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Effects of phase fluctuations on phase sensitivity and visibility of path-entangled photon Fock states

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We study effects of phase fluctuations on phase sensitivity and visibility of a class of robust path-entangled photon Fock states (known as mm' states) as compared to the maximally path-entangled NOON states in the presence of realistic phase fluctuations such as turbulence noise. Our results demonstrate that the mm' states, which are more robust than the NOON state against photon loss, perform equally well when subject to such fluctuations. We derive the quantum Fisher information with the phase-fluctuation noise and show that the phase sensitivity with parity detection for both of the above states saturates the quantum Cramér-Rao bound in the presence of such noise, suggesting that parity detection is an optimal detection strategy.

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I. INTRODUCTION

Quantum states of light such as squeezed sates or entangled states have long been known to produce greater precision, resolution, and sensitivity in metrology, imaging, and object ranging [1–4] than what is possible classically. One of the most prominent examples of such a nonclassical state is the NOON state [5–7], which is an equal coherent superposition of N photons in one path of a Mach-Zehnder interferometer with none in the other, and vice versa. This state may be written as $|N :: 0\rangle = (|N,0\rangle + |0,N\rangle)/\sqrt{2}$, and can be used to achieve Heisenberg-limited supersensitivity as well as super-resolution in quantum metrology [5,8]. In recent years, several schemes for reliable production of such states have been proposed, making them useful in superprecision measurements in optical interferometry, atomic spectroscopy, gravitational wave detection, and magnetometry, along with potential applications in rapidly evolving fields such as quantum imaging and sensing [9–15].

However, due to inevitable interactions with the surrounding environment, the NOON state tends to decohere in the presence of a noisy environment. Recently, a few authors investigated the effects of photon loss on the performance of NOON state in quantum interferometric setups [16–20] that demonstrate that NOON states undergoing loss decohere very rapidly, making it difficult to achieve supersensitivity and resolution in a lossy environment. Huver *et al.* proposed a class of generalized Fock states, known as mm' states, by introducing decoy photons to the NOON state in both paths of the interferometer, and showed that such states provide better metrological performance than NOON states in presence of photon loss [19].

In real-life applications such as a quantum sensor or radar, phase fluctuation due to different noise sources can further degrade the phase sensitivity by adding significant noise to the phase ϕ to be estimated or detected. For instance, when one considers propagation of the entangled states over distances of kilometers, through say the atmosphere, then atmosphere turbulence becomes an issue as it can cause uncontrollable noise or fluctuation in the phase. In this sense, phase fluctuation stands as the most detrimental for phase estimation, rendering the quantum metrological advantage for achieving supersensitivity and super-resolution totally useless. It is therefore imperative to investigate the impacts of such random phase fluctuations on the phase sensitivity of quantum mechanically entangled states. In particular, we consider both the mm' and NOON states, and show how the phase sensitivity and visibility of the phase signal are affected by added phase fluctuations caused by turbulence noise.

We study the parity detection [21] for the interferometry with the phase-fluctuated mm' and NOON states. This detection scheme has been shown to reach Heisenberg limited sensitivity when combined with the lossless NOON state [21–24]. Here we calculate the minimum detectable phase shift in the presence of the turbulence noise and show that the lower bound of the phase-fluctuated sensitivity for both the states saturates the quantum Cramér-Rao bound [25,26], which gives the ultimate limit to the precision of the phase measurement. This result suggests that the parity detection serves as an optimal detection strategy when the given states are subject to the phase fluctuations.

The paper is organized as follows. In Sec. II, we introduce the mm' and NOON states and describe their evolution under the phase fluctuations. We define the parity detection operator in Sec. III and calculate the phase sensitivity and the visibility with the phase noise using the parity operator. In Sec. IV, we derive lowest possible uncertainty (quantum Cramér-Rao bound) in estimating the phase ϕ for these path-entangled Fock states and show that the parity operator saturates the quantum Cramér-Rao bound for both mm' and NOON states. Using the same detection technique, we then derive the phase sensitivity and the visibility in a more general case with both the photon loss and phase fluctuations in Sec V. Section VI contains our concluding remarks and further outlook with the potential implementations of the phase estimation with fluctuating phase noise.

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II. EVOLUTION OF THE mm' AND NOON STATES UNDER PHASE FLUCTUATIONS

The states we investigate are the following:

$$|m :: m'\rangle_{a,b} = \frac{1}{\sqrt{2}}(|m,m'\rangle_{a,b} + |m',m\rangle_{a,b}), \tag{1}$$

where a and b indicate the two paths of a two-mode optical interferometer. These states are called the mm' states, and they can be produced, for example, by postselecting on the output of a pair of optical parametric oscillators [27].

The mm' state reduces to a NOON state when $m = N$ and $m' = 0$, leading to

$$|N :: 0\rangle_{a,b} = \frac{1}{\sqrt{2}}(|N,0\rangle_{a,b} + |0,N\rangle_{a,b}). \tag{2}$$

The mm' states have been shown to be more robust than the NOON states against photon loss [19,20]. In following calculations, we drop the subscripts and always assume the first number in a ket or bra corresponds to mode a while the second corresponds to mode b.

We start with the propagation of mm' and NOON states through a simplified Mach-Zehnder interferometer as shown in Fig. 1, where details of source and detection are represented by their respective boxes. The input state at stage I is presented by Eq. (1), and the photon number difference ($\Delta m = m - m'$) between the two arms is fixed.

The presence of the phase shifter in the upper path b introduces a phase shift ϕ to the photons traveling through it, so that the state at stage II becomes

$$|\psi\rangle_{\mathrm{II}} = \frac{1}{\sqrt{2}}(e^{im'\phi}|m,m'\rangle + e^{im\phi}|m',m\rangle)$$
$$= \alpha|m,m'\rangle + \beta|m',m\rangle, \tag{3}$$

where $\alpha = e^{im'\phi}/\sqrt{2}$ and $\beta = e^{im\phi}/\sqrt{2}$. Because of the different number of photons being phase shifted on the upper path b, phase shifts accumulated are different along the two paths, thus providing the possibility of interference upon detection.

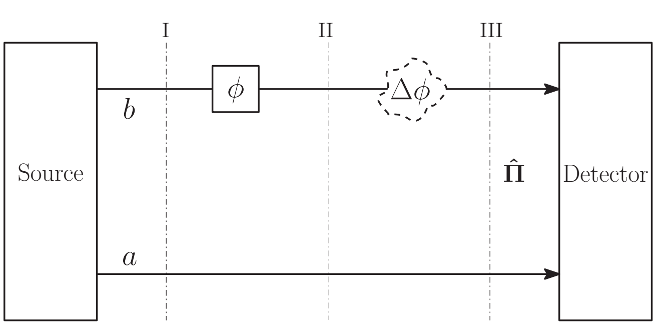

FIG. 1. Schematic diagram of a simplified Mach-Zehnder interferometer with the modes a and b for the mm' and NOON states as the input. The source and detector in the interferometer are represented by the respective boxes. Effects of the phase fluctuations due to the turbulence noise are represented by $\Delta\phi$ in the upper path b of the interferometer. The upper beam passes through a phase shifter ϕ, and the phase acquired depends on the total number of photons $\Delta m = m - m'$ (or N) passing throughout the upper path. Transformed parity detection is used as the detection scheme at both of the two modes at stage III inside the interferometer.

The combined effects of random phase fluctuations are represented by $\Delta\phi$ in the upper path in Fig. 1, and the mm' state at stage III is then given by

$$|\psi(\Delta\phi)\rangle_{\mathrm{III}} = \alpha e^{im'\Delta\phi}|m,m'\rangle + \beta e^{im\Delta\phi}|m',m\rangle. \tag{4}$$

Notice that because of the random nature of the phase fluctuations, the state of the system becomes a mixed state and the associated density matrix is then

$$\rho_{mm'} = \langle|\psi(\Delta\phi)\rangle_{\mathrm{III}}\,{}_{\mathrm{III}}\langle\psi(\Delta\phi)|\rangle. \tag{5}$$

Random fluctuations $\Delta\phi$ in the phase effectively causes the system to undergo pure dephasing. As a result, the off-diagonal terms in the density matrix will acquire decay terms, while the diagonal terms representing the population will remain intact, i.e., the photon number will be preserved along the path [28].

We can expand the exponential in Eq. (4) in a series expansion and consider the terms up to the second order in $\Delta\phi$. We assume the random-phase fluctuation $\Delta\phi$ to have Gaussian statistics described by the Wiener process, i.e., with zero mean and nonzero variance $\langle\Delta\phi^2\rangle = 2\Gamma L$ (L is the length of the dephasing region and Γ is the dephasing rate). Ensemble averaging over all realizations of the random process then gives

$$\langle e^{i\Delta m\Delta\phi}\rangle = 1 + i\Delta m\langle\Delta\phi\rangle - (\Delta m)^2\langle\Delta\phi^2\rangle/2$$
$$= 1 - (\Delta m)^2\Gamma L \approx e^{-(\Delta m)^2\Gamma L}.$$

The density matrix for the mm' state is given by

$$\rho_{mm'} = |\alpha|^2|m,m'\rangle\langle m,m'| + |\beta|^2|m',m\rangle\langle m',m|$$
$$+ \alpha^*\beta e^{-(\Delta m)^2\Gamma L}|m,m'\rangle\langle m',m|$$
$$+ \alpha\beta^* e^{-(\Delta m)^2\Gamma L}|m',m\rangle\langle m,m'|. \tag{6}$$

This result agrees with the density matrix obtained from solving the master equation in Ref. [28]. The similar equation for the NOON state can be obtained from Eq. (2) as

$$\rho_{\mathrm{NOON}} = |\alpha|^2|N,0\rangle\langle N,0| + |\beta|^2|0,N\rangle\langle 0,N|$$
$$+ \alpha^*\beta e^{-N^2\Gamma L}|N,0\rangle\langle 0,N| + \alpha\beta^* e^{-N^2\Gamma L}|0,N\rangle\langle N,0|. \tag{7}$$

III. PARITY OPERATOR

Achieving super-resolution and supersensitivity depends not only on the state preparation but also on the optimal detection schemes with specific properties. In this paper, we study parity detection, which was originally proposed by Bollinger *et al.* in the context of trapped ions [29] and was later adopted for optical interferometry by Gerry [21]. The original parity operator can be expressed as $\hat{\pi} = \exp(i\pi\hat{n})$, which distinguishes states with even and odd numbers of photons without having to know the full photon number counting statistics. Usually the parity detection is only applied to one of two output modes of the Mach-Zehnder interferometer. In our case, the parity operator inside the interferometer, following Ref. [30], can be written as

$$\hat{\Pi} = i^{(m+m')}\sum_{k=0}^{m}(-1)^k|k,n-k\rangle\langle n-k,k|, \tag{8}$$

where $\hat{\Pi}^2 = 1$ and $n = m + m'$, is the total number of photons. It should be noticed that the parity operator inside the interferometer detects both modes a and b of the field.

The expectation value of the parity for the mm' state is then calculated as

$$\langle\hat{\Pi}\rangle_{mm'} = \text{Tr}[\hat{\Pi}\rho_{mm'}] = (-1)^{(m+m')}e^{-(\Delta m)^2\Gamma L}\cos[\Delta m(\phi - \pi/2)], \tag{9}$$

where the density matrix $\rho_{mm'}$ is given by Eq. (6). If we put a half-wave plate in front of the phase shifter, which amounts to replacing ϕ by $\phi + \pi/2$, the expectation value becomes

$$\langle\hat{\Pi}\rangle_{mm'} = (-1)^{(m+m')}e^{-(\Delta m)^2\Gamma L}\cos[\Delta m\phi]. \tag{10}$$

Using the density matrix ρ_{NOON} in Eq. (7) for the NOON state, we can also obtain the expectation value of the parity operator for the NOON state as

$$\langle\hat{\Pi}\rangle_{\text{NOON}} = \text{Tr}[\hat{\Pi}\rho_{\text{NOON}}] = (-1)^N e^{-N^2\Gamma L}\cos[N\phi]. \tag{11}$$

A. Phase sensitivity

In quantum optical metrology, the precision of the phase measurement is given by the phase sensitivity. We now calculate the phase sensitivity for both the mm' and NOON states using the expectation values of the parity operator obtained above.

Phase sensitivity using the parity detection is defined by the linear error propagation method [31]

$$\delta\phi = \frac{\Delta\hat{\Pi}}{|\partial\langle\hat{\Pi}\rangle/\partial\phi|}, \tag{12}$$

where $\Delta\hat{\Pi} = \sqrt{\langle\hat{\Pi}^2\rangle - \langle\hat{\Pi}\rangle^2}$. Given $\langle\hat{\Pi}^2_{mm'}\rangle = 1$ the phase sensitivity with the parity detection for the mm' state is

$$\delta\phi_{mm'} = \sqrt{\frac{1 - e^{-2(\Delta m)^2\Gamma L}\cos^2(\Delta m\phi)}{(\Delta m)^2 e^{-2(\Delta m)^2\Gamma L}\sin^2(\Delta m\phi)}}. \tag{13}$$

For the NOON state the phase sensitivity with the parity detection is similarly obtained as

$$\delta\phi_{\text{NOON}} = \sqrt{\frac{1 - e^{-2N^2\Gamma L}\cos^2 N\phi}{N^2 e^{-2N^2\Gamma L}\sin^2 N\phi}}. \tag{14}$$

We note that in the limit of no dephasing ($\Gamma \to 0$), $\delta\phi_{mm'} \to 1/(\Delta m)$. For the NOON state, $\Gamma \to 0$ case leads to $\delta\phi_{\text{NOON}} \to 1/N$ (Heisenberg limit of the phase sensitivity for the NOON state).

In Fig. 2, we plot the phase sensitivities $\delta\phi_{mm'}$ and $\delta\phi_{\text{NOON}}$ for the various dephasing rates Γ choosing $\Delta m = N$, so that the amount of phase information is the same for either state. For $\Delta m = N$, Eqs. (13) and (14) show that the mm' and NOON states give rise to the same phase sensitivity. In particular, we show the phase sensitivity for the states $|4 :: 0\rangle$ and $|5 :: 1\rangle$ and find that both the states perform equally well in the presence of phase fluctuations when parity detection is used, although the former has been shown to outperform NOON states in the presence of photon loss [19,20].

The minimum phase sensitivities $\delta\phi_{\text{min}}$ can be obtained from Eqs. (13) and (14) for $\phi = \pi/(2\Delta m)$, or $\phi = \pi/(2N)$ for the mm' or NOON states, respectively. For the $|4 :: 0\rangle$ and $|5 :: 1\rangle$ states, we plot the minimum phase sensitivity $\delta\phi_{\text{min}}$ in Fig. 3 for as a function of Γ and compare it with the SNL and HL for both the states.

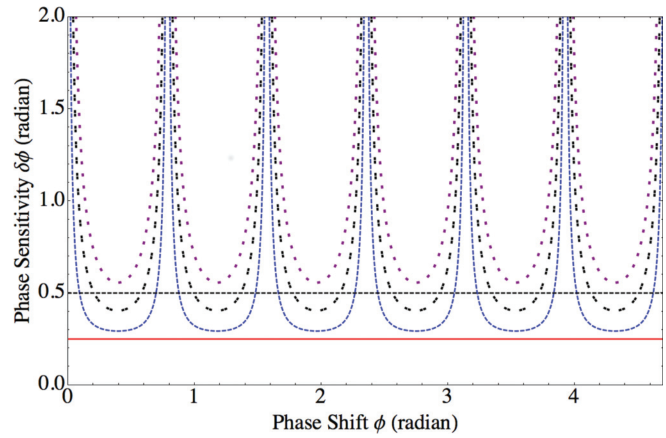

FIG. 2. (Color online) Phase sensitivity $\delta\phi$ of the mm' state $|5 :: 1\rangle$, or the NOON state $|4 :: 0\rangle$, having the same phase information, as a function of phase shift ϕ from a two-mode interferometer for different values of Γ: $\Gamma = 0.1$ (curved dashed line, blue online), $\Gamma = 0.3$ (curved black double-dotted line), and $\Gamma = 0.5$ (curved dotted line, purple online). The Heisenberg limit ($1/N$) and the shot noise limit ($1/\sqrt{N}$) of the phase sensitivity for the NOON state are shown by the red (gray) solid line and the black dashed line, respectively, for comparison.

The HL for a general mm' state is $1/(m + m')$ in terms of the total number of photons available and is equal to $1/N$ for the NOON state. The SNLs for these two states are given by $1/(\sqrt{m + m'})$ and $1/\sqrt{N}$, respectively. In Fig. 3, we see that the minimum phase sensitivity $\delta\phi_{\text{min}}$ hits the HL for the NOON state for $\Gamma = 0$ only, while it never reaches the HL for the mm' state. However, $\delta\phi_{\text{min}}$ is below the SNL for both of the states for small values of Γ, but increase in the phase fluctuation, i.e., Γ, leads to the phase sensitivity above the SNL, as shown in Fig. 3.

B. Visibility

We use the parity operator for the detection, and to quantify the degree of measured phase information we define the

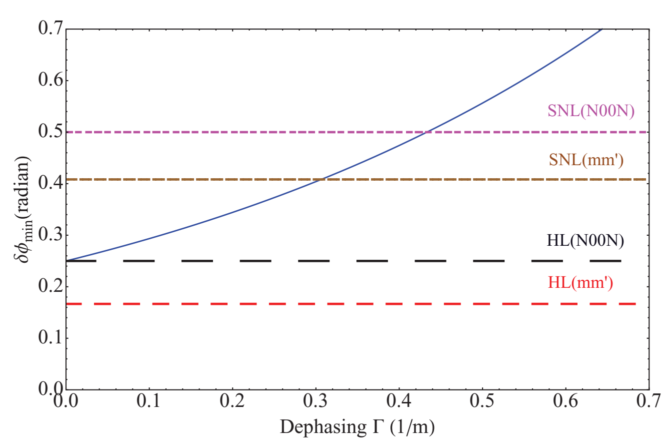

FIG. 3. (Color online) Minimum phase sensitivity $\delta\phi_{\text{min}}$ of the mm' state $|5 :: 1\rangle$ or the NOON state $|4 :: 0\rangle$ as a function of Γ. The shot noise limits (SNL) and the Heisenberg limits (HL) of the phase sensitivity for both the states are also shown for comparison.

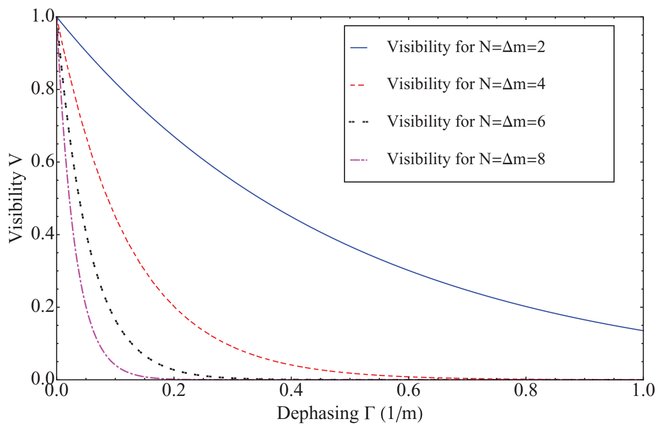

FIG. 4. (Color online) Visibility V of the mm' state for different Δm (for different N in case of NOON states with the same phase information) as a function of Γ. The visibility V is plotted for N(or $\Delta m) = 2$ (solid line, blue online), N(or $\Delta m) = 4$ (dashed line, red online), N(or $\Delta m) = 6$ (double dotted black line), and N(or $\Delta m) = 8$ (dot-dashed line, purple online). We see that the visibility drops faster for larger values of Δm (or N).

relative visibility as

$$V_{mm'} = \frac{\langle \hat{\Pi}_{mm'} \rangle_{\max} - \langle \hat{\Pi}_{mm'} \rangle_{\min}}{\langle \hat{\Pi}_{mm'}(\Gamma = 0) \rangle_{\max} - \langle \hat{\Pi}_{mm'}(\Gamma = 0) \rangle_{\min}}, \tag{15}$$

where the numerator corresponds to the difference in the maximum and minimum parity signals in the presence of phase fluctuations, while the denominator corresponds to the one with no dephasing, i.e., $\Gamma = 0$. Visibility for the NOON state is similarly defined as

$$V_{\text{NOON}} = \frac{\langle \hat{\Pi}_{\text{NOON}} \rangle_{\max} - \langle \hat{\Pi}_{\text{NOON}} \rangle_{\min}}{\langle \hat{\Pi}_{\text{NOON}}(\Gamma = 0) \rangle_{\max} - \langle \hat{\Pi}_{\text{NOON}}(\Gamma = 0) \rangle_{\min}}. \tag{16}$$

Using Eqs. (10) and (11), we then obtain the visibilities for the mm' state

$$V_{mm'} = e^{-(\Delta m)^2 \Gamma L} \tag{17}$$

and for the NOON state

$$V_{\text{NOON}} = e^{-N^2 \Gamma L}. \tag{18}$$

We note that the visibility of the NOON state with the parity detection in Eq. (18) agrees with the visibility in Ref. [28].

The visibility in Eqs. (17) and (18) depends on the value of the dephasing rate Γ and N (or $\Delta m = m - m'$), and for a given value of Γ, the visibility falls down faster as N increases. Hence, high-NOON states (or mm' states) with large number of photons are very much susceptible to the phase fluctuations compared to the low-NOON states and hence are not suitable to achieve metrological advantage with robustness in the presence of phase noise. This is shown in Fig. 4, where we plotted the visibility for different N (or Δm) with respect to the dephasing rate Γ.

IV. QUANTUM FISHER INFORMATION: BOUNDS FOR PHASE SENSITIVITY

In order to minimize the uncertainty $\delta\phi$ of the measured phase, we now seek to provide the lowest bound on the uncertainty of the phase. This bound is given by the quantum Cramér-Rao bound $\delta\phi_{\text{QCRB}}$, and is inversely proportional to the quantum Fisher information $F(\phi)$ [25,26,32,33]

$$\delta\phi_{\text{QCRB}} \geqslant \frac{1}{\sqrt{F(\phi)}}. \tag{19}$$

A general framework for estimating the ultimate precision limit in noisy quantum-enhanced metrology has been studied by Escher *et al.* [34]. In the following, we first obtain the quantum Fisher information, leading to the quantum Cramér-Rao bound for both the mm' and NOON states in the presence of the phase fluctuations, and show that the parity detection attains the quantum Cramér-Rao bound for both of these states subject to the dephasing.

The quantum Cramér-Rao bound has been shown to be always reached asymptotically by maximum likelihood estimations and a projective measurement in the eigenbasis of the symmetric logarithmic derivative L_ϕ [25,26,35], which is a self-adjoint operator satisfying the equation

$$\frac{L_\phi \rho_\phi + \rho_\phi L_\phi}{2} = \frac{\partial \rho_\phi}{\partial \phi}, \tag{20}$$

where ρ_ϕ is given by Eq. (6) for the mm' state and by Eq. (7) for the NOON state. The quantum Fisher information $F(\rho_\phi)$ is then expressed as [36]

$$F(\rho_\phi) = \text{Tr}(\rho_\phi L_\phi L_\phi^\dagger) = \text{Tr}(\rho_\phi L_\phi^2). \tag{21}$$

The symmetric logarithmic operator L_ϕ is given by

$$\frac{\lambda_i + \lambda_j}{2} \langle i | L_\phi | j \rangle = \langle i | \frac{\partial \rho_\phi}{\partial \phi} | j \rangle, \tag{22}$$

for all i and j, where λ_i and $|i\rangle$ are the eigenvalue and the corresponding eigenvector of ρ_ϕ. By evaluating ρ_ϕ and $\partial\rho_\phi/\partial\phi$ from Eq. (6) and then using Eqs. (21) and (22), we obtain the quantum Fisher information for the mm' state

$$F_{mm'} = (\Delta m)^2 e^{-2(\Delta m)^2 \Gamma L}, \tag{23}$$

leading to the quantum Cramér-Rao bound

$$\delta\phi_{\text{QCRB},mm'} \geqslant \frac{1}{\sqrt{F_{mm'}}} = \frac{1}{\Delta m e^{-(\Delta m)^2 \Gamma L}}. \tag{24}$$

For the NOON states, a similar calculation with Eq. (7) yields

$$F_{\text{NOON}} = N^2 e^{-2N^2 \Gamma L} \tag{25}$$

and

$$\delta\phi_{\text{QCRB,NOON}} \geqslant \frac{1}{\sqrt{F_{\text{NOON}}}} = \frac{1}{N e^{-N^2 \Gamma L}}. \tag{26}$$

Equations (24) and (26) represent the lowest bound on the uncertainty of the phase measurement for the mm' and NOON states, respectively.

For a detection scheme to be optimal, it has to saturate the quantum Cramér-Rao bound. Equations (13) and (14) represent phase sensitivity for the mm' and NOON states respectively, and these expressions can be shown to be identical to the quantum Cramér-Rao bounds in Eqs. (24) and (26) for $\phi = \pi/(2\Delta m)$ or $\phi = \pi/(2N)$ for the mm' or NOON states respectively. Thus, parity detection saturates the quantum Cramér-Rao bounds and is optimal for both the states in the presence of the phase fluctuations.

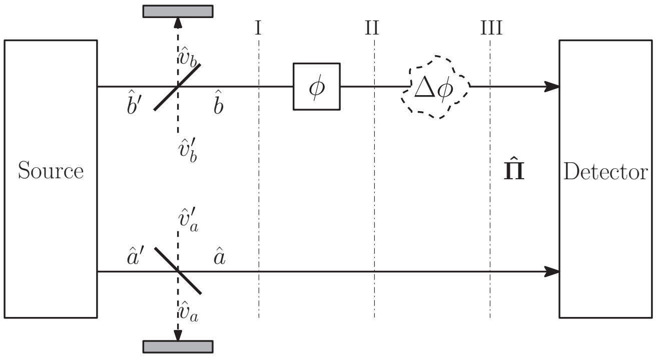

FIG. 5. Two fictitious beam splitters are introduced to Fig. 1 to mimic the loss of photons from the system into the environment. After tracing out the environment modes v_b and v_a, the system results in a mixed state at stage I.

V. EFFECTS OF BOTH PHOTON LOSS AND PHASE FLUCTUATIONS

A. Evolution

Following Ref. [20], two fictitious beam splitters are added before stage I of our previous configuration to model photon loss from the system into the environment, as shown in Fig. 5. The two fictitious beam splitters have transmittance T_a and T_b, and reflectance $R_a = 1 - T_a$ and $R_b = 1 - T_b$, respectively. General T_a and T_b are used in the following derivation of the density matrix, but later we assume $R_a = 0$ to mimic the local path, which is well isolated from the environment.

The photon loss entangles the system with the environment and leaves the system in a mixed state. For a general mm' input state, the density matrix of the system at stage II can be easily deducted from Ref. [20] as

$$\begin{aligned}\rho_{mm'}(t) = &\sum_{k=0}^{m}\sum_{k'=0}^{m'}\{|\alpha|^2 d_1(t)|k,k'\rangle\langle k,k'| \\ &+ |\beta|^2 d_2(t)|k',k\rangle\langle k',k|\} \\ &+ \sum_{k=0}^{m'}\sum_{k'=0}^{m'}\{\alpha\beta^* d_3(t)|\Delta m+k,k'\rangle\langle k,\Delta m+k'| \\ &+ \alpha^*\beta d_4(t)|k',\Delta m+k\rangle\langle\Delta m+k',k|\},\end{aligned} \tag{27}$$

where $\alpha = e^{im'\phi}/\sqrt{2}$, $\beta = e^{im\phi}/\sqrt{2}$ as before, and the coefficients $d_i (i = 1,2,3,4)$ are defined as

$$\begin{aligned}d_1(k,k',t=0) &= \binom{m}{k}\binom{m'}{k'}|T_a|^k|R_a|^{m-k}|T_b|^{k'}|R_b|^{m'-k'}, \\ d_2(k,k',t=0) &= \binom{m}{k}\binom{m'}{k'}|T_a|^{k'}|R_a|^{m'-k'}|T_b|^k|R_b|^{m-k}, \\ d_3(k,k',t=0) &= \binom{m}{\Delta m+k}^{\frac12}\binom{m}{\Delta m+k'}^{\frac12}\binom{m'}{k}^{\frac12}\binom{m'}{k'}^{\frac12} \\ &\quad\times T_a^{\frac12(\Delta m+2k)}R_a^{m'-k}T_b^{\frac12(\Delta m+2k')}R_b^{m'-k'}, \\ d_4(k,k',t=0) &= \binom{m}{\Delta m+k}^{\frac12}\binom{m}{\Delta m+k'}^{\frac12}\binom{m'}{k}^{\frac12}\binom{m'}{k'}^{\frac12} \\ &\quad\times T_a^{\frac12(\Delta m+2k')}R_a^{m'-k'}T_b^{\frac12(\Delta m+2k)}R_b^{m'-k}.\end{aligned} \tag{28}$$

Given that the system undergoes pure dephasing after stage II, we may use the previous result and show that the evolution of the density matrix $\rho_{mm'}(t)$ is

$$\begin{aligned}\dot{\rho}_{mm'}(t) = &-\Delta m^2\Gamma\sum_{k,k'=0}^{m'}\{\alpha\beta^* d_3(t)|\Delta m+k,k'\rangle\langle k,\Delta m+k'| \\ &+ \alpha^*\beta d_4(t)|k',\Delta m+k\rangle\langle\Delta m+k',k|\}.\end{aligned} \tag{29}$$

It is then easy to see that $d_1(t) = d_1(0)$, $d_2(t) = d_2(0)$, $d_3(t) = e^{-\Delta m^2\Gamma L}d_3(0)$, and $d_4(t) = e^{-\Delta m^2\Gamma L}d_4(0)$.

B. Phase sensitivity and visibility

Similar to what was done in Ref. [20], we define

$$\begin{aligned}K_1(t) &= \sum_{k=0}^{m'}[d_1(k,k,t) + d_2(k,k,t)], \\ K_2(t) &= \sum_{k=0}^{m'}[d_3(k,k,t) + d_4(k,k,t)],\end{aligned} \tag{30}$$

and it is straightforward to show that $K_1(t) = K_1(0)$ and $K_2(t) = K_2(0)e^{-\Delta m^2\Gamma L}$. From Eqs. (10) and (27), the parity signal of a mm' state under both photon loss and phase fluctuation can be shown to be

$$\langle\hat{\Pi}_{mm'}\rangle = K_1(t) + (-1)^{m+m'}K_2(t)\cos(\Delta m\phi). \tag{31}$$

This gives rise to the phase sensitivity for the parity detection for a mm' state under both photon loss and phase fluctuations as

$$\delta\phi_{mm'} = \sqrt{\frac{1-\{K_1(t)+(-1)^{m+m'}K_2(t)\cos(\Delta m\phi)\}^2}{\{\Delta m K_2(t)\sin(\Delta m\phi)\}^2}}, \tag{32}$$

where linear error propagation method in Eq. (12) is employed. Notice that when loss is negligible this sensitivity recovers Eq. (13).

A relative visibility with respect to both loss and phase fluctuations can be defined as

$$\begin{aligned}V_{mm'} &= \frac{\langle\hat{\Pi}_{mm'}\rangle_{\max} - \langle\hat{\Pi}_{mm'}\rangle_{\min}}{\langle\hat{\Pi}_{mm'}(\Gamma=0,L=0)\rangle_{\max} - \langle\hat{\Pi}_{mm'}(\Gamma=0,L=0)\rangle_{\min}}, \\ &= K_2(0)e^{-\Delta m^2\Gamma L}\end{aligned} \tag{33}$$

where $L = R_b$ characterizes the loss in the upper path and R_a is set to be zero as mentioned previously. In the limit of $L \to 0$, $K_2(0)$ approaches one and the visibility reduces to the previous result. Notice the dephasing only affects the off-diagonal terms of the density matrix while loss affects both diagonal and off-diagonal terms. However, because of the linearity of the Mach-Zehnder interferometer, the effect from photon loss is independent of that from phase fluctuation, as expected. All results in this section apply to NOON states with $N = m$ and $m' = 0$.

VI. SUMMARY

In this work, we studied the effects of phase fluctuations on the phase sensitivity and visibility of mm' and NOON states in an optical interferometric setup. Although mm' states are more robust than NOON states against photon loss, we showed that they do not provide any better performance in the presence of such phase fluctuations than their equivalent NOON counterpart. We have used the parity-detection technique for phase estimation, which can be readily implemented using photon-number-resolving detectors [37] in the low-power regime and using optical nonlinearities and homodyning in the high-power regime [21,38–40]. Using the same detection technique, we explicitly derived the phase sensitivity and the visibility in a more general case with both the photon loss and phase fluctuations. We have also presented a brief study on the quantum Fisher information for both the mm' and NOON states and shown that the parity detection serves as the optimal detection strategy in both cases as it saturates the quantum Cramér-Rao bound of the interferometric scheme.

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