho_t^{sa} = \mathcal{Q}\mathcal{L}(\mathcal{P} + \mathcal{Q})\rho_t^{sa}. \tag{14}$$

On the other hand, in consistency with the projector definition (12), a separable state defines the bipartite initial condition,

$$\rho_0^{sa} = \rho_0^s \otimes \bar{\rho}_a, \tag{15}$$

where ρ_0^s is an arbitrary system state. With this initial state, it follows that $\mathcal{Q}\rho_0^{sa} = 0$. Therefore, Eq. (14) can be integrated [3] as $\mathcal{Q}\rho_t^{sa} = \int_0^t dt' \exp[\mathcal{Q}\mathcal{L}(t - t')]\mathcal{Q}\mathcal{L}\mathcal{P}\rho_{t'}^{sa}$, which in turn, after replacing in Eq. (13), leads to the convoluted evolution

$$\frac{d}{dt}\mathcal{P}\rho_t^{sa} = \mathcal{P}\mathcal{L}\mathcal{P}\rho_t^{sa} + \mathcal{P}\mathcal{L}\int_0^t dt' \exp[\mathcal{Q}\mathcal{L}(t - t')]\mathcal{Q}\mathcal{L}\mathcal{P}\rho_{t'}^{sa}. \tag{16}$$

The superoperator $\mathcal{L}$ is defined by Eq. (5). From Eqs. (6) and (7), it can be rewritten as

$$\begin{aligned}\mathcal{L}[\bullet] = (\mathcal{L}_s + \mathcal{L}_a)[\bullet] - \tfrac{1}{2}\gamma\{|a_0\rangle\langle a_0|,\bullet\}_+ \\ + \gamma\mathcal{E}_s[\langle a_0|\bullet|a_0\rangle] \otimes \bar{\rho}_a,\end{aligned} \tag{17}$$

where the collision superoperator $\mathcal{E}_s$ and the ancilla state $\bar{\rho}_a$ are defined by Eqs. (1) and (11), respectively. Equations (12) and (17) lead to

$$\mathcal{P}\mathcal{L}[\bullet] = \{\mathcal{L}_s(\mathrm{Tr}_a[\bullet]) + \gamma\mathcal{C}_s(\langle a_0|\bullet|a_0\rangle)\} \otimes \bar{\rho}_a, \tag{18}$$

where $\mathcal{C}_s$ follows from Eq. (3). With these last two expressions it is possible to evaluate all contributions in Eq. (16). By using that $\langle a_0|\bar{\rho}_a|a_0\rangle = 0$, we get $\mathcal{P}\mathcal{L}\mathcal{P}\rho_t^{sa} = \mathcal{L}_s[\rho_t^s] \otimes \bar{\rho}_a$, and $\mathcal{Q}\mathcal{L}\mathcal{P}\rho_{t'}^{sa} = \rho_{t'}^s \otimes \mathbb{L}_a[\bar{\rho}_a]$, where the ancilla superoperator $\mathbb{L}_a$ follows from Eq. (8). We have also used that $\mathcal{C}_a[\bar{\rho}_a] = 0$ [see Eqs. (10) and (11)]. Similarly, it is possible to demonstrate that $\mathcal{Q}\mathcal{L}(\rho_{t'}^s \otimes \mathbb{L}_a[\bar{\rho}_a]) = (\mathcal{L}_s + \mathbb{L}_a)(\rho_{t'}^s \otimes \mathbb{L}_a[\bar{\rho}_a])$, which by induction implies the expression

$$\exp[\mathcal{Q}\mathcal{L}t]\mathcal{Q}\mathcal{L}\mathcal{P}\rho_{t'}^{sa} = \exp[(\mathcal{L}_s + \mathbb{L}_a)t]\big(\rho_{t'}^s \otimes \mathbb{L}_a[\bar{\rho}_a]\big). \tag{19}$$

By introducing the previous results in Eq. (16), using that $\mathrm{Tr}_a[\mathbb{L}_a(\bullet)] = 0$, straightforwardly we recover the convoluted evolution (2) with the kernel function

$$k(t) = \gamma\,\langle a_0|\exp(t\mathbb{L}_a)\mathbb{L}_a[\bar{\rho}_a]\,|a_0\rangle\,, \tag{20a}$$

$$= \gamma\frac{d}{dt}\,\langle a_0|\exp(t\mathbb{L}_a)[\bar{\rho}_a]\,|a_0\rangle\,. \tag{20b}$$

This is the main result of this section. It demonstrates that the non-Markovian evolution (2) also arises as the marginal dynamics of a Markovian bipartite dynamics. In addition, here the kernel function is not arbitrary. In fact, it is completely determined from the ancilla dynamics [see Eqs. (8) and (20)]. Notice that the solution map $\rho_0^s \to \rho_t^s$ associated with the evolution (2) with the kernel (20) is, by construction, completely positive.

III. QUANTUM MEASUREMENT THEORY

In the previous section we have found an underlying bipartite Markovian dynamics that leads to the non-Markovian system dynamics. Here, over the same basis we find a clear physical interpretation to the ensemble of realizations [5,6] associated with the master equation (2).

A. Quantum jumps in the bipartite dynamics

The realizations of the collision model do not rely on a quantum measurement theory. Nevertheless, this link can be established by studying the bipartite dynamics when a measurement process is performed over the ancilla system. Specifically, we assume that the apparatus is sensitive to all ancilla transitions $|a_0\rangle \rightsquigarrow |a_l\rangle$. As the bipartite dynamics is Markovian, from a standard quantum jump approach [27,28] it is possible to associate each realization of the monitoring process with a realization in the system-ancilla Hilbert space such that

$$\rho_t^{sa} = \overline{\rho_{sa}^{\mathrm{st}}(t)}. \tag{21}$$

Here, $\rho_{sa}^{\mathrm{st}}(t)$ is a stochastic density matrix and the overbar denotes an ensemble average. The time evolution of ρ_t^{sa} is defined by Eq. (5). As usual, the stochastic dynamics of $\rho_{sa}^{\mathrm{st}}(t)$ consists of disruptive transformations associated to each recording event, while in the intermediate time intervals it is smooth and nonunitary [27,28].

Consistently with a quantum measurement theory [2], in each detection event the bipartite state suffers the (measurement) transformation

$$\rho \to \mathcal{M}\rho = \frac{\mathcal{J}\rho}{\mathrm{Tr}_{sa}[\mathcal{J}\rho]}, \tag{22}$$

where the superoperator $\mathcal{J}$ takes into account all possible transitions $|a_0\rangle \rightsquigarrow |a_l\rangle$ that lead to a detection event. Assuming that $\mathcal{L}_a$ does not induce these kinds of transitions, from Eq. (6) we write

$$\mathcal{M}\rho = \frac{\sum_{\alpha l}\gamma_l V_{\alpha l}\rho V_{\alpha l}^\dagger}{\{\mathrm{Tr}_{sa}[\sum_{\alpha l}\gamma_l V_{\alpha l}^\dagger V_{\alpha l}\rho]\}}, \tag{23a}$$

$$= \frac{\mathcal{E}_s\,\langle a_0|\,\rho\,|a_0\rangle}{\mathrm{Tr}_s[\langle a_0|\,\rho\,|a_0\rangle]} \otimes \bar{\rho}_a. \tag{23b}$$

This last expression follows from the definition of the operators $V_{\alpha l}$ [Eq. (7)]. On the other hand, the conditional evolution of $\rho_{sa}^{\mathrm{st}}(t)$ between detection events is given by the normalized propagator [27,28]

$$\mathcal{T}_c(t)\rho = \frac{\mathcal{T}(t)\rho}{\mathrm{Tr}_{sa}[\mathcal{T}(t)\rho]}, \tag{24}$$

where the un-normalized propagator $\mathcal{T}(t)$ is

$$\mathcal{T}(t)\rho = \exp[t\mathcal{D}]\rho. \tag{25}$$

Here, the exponential superoperator is defined by the generator $\mathcal{D}$, which is the complement of $\mathcal{J}$, that is, $\mathcal{L} = \mathcal{D} + \mathcal{J}$. Hence, from Eq. (6) it reads

$$\mathcal{D}\rho = (\mathcal{L}_s + \mathcal{L}_a)\rho - \frac{\gamma}{2}\{|a_0\rangle\langle a_0|,\rho\}_+. \tag{26}$$

The measurement transformation $\mathcal{M}$ and the propagator $\mathcal{T}_c(t)$ completely define the structure of the realizations of $\rho_{sa}^{\mathrm{st}}(t)$. It only remains to define an algorithm that allows the random detection times to be obtained. Here they are characterized through a survival probability function $P_0(t|\rho)$ [27]. Given that at time τ the bipartite system state is ρ, the probability of not having any detection up to time t is [29]

$$P_0(t-\tau|\rho) = \mathrm{Tr}_{sa}[\mathcal{T}(t-\tau)\rho] = \mathrm{Tr}_{sa}[e^{t\mathcal{D}}\rho]. \tag{27}$$

With this function the realizations can be obtained as follows: Given the initial state ρ_0^{sa}, the time t_1 of the first detection event follows by solving the equation $P_0(t_1 - 0|\rho_0^{sa}) = r$, where r is a random number in the interval $(0,1)$. The dynamic of $\rho_{sa}^{\mathrm{st}}(t)$ in the interval $(0,t_1)$ is defined by Eq. (24). At $t = t_1$ the disruptive transformation [Eq. (23)] $\rho_s^{\mathrm{st}}(t_1) \to \mathcal{M}\rho_{sa}^{\mathrm{st}}(t_1)$ is applied. The subsequent dynamics is the same. In fact, after the nth measurement event at time t_n, $\rho_{sa}^{\mathrm{st}}(t_n) \to \mathcal{M}\rho_{sa}^{\mathrm{st}}(t_n)$, the time t_{n+1} for the next detection event follows from $P_0(t_{n+1} - t_n|\mathcal{M}\rho_{sa}^{\mathrm{st}}(t_n)) = r$, where again r is a random number in the interval $(0,1)$. The dynamic in the interval (t_n,t_{n+1}) is defined by the conditional propagator (24). The realizations generated with this algorithm fulfill Eq. (21) (see, for example, Appendix A of Ref. [22]).

B. Stochastic realizations

The standard quantum jump approach allows the realizations of $\rho_{sa}^{\mathrm{st}}(t)$ to be defined. Straightforwardly from this it is possible to obtain the partial stochastic dynamics of each system,

$$\rho_s^{\mathrm{st}}(t) = \mathrm{Tr}_a\big[\rho_{sa}^{\mathrm{st}}(t)\big], \quad \rho_a^{\mathrm{st}}(t) = \mathrm{Tr}_s\big[\rho_{sa}^{\mathrm{st}}(t)\big]. \tag{28}$$

Furthermore, from Eq. (21), the relations $\rho_t^s = \overline{\rho_s^{\mathrm{st}}(t)}$ and $\rho_t^a = \overline{\rho_a^{\mathrm{st}}(t)}$ are also valid. Given the separable initial condition (15), from Eqs. (23) and (26) it is simple to realize that $\rho_{sa}^{\mathrm{st}}(t)$ becomes separable at all times:

$$\rho_{sa}^{\mathrm{st}}(t) = \rho_s^{\mathrm{st}}(t) \otimes \rho_a^{\mathrm{st}}(t). \tag{29}$$

In fact, given the absence of initial correlations, the conditional dynamic (24) remains separable [see Eq. (26)]. Furthermore, in each detection event, given a separable input, the postmeasurement state also becomes separable. Nevertheless, notice that $\rho_s^{\mathrm{st}}(t)$ and $\rho_a^{\mathrm{st}}(t)$ are statistically correlated. Below, we describe their dynamics.

1. Ancilla realizations

After taking a partial trace over Eq. (23), from Eq. (29) we deduce that in each measurement event the ancilla state suffers the transformation

$$\rho_a^{\mathrm{st}}(t) \to \mathrm{Tr}_s\big[\mathcal{M}\rho_{sa}^{\mathrm{st}}(t)\big] = \frac{\mathcal{J}_a\rho_a^{\mathrm{st}}(t)}{\mathrm{Tr}_a\big[\mathcal{J}_a\rho_a^{\mathrm{st}}(t)\big]} = \bar{\rho}_a, \tag{30}$$

where the ancilla superoperator $\mathcal{J}_a$ is

$$\mathcal{J}_a[\rho] = \gamma\,\langle a_0|\,\rho\,|a_0\rangle\,\bar{\rho}_a. \tag{31}$$

Hence, the collapsed ancilla state is always the same [Eq. (11)]. Similarly, from Eqs. (25) and (26), we deduce that between detection events the conditional ancilla dynamics is defined by the (un-normalized) superoperator $\mathcal{T}_a(t)\rho = \exp[t\mathcal{D}_a]\rho$, where

$$\mathcal{D}_a\rho = \mathcal{L}_a\rho - \frac{\gamma}{2}\{|a_0\rangle\langle a_0|,\rho\}_+. \tag{32}$$

For separable initial conditions, this propagator also applies at the initial time. This simplification explains the chosen initial state (15) and the projectors (12).

From Eqs. (26) and (27), we notice that the survival probability can be rewritten as $[P_0(t-\tau|\rho) \to P_0(t-\tau)]$:

$$P_0(t-\tau) = \mathrm{Tr}_a\{\exp[\mathcal{D}_a(t-\tau)]\bar{\rho}_a\}. \tag{33}$$

In fact, the ancilla state is always the same after a detection event. Consequently, the measurement statistics correspond to a renewal process, that is, the interevent probability distribution is always the same. On the other hand, it is simple to realize that the measurement transformation (30), the conditional ancilla dynamics defined by Eq. (32), and the survival probability (33) also arise by formulating the quantum jump approach over the basis of Eq. (8). In fact, $\mathbb{L}_a = \mathcal{D}_a + \mathcal{J}_a$.

2. System realizations

Given the separability property (29), from Eq. (23) it follows that in each detection event (ancilla measurement apparatus) the system suffers the transformation

$$\rho_s^{\mathrm{st}}(t) \to \mathrm{Tr}_a\big[\mathcal{M}\rho_{sa}^{\mathrm{st}}(t)\big] = \mathcal{E}_s\big[\rho_s^{\mathrm{st}}(t)\big], \tag{34}$$

that is, the transformation associated to a collision event. On the other hand, given that a measurement event happened at time τ, from Eqs. (24) and (26) we deduce that the posterior system conditional evolution is given by

$$\rho_s^{\mathrm{st}}(t) = \mathrm{Tr}_a\big[\mathcal{T}_c(t-\tau)\rho_{sa}^{\mathrm{st}}(\tau)\big] = \exp[\mathcal{L}_s(t-\tau)]\rho_s^{\mathrm{st}}(\tau). \tag{35}$$

This interevent evolution also corresponds to the dynamics of the QCM. Therefore, by assuming that the measurement process is performed over the ancilla system, the realizations of the system of interest have the same structure than in the phenomenological QCM. This is the main result of this section. Notice that each system collisional event happens when the measurement apparatus detects an ancilla transition.

The renewal property of the realizations was proven previously. In fact, from the survival probability (33) we define the waiting time distribution $w(t) = -(d/dt)P_0(t)$, which delivers

$$w(t) = -\mathrm{Tr}_a\{\mathcal{D}_a\exp[t\mathcal{D}_a]\bar{\rho}_a\}, \tag{36a}$$

$$= \gamma\,\langle a_0|\exp(t\mathcal{D}_a)[\bar{\rho}_a]\,|a_0\rangle. \tag{36b}$$

In deriving this expression we used Eq. (32) and that $\mathrm{Tr}_a[\mathcal{L}_a\rho] = 0$. Hence, in the present modeling the quantum jump approach allows the waiting time distribution to be written in terms of the ancilla dynamics. Indeed, from Eqs. (34) and (35), we deduce that the ancilla dynamics mainly determine the statistics of the system realizations.

C. Consistence between master equation and ensemble of realizations

For showing the consistency of the developed results, it remains to demonstrate that the waiting time distribution (36), which determines the realizations statistics, and the kernel (20), which determines the density matrix evolution, fulfill in the Laplace domain the relation (3).

The Laplace transform of Eq. (36) reads

$$w(u) = \gamma \langle a_0| \frac{1}{u - \mathcal{D}_a}[\bar{\rho}_a] |a_0\rangle , \quad (37)$$

while from Eq. (20) we obtain

$$\frac{k(u)}{u} = \gamma \langle a_0| \frac{1}{u - \mathbb{L}_a}[\bar{\rho}_a] |a_0\rangle . \quad (38)$$

In deriving this expression we used that $\langle a_0| \bar{\rho}_a |a_0\rangle = 0$. On the other hand, using that $\mathbb{L}_a = \mathcal{D}_a + \mathcal{J}_a$, it follows that

$$\frac{1}{u - \mathbb{L}_a} = \sum_{n=0}^{\infty} \left[\frac{1}{u - \mathcal{D}_a}\mathcal{J}_a \right]^n \frac{1}{u - \mathcal{D}_a}. \quad (39)$$

By introducing this expression in Eq. (38) and by using the definition (31) we get

$$\frac{k(u)}{u} = \sum_{n=1}^{\infty} w(u) = \frac{w(u)}{1 - w(u)}, \quad (40)$$

which recovers the relation (3) associated to the phenomenological approach.

IV. EXAMPLE

In this section, we study the dynamics of a two-level system, which in turn may be read, for example, as a qubit unit. In quantum information arrangements it is expected that decoherence and dissipation are "mediated" by interactions with extra quantum subunits. Therefore, as ancilla we consider another system whose dynamics is able to develop quantum coherent effects. For simplicity it is also taken as a two-level system.

In the approach developed in the previous sections, the collision statistics is completely defined by the ancilla dynamics. Hence, in the next example, its structure depends on underlying quantum coherent effects. We remark that this feature is foreign in phenomenological formulations, where the waiting time distribution is usually defined by a linear combination of decaying exponential functions [5–8]. We demonstrate that these kinds of distributions arise when the ancilla dynamics is completely incoherent. This property motivates the dynamics studied below. Both dephasing and dissipative channels are formulated.

A. Dephasing channel

As system we consider a two-level system whose Hamiltonian reads $H_s = \hbar\omega_s\sigma_z/2$, where ω_s is the transition frequency between its eigenstates, denoted as $|\pm\rangle$, while σ_z is the z-Pauli matrix. The ancilla system is also a two-level system. In an interaction representation with respect to H_s, the evolution of the bipartite state ρ_t^{sa} reads

$$\frac{d\rho_t^{sa}}{dt} = \frac{-i\Delta}{2}\left[\mathrm{I}_s \otimes \sigma_x, \rho_t^{sa}\right] + \frac{\gamma}{2}\left(\left[V, \rho_t^{sa} V^\dagger\right] + \left[V\rho_t^{sa}, V^\dagger\right]\right). \quad (41)$$

The first unitary contribution defines the ancilla Hamiltonian. It is given by the x-Pauli matrix σ_x, written in the basis of σ_z eigenstates: $|\pm\rangle$. The Lindblad contribution is written in terms of the operator [see Eq. (7)]

$$V = \sigma_z \otimes \sigma. \quad (42)$$

Here, $\sigma = |-\rangle\langle+|$ is the lowering operator acting on the ancilla states $|\pm\rangle$. Hence, V leads to a dissipative coupling between both systems. The initial bipartite state [see Eq. (15)] is taken as

$$\rho_0^{sa} = \rho_0^s \otimes |-\rangle\langle-|, \quad (43)$$

where ρ_0^s is an arbitrary system state. The ancilla begins in its lower state.

Performing the partial trace $\rho_t^a = \mathrm{Tr}_s[\rho_t^{sa}]$, the bipartite evolution (41) leads to

$$\frac{d\rho_t^a}{dt} = \frac{-i\Delta}{2}\left[\sigma_x, \rho_t^a\right] + \frac{\gamma}{2}\left(\left[\sigma, \rho_t^a \sigma^\dagger\right] + \left[\sigma\rho_t^a, \sigma^\dagger\right]\right). \quad (44)$$

This marginal ancilla dynamics corresponds to a quantum fluorescent system [2,28], where γ defines its natural decay rate while Δ is the Rabi frequency. On the other hand, the interaction defined by Eq. (42) leads to a decoherence system channel [33]. Hence, only the system coherences are affected by the undesirable interaction.

1. System stochastic realizations

The measurement apparatus records the ancilla transitions $|+\rangle \rightsquigarrow |-\rangle$. Therefore, from Eqs. (23) and (42) we deduce that in each measurement event the ancilla collapses to its ground state $\bar{\rho}_a = |-\rangle\langle-|$, while the system suffers the completely positive transformation

$$\mathcal{E}_s[\rho] = \sigma_z\rho\sigma_z. \quad (45)$$

As is well known, this superoperator leads to a change of sign in the system coherences [5]. On the other hand, during the successive measurement events the system dynamics is frozen, that is, it does not evolve. This conclusion follows from Eqs. (35) and (41).

The statistics of the time interval between successive detection events define a renewal process. Its probability distribution is given by Eq. (36). Under the associations $|a_0\rangle \rightarrow |+\rangle$, and $\mathcal{D}_a[\rho] = -(i\Delta/2)[\sigma_x,\rho] - (1/2)\gamma\{\sigma^\dagger\sigma,\rho\}_+$, we get the waiting time distribution

$$w(t) = 4\gamma\Delta^2 e^{-\gamma t/2}\left\{\frac{\sinh[(t/4)\sqrt{\gamma^2 - 4\Delta^2}]}{\sqrt{\gamma^2 - 4\Delta^2}}\right\}^2. \quad (46)$$

Notice that Eqs. (45) and (46) completely define the system realizations.

In Fig. 1 we show a realization of the system coherence $\langle+|\rho_s^{\mathrm{st}}(t)|-\rangle$. In order to show the consistence of the developed approach, it was obtained from the realizations of

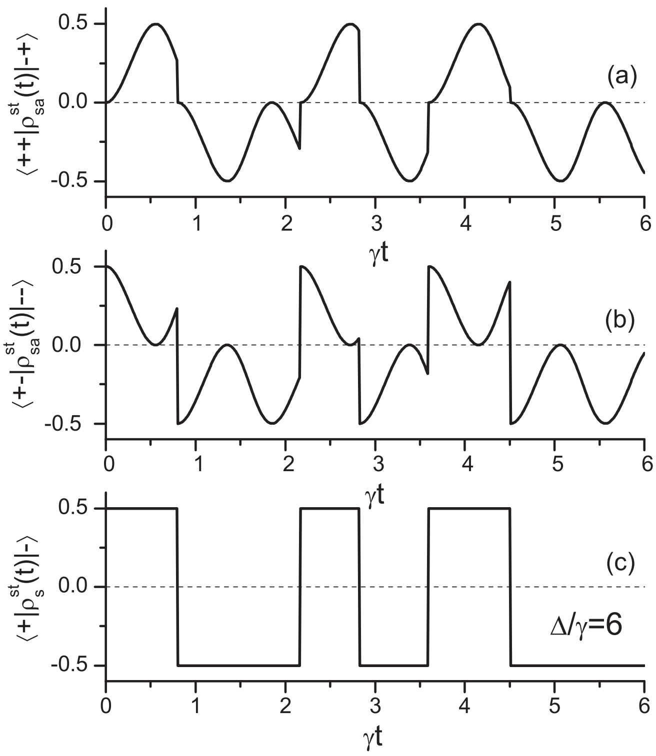

FIG. 1. Realizations of matrix elements of the stochastic density matrix $\rho_{st}^{sa}(t)$ and $\rho_{st}^{s}(t)$: (a) $\langle ++|\,\rho_{sa}^{st}(t)\,|-+\rangle$, (b) $\langle +-|\,\rho_{sa}^{st}(t)\,|--\rangle$, and (c) $\langle +|\,\rho_{s}^{st}(t)\,|-\rangle$. The characteristic parameters of the bipartite evolution (41) satisfy $\Delta/\gamma = 6$.

the underlying bipartite dynamics,

$$\langle +|\rho_s^{st}(t)|-\rangle = \langle ++|\,\rho_{sa}^{st}(t)|-+\rangle + \langle +-|\rho_{sa}^{st}(t)|--\rangle, \tag{47}$$

that is, from the partial trace of $\rho_{sa}^{st}(t)$. The states $\{|ij\rangle\}$, $i,j = +,-$, provide a complete basis of the bipartite Hilbert space. The realizations of $\rho_{sa}^{st}(t)$ follow from a "standard Markovian quantum jump approach" formulated on the basis of Eq. (41). We have taken the initial condition $\rho_{sa}^{st}(0) = |x_+\rangle\langle x_+|\otimes|-\rangle\langle -|$, where $|x_+\rangle = (1/\sqrt{2})(|+\rangle + |-\rangle)$ is an eigenstate of σ_x. In Fig. 1(a), we see that in each recording event the bipartite coherence $\langle ++|\,\rho_{sa}^{st}(t)\,|-+\rangle$ collapses to zero:

$$\langle ++|\,\mathcal{M}\rho_{sa}^{st}(t)\,|-+\rangle = 0. \tag{48}$$

This result follows from the action of the operator (42), which induces the ancilla transitions $|+\rangle \rightsquigarrow |-\rangle$. On the other hand, the bipartite coherence $\langle +-|\,\rho_{sa}^{st}(t)\,|--\rangle$ suffers the disruptive changes $\langle +-|\,\rho_{sa}^{st}(t)\,|--\rangle \to -\langle +-|\,\rho_{sa}^{st}(0)\,|--\rangle$ [Fig. 1(b)]. By calculating the measurement transformation (23), from Eq. (42) we get

$$\langle +-|\,\mathcal{M}\rho_{sa}^{st}\,|--\rangle = \frac{-\langle ++|\,\rho_{sa}^{st}\,|-+\rangle}{\langle ++|\,\rho_{sa}^{st}\,|++\rangle + \langle -+|\,\rho_{sa}^{st}\,|-+\rangle}.$$

By an explicit calculation of the conditional evolution defined by the operator $\mathcal{D}$ [Eq. (26)], it follows that the quotient of the previous bipartite matrix elements is an invariant of the conditional evolution, delivering the observed property

$$\langle +-|\,\mathcal{M}\rho_{sa}^{st}(t)\,|--\rangle = -\langle +|\,\rho_s^{st}(0)\,|-\rangle\,, \tag{49}$$

where we have used that $\langle +-|\,\rho_{sa}^{st}(0)\,|--\rangle = \langle +|\,\rho_s^{st}(0)\,|-\rangle$ [Eq. (43)]. Therefore, in each measurement event the coherence $\langle +-|\,\rho_{sa}^{st}(t)\,|--\rangle$, besides a change of sign, recovers its initial value.

In Fig. 1(c) we plot the realization of $\langle +|\,\rho_s^{st}(t)\,|-\rangle$ obtained from Eq. (47), that is, by adding the two bipartite coherences. As both coherences $\langle ++|\,\mathcal{M}\rho_{sa}^{st}(t)\,|-+\rangle$ and $\langle +-|\,\mathcal{M}\rho_{sa}^{st}(t)\,|--\rangle$ always oscillate in a complementary way, during the interevent time intervals $\langle +|\,\rho_s^{st}(t)\,|-\rangle$ is constant, while in the measurement events it changes sign. In this way, we explicitly show that the underlying quantum jump approach led to the realizations of the phenomenological collision model. In fact, the action of the superoperator (45) only introduces a change of sign in the system coherences. In a similar way, it is possible to show that the system populations are not affected by the dynamics, that is, $\langle \pm|\,\rho_s^{st}(t)\,|\pm\rangle = \langle \pm|\,\rho_s^{st}(0)\,|\pm\rangle$.

2. *Density matrix evolution*

In Fig. 2 we show the average coherence behavior obtained from the ensemble of realizations shown in Fig. 1 (noisy curve). Furthermore, we present the exact solution of the coherence that follows from the master equation (2) (black full line). Taking into account the underlying Lindblad equation (41), it can be written as

$$\frac{d}{dt}\rho_t^s = \int_0^t dt' k(t-t')\mathcal{C}_s[\rho_{t'}^s]. \tag{50}$$

The superoperator $\mathcal{C}_s = (\mathcal{E}_s - \mathrm{I}_s)$, from Eq. (45), reads

$$\mathcal{C}_s[\bullet] = \tfrac{1}{2}([\sigma_z,\bullet\,\sigma_z] + [\sigma_z\,\bullet\,,\sigma_z]). \tag{51}$$

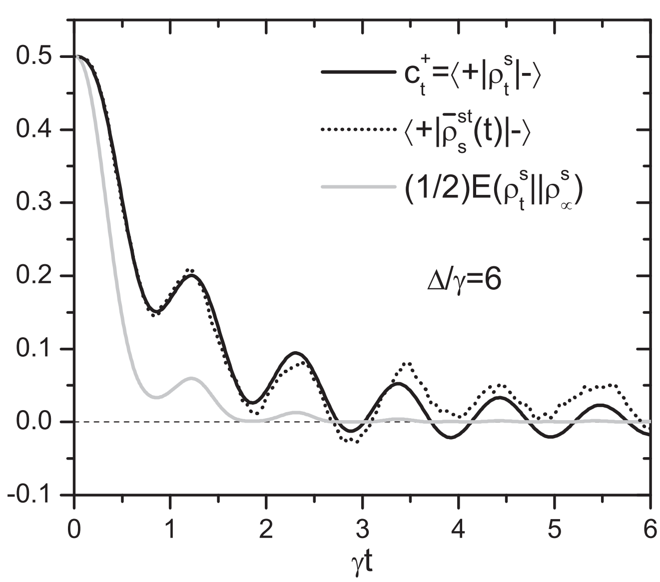

FIG. 2. System coherence. Full line, exact solution Eq. (53). Dotted (noisy) line, average coherence $\langle +|\,\overline{\rho_s^{st}(t)}\,|-\rangle$ obtained by averaging 10^3 realizations. Gray line, relative entropy $E(\rho_t^s||\rho_\infty^s)$, Eq. (55). The parameters are the same as in Fig. 1, $\Delta/\gamma = 6$.

On the other hand, the kernel is determined by the general expression (20). From Eq. (44) it follows that

$$k(t) = 2\gamma\Delta^2 e^{-(3/4)\gamma t}\left\{\frac{\sinh[(t/4)\sqrt{\gamma^2-16\Delta^2}]}{\sqrt{\gamma^2-16\Delta^2}}\right\}. \quad (52)$$

This kernel and the waiting time distribution (46) fulfill the Laplace relation (3).

Consistently with the system stochastic realizations, Eq. (50) does not modify the populations, $\langle\pm|\,\rho_t^s\,|\pm\rangle = \langle\pm|\,\rho_0^s\,|\pm\rangle$. On the other hand, working in a Laplace domain, the coherences $c_t^{\pm} \equiv \langle\pm|\,\rho_t^s\,|\mp\rangle$ read

$$c_t^{\pm} = c_0^{\pm}\left\{e^{-\gamma t}\frac{2\Delta^2}{\gamma^2+2\Delta^2} + e^{-\gamma t/4}\left[\frac{\gamma^2}{\gamma^2+2\Delta^2}\cosh(\varphi t) + \frac{\gamma(\gamma^2+8\Delta^2)}{4(\gamma^2+2\Delta^2)}\frac{\sinh(\varphi t)}{\varphi}\right]\right\}, \quad (53)$$

where for shortening the expression we introduced the "frequency,"

$$\varphi = \sqrt{(\gamma/4)^2-\Delta^2}. \quad (54)$$

The same expression follows from the alternative solution $c_t^{\pm} = \langle\pm|\,\rho_t^s\,|\mp\rangle = \langle\pm\pm|\,\rho_t^{sa}\,|\mp\pm\rangle + \langle\pm\mp|\,\rho_t^{sa}\,|\mp\mp\rangle$, where ρ_t^{sa} is the solution of the bipartite evolution (41). Notice that in Eq. (53), besides a monotonic decaying contribution, the two remaining terms may develop an oscillatory behavior. As shown in Fig. 2, Eq. (53) correctly fits the average ensemble behavior.

3. *Environment-to-system backflow of information*

The analysis of Refs. [25,26] demonstrate that QCMs may lead to non-Markovian effects such as an environment-to-system backflow of information [23]. This property or phenomenon can be defined on the basis of "any measure" that in the Markovian case presents a monotonic time decay behavior [2]. One well-known example is the relative entropy between two states [24,25]. As we are not interested in quantifying the non-Markovian effects, for simplicity here we consider the relative entropy with respect to the stationary state,

$$E\left(\rho_t^s\|\rho_\infty^s\right) = \mathrm{Tr}_s\left[\rho_t^s\left(\ln_2\rho_t^s - \ln_2\rho_s^\infty\right)\right], \quad (55)$$

where $\rho_s^\infty = \lim_{t\to\infty}\rho_t^s$. Hence, the backflow of information arises if there exist times $t_2 > t_1$ such that $E(\rho_{t_2}^s\|\rho_\infty^s) > E(\rho_{t_1}^s\|\rho_\infty^s)$. Below we show that this feature arises in the dynamics described previously.

In Fig. 2 we also plotted $E(\rho_t^s\|\rho_\infty^s)$ (gray full line) where ρ_t^s is the solution of Eq. (50). The stationary state is the diagonal matrix $\rho_\infty^s = \mathrm{diag}\{\langle+|\,\rho_0^s\,|+\rangle, \langle-|\,\rho_0^s\,|-\rangle\}$. Clearly the time behavior is nonmonotonous, indicating a backflow of information. Furthermore, the oscillatory behavior of $E(\rho_t^s\|\rho_\infty^s)$ is correlated with the oscillatory behavior of the coherences, which arise whenever φ is a complex quantity, that is, from Eq. (54), $\Delta > (\gamma/4)$.

4. *Incoherent ancilla dynamics*

For the dynamics (41), the ancilla dynamics develop quantum coherent effects [Eq. (44)], which in turn determine the waiting time distribution [Eq. (46)]. Here, we introduce an alternative ancilla dynamics which only induces incoherent transitions. Instead of Eq. (41), for the same system S, we take the bipartite evolution as

$$\frac{d\rho_t^{sa}}{dt} = \frac{\gamma}{2}\left(\left[V,\rho_t^{sa}V^\dagger\right] + \left[V\rho_t^{sa},V^\dagger\right]\right) + \frac{\beta}{2}\left(\left[A,\rho_t^{sa}A^\dagger\right] + \left[A\rho_t^{sa},A^\dagger\right]\right), \quad (56)$$

with initial condition $\rho_0^{sa} = \rho_0^s \otimes |-\rangle\langle-|$, while

$$V = \sigma_z\otimes\sigma, \quad A = \mathrm{I}_s\otimes\sigma^\dagger. \quad (57)$$

Hence, the ancilla dynamics [Eq. (8)] only lead to the incoherent (classical) transitions $|+\rangle \overset{\gamma}{\rightsquigarrow} |-\rangle$ and $|-\rangle \overset{\beta}{\rightsquigarrow} |+\rangle$. The corresponding statistical behavior is defined by a (two-level) classical rate master equation.

We assume that the recording apparatus is only sensitive to the ancilla transition $|+\rangle \rightsquigarrow |-\rangle$, that is, the transition induced by the operator V. In this situation, from Eqs. (23) and (57), we deduce that the collisional superoperator again reads $\mathcal{E}_s[\rho] = \sigma_z\rho\sigma_z$ [Eq. (45)]. Thus, the system evolution is given by Eq. (50). Nevertheless, the kernel follows from Eq. (3), where the waiting time distribution can be calculated from Eq. (36). We get

$$w(u) = \left(\frac{\gamma}{u+\gamma}\right)\left(\frac{\beta}{u+\beta}\right). \quad (58)$$

In the time domain $w(t)$ is the convolution of two exponential functions. The system coherences become $c_u^{\pm} = c_0^{\pm}(u+\gamma+\beta)/[2u^2+2u(\gamma+\beta)+\gamma\beta]$, which can also be written as a lineal combination of decaying exponential functions. Independently of the initial conditions, in this case the dynamics does not present an environment-to-system backflow of information, suggesting that underlying coherent effects may be necessary for the development of this phenomenon.

Taking an ancilla system with a higher number of states, all of them coupled via incoherent transitions, the waiting time distribution becomes defined by more complex expressions which in the time domain are linear combinations of decaying exponential functions. For example, taking a unidirectional coupling $|a_0\rangle \rightsquigarrow |a_1\rangle \rightsquigarrow \cdots |a_m\rangle \rightsquigarrow |a_0\rangle$, all of them with rate γ, the waiting time distribution becomes $w(u) = [\gamma/(u+\gamma)]^{m+1}$. These kinds of distributions, which rely on incoherent ancilla dynamics, were considered, for example, in Ref. [8].

B. Dissipative channels

In the previous example, Eqs. (50) and (51) define a non-Markovian decoherence channel. One may also consider interactions that lead to dissipative channels. For example, by maintaining the ancilla dynamics (44), a depolarizing [31] non-Markovian channel arises by introducing two bipartite Lindblad terms [$\alpha = x,y$ in Eq. (6)] defined by the operators $V_x = \sqrt{p}\sigma_x\otimes\sigma$, and $V_y = \sqrt{1-p}\sigma_y\otimes\sigma$, where the parameter p satisfies $0 < p < 1$. With the same collision statistics [Eq. (46)], in this case the stationary system state becomes $\rho_\infty^s = (1/2)\,\mathrm{I}_s$. A thermal stationary state can be obtained by considering a generalized amplitude damping superoperator [31].

V. GENERALIZED COLLISIONAL MODELS

In the previous sections we associated the basic master equation of the collision model [Eq. (2)] with an underlying Markovian microscopic dynamics [Eq. (5)]. Furthermore, the realizations of the model, given that the ancilla system is continuously monitored in time, were established on the basis of the quantum jump approach. In this section, we show that these results also apply in different possible generalizations of the basic approach.

A. Nonstationary renewal collision dynamics

The basic ingredients of the present approach remain valid when the evolution of the ancilla system, in the bipartite Lindblad dynamics (5), depends explicitly on time, $\mathcal{L}_a \to \mathcal{L}_a(t)$. Under this situation, the main change is the measurement statistics. While it remains a renewal process, the waiting time distribution explicitly depends on the observation time. This case can be worked out with the elements introduced in the previous sections.

B. Nonrenewal collision statistics

With the same system realizations, the formalism may become nonrenewal when the measurement process is nonrenewal. Basically this situation occurs whenever the ancilla resetting state is not always the same. This case arises, for example, when the operators (7) are generalized as

$$V_{\alpha \mathrm{lk}} = V_\alpha \otimes |a_l\rangle \langle a_0^k| . \quad (59)$$

Hence, instead of a unique state $|a_0\rangle$, here many of them play the same role. Assuming that the measurement apparatus is sensitive to "all transitions" $|a_0^k\rangle \rightsquigarrow |a_l\rangle$, the stochastic ancilla becomes nonrenewal. This case may correspond, for example, to optical cascade systems [29].

While the structure of the system realizations remains the same, the statistics of the interevent time intervals can only be determined by knowing the ancilla state at all times. Therefore, for generating the system realizations, unavoidably one must also generate the ancilla realizations.

C. Non-Markovian intercollision dynamics

Maintaining the renewal property, in Ref. [8] Vacchini introduced an interesting generalization that consists in assuming that the interevent dynamics is non-Markovian. This situation naturally arises when considering a system interacting successively with a string of qubits systems [35–37].

Instead of the Markovian evolution defined by Eq. (35), it is taken as

$$\rho_s^{\mathrm{st}}(t) = \mathcal{G}(t-\tau)\big[\rho_s^{\mathrm{st}}(\tau)\big], \quad (60)$$

where $\mathcal{G}(t)$ is an arbitrary (trace preserving), completely positive propagator that cannot be written as a semigroup, $\mathcal{G}(t) \neq \exp[t\mathcal{L}_s]$ [8]. Here, we demonstrate that this case can also be covered with the present formalism.

The generalized QCM can be embedded in a tripartite underlying Lindblad equation. Hence, besides the system of interest S, the ancilla system A, we consider an extra auxiliary system B. The evolution of their joint density matrix ρ_t^{sab} is written as

$$\frac{d}{dt}\rho_t^{\mathrm{sab}} = \mathcal{L}\rho_t^{\mathrm{sab}} = (\mathcal{L}_{sb} + \mathcal{L}_a + \mathcal{C}_{\mathrm{sab}})\rho_t^{\mathrm{sab}}. \quad (61)$$

The first superoperator reads

$$\mathcal{L}_{sb} = \mathcal{L}_s + \mathcal{L}_b + \mathcal{C}_{sb}. \quad (62)$$

Here, $\mathcal{L}_s$ and $\mathcal{L}_b$ are arbitrary Lindblad equations for the systems S and B, respectively. $\mathcal{C}_{sb}$ is an extra Lindblad contribution that introduces an arbitrary interaction (unitary and dissipative) between them. As before, $\mathcal{L}_a$ defines the dynamics of the ancilla system A. The contribution $\mathcal{C}_{\mathrm{sab}}$ introduces a dissipative interaction between the three systems,

$$\mathcal{C}_{\mathrm{sab}}[\rho] = \frac{1}{2}\sum_{\alpha,l,m}\gamma_l([V_{\alpha \mathrm{lm}},\rho V_{\alpha \mathrm{lm}}^\dagger] + [V_{\alpha \mathrm{lm}}\rho, V_{\alpha \mathrm{lm}}^\dagger]), \quad (63)$$

where γ_l are the dissipative rates and the operators are

$$V_{\alpha \mathrm{lm}} = V_\alpha \otimes |a_l\rangle\langle a_0| \otimes |b_0\rangle\langle b_m| . \quad (64)$$

The system operators V_α and the states $|a_l\rangle$ are the same as in Eq. (7), where the index $l = 1,2,\ldots,\dim\{\mathcal{H}_a\}-1$ does not include the single state $|a_0\rangle$. On the other hand, the states $|b_m\rangle$, $m = 0,1,\ldots \dim \mathcal{H}_b - 1$ form a complete orthonormal basis in the Hilbert space $\mathcal{H}_b$ of B. Notice that here the state $|b_0\rangle$ must be included in the summation index m. For simplicity, the tripartite initial state is chosen to be separable,

$$\rho_0^{\mathrm{sab}} = \rho_0^s \otimes \bar{\rho}_a \otimes |b_0\rangle\langle b_0| , \quad (65)$$

where ρ_0^s is an arbitrary system state and $\bar{\rho}_a$ follow from Eq. (11).

We determine the system realizations over the basis of a standard quantum jump approach formulated on the basis of Eq. (61). As before, the measurement apparatus is only sensitive to transitions of the auxiliary system A. Therefore the transformation associated with each detection event, instead of Eq. (23), here reads

$$\mathcal{M}\rho = \frac{\mathcal{E}_s[\sum_m \langle a_0 b_m|\,\rho\,|a_0 b_m\rangle]}{\mathrm{Tr}_s[\sum_m \langle a_0 b_m|\,\rho\,|a_0 b_m\rangle]} \otimes \bar{\rho}_a \otimes |b_0\rangle\langle b_0| . \quad (66)$$

The collisional superoperator $\mathcal{E}_s$ is given by Eq. (1). On the other hand, the (tripartite) conditional dynamics can be written as in Eqs. (24) and (25). Nevertheless, here the superoperator $\mathcal{D}$ reads

$$\mathcal{D}\rho = (\mathcal{L}_{sb} + \mathcal{L}_a)\rho - \frac{\gamma}{2}\{|a_0\rangle\langle a_0|,\rho\}_+ . \quad (67)$$

In deriving this result we used that $\sum_\alpha V_\alpha^\dagger V_\alpha = \mathrm{I}_s$, and $\sum_{b=0}^{\dim \mathcal{H}_b -1} |b_m\rangle\langle b_m| = \mathrm{I}_b$. With the previous definition of $\mathcal{D}$, the expression for the survival probability [Eq. (27)] remains almost the same, $P_0(t-\tau|\rho) = \mathrm{Tr}_{\mathrm{sab}}[e^{t\mathcal{D}}\rho]$.

Over the basis of the previous two equations and the initial condition (65), it is simple to conclude that the tripartite stochastic state $\rho_{\mathrm{sab}}^{\mathrm{st}}(t)$ $[\overline{\rho_{\mathrm{sab}}^{\mathrm{st}}(t)} = \rho_t^{\mathrm{sab}}]$ can be written at all times as

$$\rho_{\mathrm{sab}}^{\mathrm{st}}(t) = \rho_{sb}^{\mathrm{st}}(t) \otimes \rho_a^{\mathrm{st}}(t). \quad (68)$$

The dynamics for the ancilla state $\rho_a^{\mathrm{st}}(t)$ remains the same as before, that is, Eqs. (30)–(32) are not modified by the introduction of system B. Consequently, the measurement statistics, defined by the survival probability (33), or equivalently, the waiting time distribution (36), are also the same.

The induced stochastic system dynamics follows from $\rho_s^{\rm st}(t)=\mathrm{Tr}_{ab}[\rho_{\rm sab}^{\rm st}(t)]$. Hence, in each recording event the state suffers the disruptive transformation

$$\rho_s^{\rm st}(t)\to \mathrm{Tr}_{ab}\big[\mathcal{M}\rho_{\rm sab}^{\rm st}(t)\big]=\mathcal{E}_s\big[\rho_s^{\rm st}(t)\big]. \quad (69)$$

This expression follows straightforwardly from Eq. (66), after using Eq. (68) and noting that $\sum_m \langle b_m|\bullet|b_m\rangle=\mathrm{Tr}_b[\bullet]$. On the other hand, the intercollision dynamic [Eq. (35)] here is $\rho_s^{\rm st}(t)=\mathrm{Tr}_{ab}[\mathcal{T}_c(t-\tau)\rho_{\rm sab}^{\rm st}(\tau)]$. Given that $\mathcal{T}_c(t)$ follows from Eqs. (24) and (25), the operator $\mathcal{D}$ [Eq. (67)] and the separability property defined by Eqs. (66) and (68) lead to

$$\rho_s^{\rm st}(t)=\mathrm{Tr}_b\{\exp[(t-\tau)\mathcal{L}_{sb}]\,|b_0\rangle\langle b_0|\}\rho_s^{\rm st}(\tau). \quad (70)$$

This conditional dynamics recovers the phenomenological proposal Eq. (60). Hence, the non-Markovian propagator $\mathcal{G}(t)$ reads

$$\mathcal{G}(t)=\mathrm{Tr}_b[\exp(t\mathcal{L}_{sb})\,|b_0\rangle\langle b_0|]. \quad (71)$$

This is the main result of this section. It implies that the generalized phenomenological approach of Ref. [8] can be described over the basis of a tripartite Markovian evolution. If $\mathcal{L}_{sb}=\mathcal{L}_s+\mathcal{L}_b$, that is, when the system S and the ancilla B do not interact, the formalism of the previous section, $\mathcal{G}(t)=\exp(t\mathcal{L}_s)$, is recovered. Hence, given the structure of the operators (64), it becomes clear that the main role of system B is to modify the intercollision system dynamics.

The realizations defined by the measurement transformation (66) and the interevent dynamics (71) are similar to those found in Ref. [22], where a non-Markovian generalization of the quantum jump approach was defined over a similar basis by assuming that the system of interest is submitted to a measurement process. Nevertheless, the present treatment explicitly demonstrates that collisional dynamics can only be linked with a quantum measurement theory if the monitoring action is performed over the auxiliary ancilla system.

The nonlocal character of the propagator $\mathcal{G}(t)$ can be shown by writing Eq. (71) in the Laplace domain as $\mathcal{G}(u)=\mathrm{Tr}_b[(u-\mathcal{L}_{sb})^{-1}|b_0\rangle\langle b_0|]$. This expression can be rewritten as $\mathcal{G}(u)=\{\mathrm{Tr}_a[(u-\mathcal{L}_{sb})^{-1}(u-\mathcal{L}_{sb})|b_0\rangle\langle b_0|]\}^{-1}\{[\mathcal{G}(u)]^{-1}\}^{-1}$. Using in the curly brackets that $X^{-1}Y^{-1}=(YX)^{-1}$, where X and Y are arbitrary matrices, it follows that $\mathcal{G}(u)=\{[\mathcal{G}(u)]^{-1}(u\mathrm{Tr}_a((u-\mathcal{L}_{sb})^{-1}|b_0\rangle\langle b_0|]-\mathrm{Tr}_a[(u-\mathcal{L}_{sb})^{-1}\mathcal{L}_{sb}|b_0\rangle\langle b_0|)]\}^{-1}$, which in turn leads to

$$\mathcal{G}(u)=\frac{1}{u+\mathcal{K}(u)}, \quad (72)$$

where the system superoperator $\mathcal{K}(u)$ is

$$\mathcal{K}(u)=\left\{\mathrm{Tr}_b\left[\frac{1}{u-\mathcal{L}_{sb}}|b_0\rangle\langle b_0|\right]\right\}^{-1}\mathrm{Tr}_b\left[\frac{1}{u-\mathcal{L}_{sb}}\mathcal{L}_{sb}|b_0\rangle\langle b_0|\right].$$

Hence, in the time domain we get

$$\frac{d}{dt}\mathcal{G}(t)=\int_0^t dt'\mathcal{K}(t-t')\mathcal{G}(t'), \quad (73)$$

where $\mathcal{K}(t-t')$ is defined by its Laplace transform $\mathcal{K}(u)$.

The evolution of ρ_t^s can be obtained from Eq. (61) by using projector techniques. A simpler way is to calculate the average behavior of the ensemble of stochastic realizations (see Ref. [8]). On the other hand, the QCM introduced by Ciccarello, Palma, and Giovannetti in Ref. [36], which relies on interaction with a qubits string, can also be recovered from the present approach. In fact, as demonstrated in Ref. [8], it arises by taking $\mathcal{E}_s\to \mathrm{I}_s$. Hence, each collision only resets the evolution induced by $\mathcal{G}(t)$. The results presented by Rybar *et al.* in Ref. [35] rely on a similar approach. All non-Markovian effects arise because the ancilla string start in a correlated state [37]. Nevertheless, in our approach that formalism seems to be equivalent to a system-ancilla dynamics coupled via a unitary evolution, which in turn leads to a randomlike superposition of Hamiltonian system propagators. Therefore, extra analyses are necessary for establishing a full mapping between both approaches.

In what follows we analyze how different underlying dynamics lead to dephasing and dissipative intercollision dynamics [8,36].

1. Dephasing intercollision dynamics

In this example, both the system and the ancillas are two-level systems. Their tripartite Markovian evolution is given by Eq. (61). In an interaction representation with respect to the system Hamiltonian, we write

$$\mathcal{L}_{sb}[\rho]=\frac{-i}{\hbar}[H_{sb},\rho]=\frac{-i\lambda}{2}[\sigma_z\otimes \mathrm{I}_a\otimes\sigma_x,\rho], \quad (74)$$

where σ_j, $j=x,y,z$, are the Pauli matrices defined in each Hilbert space. Hence, the system of interest S and the auxiliary system B are coupled via a Hamiltonian interaction. The isolated dynamics of ancilla A is unitary,

$$\mathcal{L}_a[\rho]=\frac{-i}{\hbar}[H_a,\rho]=\frac{-i\Delta}{2}[\mathrm{I}_s\otimes\sigma_x\otimes \mathrm{I}_b,\rho]. \quad (75)$$

The dissipative tripartite interaction [Eq. (63)] reads

$$\mathcal{C}_{\rm sab}[\rho]=\frac{\gamma}{2}\sum_{m=0,1}([V_m,\rho V_m^\dagger]+[V_m\rho,V_m^\dagger]). \quad (76)$$

The index $m=0,1$, runs over the basis $\{|b_0\rangle,|b_1\rangle\}$ of system B. The two operators V_m are

$$V_m=\sigma_x\otimes\sigma\otimes|b_0\rangle\langle b_m|, \quad (77)$$

where, as before, σ is the lowering operator, here defined in the Hilbert space of system A. Consistently with Eq. (65), the initial tripartite state is

$$\rho_0^{\rm sab}=\rho_0^s\otimes|-\rangle\langle-|\otimes|b_0\rangle\langle b_0|. \quad (78)$$

From the previous definitions, Eqs. (66) and (69) lead to the collision system superoperator

$$\mathcal{E}_s[\rho]=\sigma_x\rho\sigma_x. \quad (79)$$

Notice that σ_x arises from the first (system) operator contribution in Eq. (77). On the other hand, the dynamics of system A again is defined by Eq. (44). Consequently, the waiting time distribution is given by Eq. (46). The intercollision dynamic follows from Eqs. (71) and (74). By an explicit calculation, we get the completely positive (non-Markovian) dephasing superoperator

$$\mathcal{G}(t)\rho=\tfrac{1}{2}[1+d(t)]\rho+\tfrac{1}{2}[1-d(t)]\sigma_z\rho\sigma_z, \quad (80)$$

which in turn can be rewritten as

$$\mathcal{G}(t)\rho = \begin{pmatrix} \langle +|\,\rho\,|+\rangle & d(t)\,\langle +|\,\rho\,|-\rangle \\ d(t)\,\langle -|\,\rho\,|+\rangle & \langle -|\,\rho\,|-\rangle \end{pmatrix}. \tag{81}$$

The function $d(t)$ defines the system coherence behavior. It reads $d(t) = \cos(\lambda t)$.

In the first example worked out in Ref. [8], the superoperator is given by Eq. (79), while the propagator $\mathcal{G}(t)$ is given by Eq. (80) (see Supplemental Material of [8]). Hence, our results provide a clear microscopic description for that phenomenological model. The waiting time distribution, instead of Eq. (46), is a classical one like Eq. (58). That case can be recovered, replacing the ancilla dynamics (75) by

$$\mathcal{L}_a[\rho] = \frac{\beta}{2}\left(\left[A, \rho_t^{sa} A^\dagger\right] + \left[A\rho_t^{sa}, A^\dagger\right]\right) \tag{82}$$

with the operator

$$A = \mathrm{I}_s \otimes \sigma^\dagger \otimes \mathrm{I}_b. \tag{83}$$

As explained previously, diverse "underlying classical" waiting time distributions can be obtained by adding extra ancilla states, all of then coupled by incoherent transitions.

2. *Dissipative intercollision dynamics*

Instead of the dephasing evolution (80), the intercollision dynamics may also lead to dissipative effects. This property is defined by the superoperator $\mathcal{L}_{sb}$ [Eq. (74) in the previous example]. For example, $\mathcal{L}_{sb}$ may correspond to a Jaynes-Cumming interaction, which couples the system to a set of bosonic field modes initially in the vacuum state [2,24]. This case, which has been studied in Refs. [8,36], can be analyzed over the basis developed previously.

VI. SUMMARY AND CONCLUSIONS

Phenomenological QCMs provided an important theoretical tool for establishing and describing non-Markovian completely positive dynamics. In this paper we have developed a solid physics basis for understanding this approach. It relies on a Markovian embedding of the non-Markovian system density matrix evolution, which in turn allows derivation of the phenomenological trajectories from a quantum measurement theory.

First, we focused our analysis on the leading case in which the collision statistics is defined by a renewal process, while the interevent dynamics is defined by a Markovian quantum semigroup. By using projector techniques we demonstrated that the non-Markovian density matrix evolution [Eq. (2)] can be obtained, without involving any approximation, from a bipartite Markovian dynamics, where the system of interest interacts with an auxiliary ancilla system [Eq. (5)]. The memory kernel that determines the system evolution becomes defined by the ancilla dynamics [Eq. (20)]. The proposed Markovian embedding allows association of a clear microscopic dynamics to the QCM. In fact, Lindblad equations are linked with well-defined microscopic dynamics.

In a second step, we assumed that the ancilla system is continuously monitored in time. Hence, over the basis of the quantum jump approach formulated for the bipartite dynamics, we find that the realizations of the QCM are recovered from the marginal conditional stochastic system dynamics [Eq. (21)]. In fact, each recording event of the ancilla measurement apparatus leads to the collisional transformations of the phenomenological approach [Eq. (34)]. The intercollision system dynamics follows from the conditional bipartite dynamics between detection events [Eq. (35)]. The waiting time distribution of the interevent time interval also becomes defined by the ancilla dynamics [Eq. (36)]. In this way, the phenomenological realizations of the collisional approach were derived from a quantum measurement theory.

The Markovian embedding and the link with the quantum jump approach were explicitly shown through an example where the dynamics of both the system of interest and the auxiliary one develop in two-dimensional Hilbert spaces (Figs. 1 and 2). In contrast to phenomenological formulations, here the collision statistics arise from quantum coherent effects developing in the ancilla Hilbert space. A system-to-environment backflow of information characterizes the dynamics. In opposition, when the ancilla dynamic is completely incoherent, this feature is absent.

The previous finding provides a solid basis for proposing different generalizations of the QCM. For example, nonstationary renewal collision dynamics can be obtained by introducing an explicit time dependence in the ancilla dynamics. Nonrenewal collision statistics can be related to a nonrenewal ancilla measurement process. On the other hand, we showed that by introducing a second auxiliary system, the intercollision dynamics becomes defined by a non-Markovian propagator, Eq. (71). This finding allowed us to recover a recent proposed generalization of the QCM [8], which in fact can also be embedded in a Markovian evolution, and their realizations derived from a quantum measurement theory. From this result, we also concluded that some non-Markovian collisional models formulated in terms of qubit logical operations [36] can also be recovered from our formalism.

The present analysis allows us to read the phenomenological QCMs from a different perspective. Besides a solid physical basis of the corresponding non-Markovian dynamics, the developed approach provides an alternative powerful tool for describing non-Markovian memory effects in open quantum systems.

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