RAPID COMMUNICATIONS

PHYSICAL REVIEW A **88**, 031802(R) (2013)

Mechanical properties of electron vortices

S. M. Lloyd, M. Babiker, and J. Yuan

Department of Physics, University of York, Heslington, York, YO10 5DD, United Kingdom

(Received 12 March 2013; published 30 September 2013)

It is shown how the quantum mechanical mass flux and the electromagnetic fields of an electron Bessel vortex mode generate its intrinsic linear momentum and angular momentum properties. Although the corresponding volume density vectors due to the mass flux contain transverse vector components, their volume integrals are shown by explicit analysis to yield null results. The total linear and angular momenta are thus purely axial vectors. There are additional contributions associated with the vortex electric and magnetic fields and these too are shown to be purely axial vectors. Order of magnitude estimates are made in the context of a suggested experiment on the rotation of an optically levitated nanoparticle subject to an electron vortex.

DOI: 10.1103/PhysRevA.88.031802 PACS number(s): 42.50.Wk, 42.50.Tx, 37.10.Vz

There is currently much interest in the physics and applications of twisted particle beams. Recent work has highlighted electron vortices (EVs), following prediction by Bliokh *et al.* [1], but the vortex concept can indeed be generalized to any particle, including neutral atoms, ions, and molecules. Electron vortices have been created in a number of laboratories, starting with Uchida and Tonomura [2], then by Verbeeck *et al.* [3], McMorran *et al.* [4], and Gnanavel *et al.* [5]. It is now generally accepted that EVs can be readily generated inside electron microscopes. The techniques of their production, especially the one involving computer generated masks, bears a great deal of resemblance to that followed in the case of twisted light beams, also called optical vortices (OVs). Indeed it is possible to argue that the existence of EVs was inspired by developments of OVs. Optical vortices are a special form of light, exemplified by the optical Laguerre-Gaussian and Bessel beams, which have been studied extensively over the last two decades or so [6–10]. Although the optical and electron vortices are similar in some respects, most notably in that they both carry the important property of orbital angular momentum, they do differ in a number of respects. In particular, OVs are characterized by vector electromagnetic fields and their quanta are massless bosons, while EVs are finite mass fermions of spin-half characterized by a scalar field in the form of the Schrödinger wave function.

Interactions of electron vortex beams with atomic matter have been investigated in the context of the transfer of orbital angular momentum to the internal dynamics [11,12], with reference to the experiment performed by Verbeeck *et al.* [3] that had shown that the interaction leads to dichroism in a magnetic sample. This is in contrast to the case of optical vortices, which are not specific in their interactions with chiral matter [12–14]. One of the main effects that have been carefully analyzed in the case of optical vortices has been their mechanical action on matter, leading to the so-called optical spanner effect [6–10] which can be loosely defined as the rotational version of the optical tweezer mechanism. Optical manipulation, in general, using OVs has been rigorously investigated by a number of authors [6–10]. Experimental work on nanomanipulation using electron vortices has been reported recently by Gnanavel *et al.* [5] and Verbeeck *et al.* [15]. These experiments demonstrate clear indications that electron vortices rotate nanoparticles and, significantly, the sense of rotation depends on the sign of the winding number l, as in the case of optical vortices. However, as far as the authors know, no analysis has so far been carried out on the source of mechanical effects of particle vortices on nanoparticles.

Additionally, the electron vortex has associated electric and magnetic fields, due to its charge and current densities [16]. In the nonrelativistic limit of the Dirac equation these fields couple to the electron spin and are responsible for spin-orbit coupling [16]. Spin-orbit coupling was also analyzed by Bliokh *et al.* [17], but without reference to the intrinsic fields which Lloyd *et al.* point out as essential for the coupling in the absence of any external fields. These intrinsic fields also contribute to the linear and angular momenta of the electron vortex, as will be shown here.

In this Rapid Communication we explore the mechanical properties of electron vortices arising from the finite mass flux and vortex electromagnetic fields for a typical Bessel electron vortex. We find that there is only one vector component of the vortex linear and orbital angular momenta, namely, an axial component. Order of magnitude estimates of the effects of these momenta are made with reference to the anticipated manipulation of nanoparticles using electron vortices.

The electron vortex wave function $\psi(\mathbf{r},t)$ is a particular solution of the free particle Schrödinger equation in cylindrical polar coordinates $\mathbf{r} = (\rho,\phi,z)$, namely,

$$\nabla^2\psi(\mathbf{r},t) + \frac{2m\mathcal{E}}{\hbar^2}\psi(\mathbf{r},t) = 0, \tag{1}$$

where $\mathcal{E}$ is the energy eigenvalue and m is the particle rest mass. The Bessel-type mode is as follows:

$$\psi(\mathbf{r},t) = N_l J_l(k_\perp\rho)e^{ik_z z}e^{il\phi}e^{-i\mathcal{E}t/\hbar}, \tag{2}$$

where $J_l(k_\perp\rho)$ is the Bessel function of order l with l the integer winding number of the beam, and $k_\perp$ and k_z are, respectively, the in-plane and axial wave-vector components. As in [12], we assume a vortex mode extending along the axis over a length D which is much larger than a typical beam width.

The normalization factor in Eq. (2) follows straightforwardly in the form

$$N_l = \left(\frac{k_\perp^2}{2\pi D\mathcal{I}_l^{(1)}}\right)^{1/2}, \tag{3}$$

where $\mathcal{I}_l^{(1)}$ is the first moment integral of the Bessel function defined by

$$\mathcal{I}_l^{(1)} = \int_0^\infty |J_l(x)|^2 x\, dx. \tag{4}$$

This is a special case of the general nth moment integral of the Bessel function

$$\mathcal{I}_l^{(n)} = \int_0^\infty |J_l(x)|^2 x^n dx \quad (n \geqslant 0). \tag{5}$$

Associated with the vortex wave function are the mass density $\tilde{\rho}_{\rm m}(\mathbf{r},t)$ and mass current density $\tilde{\mathbf{J}}_{\rm m}(\mathbf{r},t)$ in the standard forms

$$\tilde{\rho}_{\rm m}(\mathbf{r},t) = m\psi^*(\mathbf{r},t)\psi(\mathbf{r},t), \tag{6}$$

$$\tilde{\mathbf{J}}_{\rm m}(\mathbf{r},t) = \frac{e\hbar}{2i}\{\psi^*(\mathbf{r},t)\nabla\psi(\mathbf{r},t) - \psi(\mathbf{r},t)\nabla\psi^*(\mathbf{r},t)\}. \tag{7}$$

Substituting for $\psi(\mathbf{r},t)$ we find

$$\tilde{\rho}_{\rm m}(\mathbf{r},t) = m|N_l|^2|J_l(k_\perp\rho)|^2; \tag{8}$$

$$\tilde{\mathbf{J}}_{\rm m}(\mathbf{r},t) = \hbar|N_l|^2\left(\frac{l}{\rho}\hat{\boldsymbol{\phi}} + k_z\hat{\mathbf{z}}\right)|J_l(k_\perp\rho)|^2. \tag{9}$$

The mass current density is the same as the linear momentum density (i.e., linear momentum per unit volume) and hence the total linear momentum vector carried by the electron vortex follows by volume integration

$$\mathbf{P}_{\rm m} = \int dV\, \tilde{\mathbf{J}}(\mathbf{r},t). \tag{10}$$

We find

$$\mathbf{P}_{\rm m} = \hbar|N_l|^2 D\int_0^\infty\int_0^{2\pi} d\phi\left\{k_z\hat{\mathbf{z}} + \frac{l}{\rho}\hat{\boldsymbol{\phi}}\right\}|J_l(k_\perp\rho)|^2\rho\, d\rho. \tag{11}$$

The azimuthal component when integrated over the volume leads to a null result due to a vanishing angular integral. The axial component is then the only contribution to the total linear momentum, and can straightforwardly be shown by direct integration of the z component in Eq. (11) to be

$$\mathbf{P}_{\rm m} = 2\pi\hbar k_z D\mathcal{I}_l^{(1)}|N_l|^2\hat{\mathbf{z}} = \hbar k_z\hat{\mathbf{z}}. \tag{12}$$

as the total linear momentum carried by the entire electron vortex mode, where we have made use of Eq. (3). This property does not depend on the winding number l. It is reassuring that the axial linear momentum component emerges from the formalism in the expected form, but it is also remarkable that there are no transverse components of the total linear momentum of the vortex.

The angular momentum density associated with the mass flux is the moment of the linear momentum density

$$\begin{aligned}\boldsymbol{\mathcal{L}}_{\rm m} &= \mathbf{r}\times\tilde{\mathbf{J}}(\mathbf{r},t)\\ &= \hbar|N_l|^2(\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}})\times\left\{\frac{l}{\rho}\hat{\boldsymbol{\phi}} + k_z\hat{\mathbf{z}}\right\}|J_l(k_\perp\rho)|^2.\end{aligned} \tag{13}$$

The vortex orbital angular momentum vector is obtained by volume integration. We find

$$\begin{aligned}\mathbf{L}_{\rm m} &= \int\boldsymbol{\mathcal{L}}_{\rm m}\, dV = \hbar|N_l|^2\int_{-D/2}^{D/2}\int_0^{2\pi}\int_0^\infty\left\{l\hat{\mathbf{z}} - \rho k_z\hat{\boldsymbol{\phi}} - \frac{l}{\rho}z\hat{\boldsymbol{\rho}}\right\}\\ &\quad\times|J_l(k_\perp\rho)|^2\rho\, d\rho\, d\phi\, dz.\end{aligned} \tag{14}$$

Once again only the axial component leads to a contribution to the total angular momentum, as the angular and radial integrals lead to a vanishing result for the transverse components. We have, using Eq. (3), that

$$\mathbf{L}_{\rm m} = 2\pi l\hbar D\mathcal{I}_l^{(1)}|N_l|^2\hat{\mathbf{z}} = \hbar l\hat{\mathbf{z}}. \tag{15}$$

The result in Eq. (15) indicates that, in general, the electron vortex carries orbital angular momentum due to the mass flux only with an axial component. This confirms the main feature of the electron vortex, namely, that it carries a total orbital angular momentum about its axis equal to $\hbar l$ due to its mass flux and, as is the case with the total linear momentum of the vortex, there are no transverse total orbital angular momentum components. In other words the angular momenta L_x and L_y of the electron vortex are both zero as well as its linear momentum components P_x and P_y.

Equations (12) and (15) are two of our main results, explicitly displaying expressions for the linear and angular momentum vectors of the vortex arising from the mass flux. It turns out that the vanishing transverse components of the total linear and orbital angular momentum is not a preserve of electron vortices. We have verified by explicit evaluation [18] that this feature holds for both the Bessel- and Laguerre-Gaussian-type optical vortices. The momentum density in the Laguerre-Gaussian OV beam [19] has been shown to have axial as well as transverse components—the Bessel type has only z and ϕ components, like the electron Bessel vortex—as do the orbital angular momentum densities for both the Laguerre-Gaussian- [20] and Bessel-type optical beams. However, what has not been shown explicitly before is that the volume integrals of the densities lead to identically null values for the transverse linear momentum as well as the orbital angular momentum of these optical vortices, and it appears that this is a general feature of all vortex beams. In other words, this result is independent of the nature of the vortex beam, or of the particular distribution—so long as the phase factor $e^{il\phi}$ is the only source of ϕ dependence, the beam will contain linear and angular momentum in the z direction only.

We have so far concentrated on the mechanical properties of the electron vortex arising from the finite electron mass. This would be true for any electrically neutral particle vortex. An electron vortex is also endowed with electromagnetic fields $\mathbf{E}(\rho)$ and $\mathbf{B}(\rho)$ associated with its electric charge which also depend on the winding number l, which we omit here for ease of notation. These vortex fields have been evaluated in [16] for an electron vortex generated inside an electron microscope. In general, the vortex electric field has only one component,

$$\mathbf{E}(\rho) = \hat{\rho}E_\rho(\rho), \tag{16}$$

while the vortex magnetic field has two orthogonal components, one axial and another azimuthal,

$$\mathbf{B}(\rho) = \hat{\mathbf{z}}B_z(\rho) + \hat{\boldsymbol{\phi}}B_\phi(\rho). \tag{17}$$

The linear momentum density due to these vortex fields emerges straightforwardly as follows:

$$\boldsymbol{\mathcal{P}}_{\rm em} = \epsilon_0 \mathbf{E} \times \mathbf{B} = \epsilon_0\{\hat{\mathbf{z}} E_\rho B_\phi - \hat{\boldsymbol{\phi}} E_\rho B_z\}. \quad (18)$$

The total linear momentum of the vortex due to the vortex fields follows by volume integration. We have, evaluating the z integral first,

$$\mathbf{P}_{\rm em} = \epsilon_0 D \int_0^\infty \rho\, d\rho \int_0^{2\pi} d\phi \{\hat{\mathbf{z}} E_\rho B_\phi - \hat{\boldsymbol{\phi}} E_\rho B_z\}. \quad (19)$$

Since the fields are functions only of ρ [16], once again we see that the ϕ integral leads to a vanishing result of the azimuthal component and we are left only with the axial component. We have

$$\mathbf{P}_{\rm em} = \hat{\mathbf{z}} 2\pi \epsilon_0 D \int_0^\infty E_\rho B_\phi \rho\, d\rho. \quad (20)$$

The evaluation of the ρ integral is, in principle, straightforward since we have expressions for the variation of the fields with ρ.

The angular momentum contributions due to the vortex fields now follow as the integral of the moment of the linear momentum density. We have

$$\mathbf{L}_{\rm em} = \int dV\, \mathbf{r} \times \boldsymbol{\mathcal{P}}_{\rm em}$$
$$= \epsilon_0 \int dV \{\hat{\boldsymbol{\rho}} z E_\rho B_z - \hat{\boldsymbol{\phi}} \rho E_\rho B_\phi - \hat{\mathbf{z}} \rho E_\rho B_z\}. \quad (21)$$

Once again the integral over the azimuthal component vanishes due to a vanishing ϕ integral; the integral of the ρ component also results in a null value and we are left only with the axial component, namely,

$$\mathbf{L}_{\rm em} = -\hat{\mathbf{z}} 2\pi \epsilon_0 D \int_0^\infty E_\rho B_z \rho^2\, d\rho. \quad (22)$$

Equations (20) and (22) are two further results of this Rapid Communication to be added to the main results given by Eqs. (12) and (15). The set of equations shows the linear and angular momentum vector components associated with the vortex mass flux and electromagnetic fields. Though we have evaluated these quantities with reference to the normalized beam of Eq. (2), we stress that these results—that the vector components of total linear and orbital angular momentum exist only in the z direction—are general, and not dependent on the particular normalization of Eq. (3).

For orientation as to orders of magnitude arising in a feasible experimental arrangement, we assume that we are dealing with electron vortices created inside a 1 nA electron microscope of accelerating voltage 200 keV. For such a beam, we assume a typical value of the transverse wave-vector component $k_\perp = 0.01 k_z$, and proceed to estimate values for the linear momentum and orbital angular momentum of this typical electron vortex created in an electron microscope. We find

$$P_{\rm em} \approx 10^{-34} \text{ kg,m s}^{-1}, \quad (23)$$

$$L_{\rm em} \approx 10^{-48} \text{ J s}. \quad (24)$$

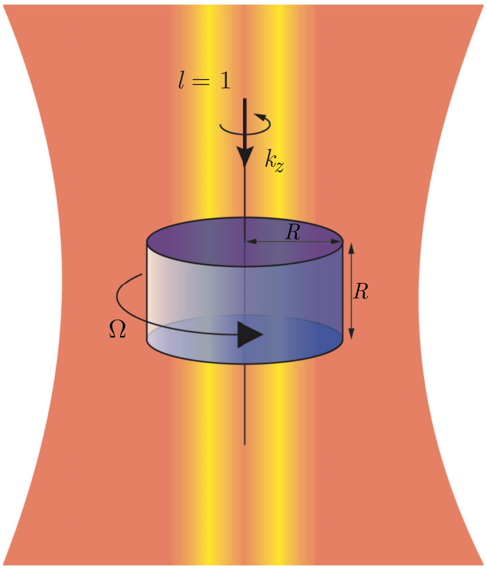

FIG. 1. (Color online) Schematic drawing (not to scale) showing a disk-shaped nanoparticle in the field of a downward electron vortex represented by the inner beam (yellow) and an upward much wider Gaussian laser beam (red) which acts to levitate the nanoparticle while the electron vortex rotates it about its axis. See the text for the parameters used to estimate the angular frequency in Eq. (29).

These are both extremely small compared to the mass counterparts. The approximate ratios found are

$$\frac{P_{\rm em}}{P_{\rm m}} \approx 10^{-12}; \quad (25)$$

$$\frac{L_{\rm em}}{L_{\rm m}} \approx 10^{-14}, \quad (26)$$

so for practical purposes the electromagnetic linear and orbital angular momenta in such a vortex, as created in an electron microscope, are negligibly small and in such a context the mechanical properties of the electron vortex stem primarily from the finiteness of the electron mass, as contained in $P_{\rm m}$ and $L_{\rm m}$. However, in other contexts these contributions could be significant when the vortex is created outside an electron microscope (as, for example, in a linear accelerator).

Finally we consider the orders of magnitude when vortices are used to rotate nanoparticles. For illustration we focus on the effects of the vortex angular momentum on a nanoparticle in the form of a small cylinder of radius R and length $d = R$ with the cylinder axis coinciding with the electron vortex axis, as depicted in Fig. 1. If we are interested only in imparting rotational motion on the nanoparticle due to the electron vortex, we will need to eliminate the axial forces as well as friction. This can be done by setting up a stable vertically oriented optical trap [21] whereby a suitable laser beam acts to levitate the nanoparticle against the downward force of gravity and the downward axial force of a single electron vortex. With the axial forces eliminated the electron vortex should only rotate the nanoparticle. Assuming that the minimum angular momentum of $\hbar$ is transferred to the nanoparticle, the nanoparticle will rotate at an angular frequency given by

$$\Omega = \frac{2\hbar}{MR^2}, \quad (27)$$

where $MR^2/2$ is the moment of inertia of the cylindrical nanoparticle of mass M, radius R, and length equal to R. For illustration we consider the case of a fused silica nanoparticle with

$$R = 10^{-8}\ \mathrm{m}; \quad M = \rho_{\mathrm{m}}\pi R^3, \tag{28}$$

and the mass density of fused silica is approximately $\rho_{\mathrm{m}} = 2.2 \times 10^3\ \mathrm{kg\,m^{-3}}$. The angular frequency at which the above fused silica nanoparticle is subject to a single electron vortex is

$$\Omega \approx 87.6\ \mathrm{Hz}. \tag{29}$$

This angular frequency is much higher than that observed by [5] in the case of a gold nanoparticle on a support. Observations suggest that friction between the nanoparticle and the support leads to damping of the rotation. The levitation setup suggested above could be a suitable arrangement towards eliminating the detrimental effects of the support. A fused silica nanoparticle should be an easier nanoparticle to rotate in the field of the electron vortex as optical levitation of a metallic nanoparticle would be more difficult to control.

The processes underlying the mechanical action of the electron vortex on a nanoparticle leading to rotation form a complex issue which is beyond the scope of this Rapid Communication, which is focused primarily on finding the sources and formalism leading to the determination of the contributions to the momentum and angular momentum carried by the electron vortex. Here we have only sought to determine the order of magnitude of the rotational frequency for a typical nanoparticle immersed in an electron vortex created inside an electron microscope. Verbeeck *et al.* [15] explain the mechanical action observed as a result of the breaking of the circular symmetry of the vortex beam by the nanoparticle and in the case of a metallic nanoparticle the interaction with the electron vortex would also involve plasmonic effects [22]. A rigorous analysis is needed which must take into account the shape of the nanoparticle as well as the surface matching of the vortex wave function on entry and exit and the interaction of the vortex inside the body of the nanoparticle.

In conclusion, we have evaluated for the first time the mechanical properties of electron vortices, analyzing expressions for the various contributions of the linear and orbital angular momenta carried by a Bessel electron vortex. We have examined the transverse components of the vortex linear and angular momentum vectors and found these to vanish identically in all cases. We have thus shown here that the mechanical properties in the form of linear and angular momentum vectors of an electron Bessel vortex are purely axial vectors and are dominated by the contributions due to the finite electron mass. We have outlined a possible experimental scenario which has the advantage of eliminating friction and would demonstrate the rotational influence of an electron vortex.

The authors are grateful to Professor Steve Barnett for useful discussion on the mechanical properties of vortex beams and for suggesting optical levitation of nanoparticles in the context described here. We wish to thank the UK Engineering and Physical Sciences Research Council (EPSRC) for financially supporting this research by the award of Grant No. EP/J022098/1. Financial support for S.M.L. from the University of York is also gratefully acknowledged.

---

[1] K. Y. Bliokh, Y. P. Bliokh, S. Savel'ev, and F. Nori, Phys. Rev. Lett. **99**, 190404 (2007).
[2] M. Uchida and A. Tonomura, Nature (London) **464**, 737 (2010).
[3] J. Verbeeck, H. Tian, and P. Schattschneider, Nature (London) **467**, 301 (2010).
[4] B. J. McMorran, A. Agrawal, I. Anderson, A. Herzing, H. Lezec, J. McClelland, and J. Unguris, Science **331**, 192 (2011).
[5] T. Gnanavel, J. Yuan, and M. Babiker, in *Proceedings of the 15th European Microscopy Congress*, edited by D. J. Stokes and J. Hutchison, Vol. 2 (Royal Microscopical Society, Oxford, 2012).
[6] L. Allen, M. J. Padgett, and M. Babiker, Prog. Opt. **39**, 291 (1999).
[7] L. Allen, S. M. Barnett, and M. J. Padgett, *Optical Angular Momentum* (Institute of Physics, Bristol, 2003).
[8] D. L. Andrews, *Structured Light and Its Applications: An Introduction to Phase-Structured Beams and Nanoscale Optical Forces* (Academic, Burlington, 2008).
[9] D. G. Grier, Nature (London) **424**, 810 (2003).
[10] D. L. Andrews and M. Babiker, *The Angular Momentum of Light* (Cambridge University Press, Cambridge, 2012).
[11] S. M. Lloyd, M. Babiker, and J. Yuan, Phys. Rev. Lett. **108**, 074802 (2012).
[12] S. M. Lloyd, M. Babiker, and J. Yuan, Phys. Rev. A **86**, 023816 (2012).
[13] M. Babiker, C. R. Bennett, D. L. Andrews, and L. C. Dávila Romero, Phys. Rev. Lett. **89**, 143601 (2002).
[14] F. Araoka, T. Verbiest, K. Clays, and A. Persoons, Phys. Rev. A **71**, 055401 (2005).
[15] J. Verbeeck, H. Tian, and G. Van Tendeloo, Adv. Mater. **25**, 1114 (2012).
[16] S. M. Lloyd, M. Babiker, J. Yuan, and C. Kerr-Edwards, Phys. Rev. Lett. **109**, 254801 (2012).
[17] K. Y. Bliokh, M. R. Dennis, and F. Nori, Phys. Rev. Lett. **107**, 174802 (2011).
[18] S. M. Lloyd, Ph.D. thesis, University of York, 2013.
[19] L. Allen and M. J. Padgett, Opt. Commun. **184**, 67 (2000).
[20] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Phys. Rev. A **45**, 8185 (1992).
[21] A. Ashkin, Biophys. J. **61**, 569 (1992).
[22] P. E. Batson, A. Reyes-Coronado, R. G. Barrera, A. Rivacoba, P. M. Echenique, and J. Aizpurua, Nano Lett. **11**, 3388 (2011).