Polytropic equilibrium and normal modes in cold atomic traps

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The compressibility limit of a cold gas confined in a magneto-optical trap due to multiple scattering of light is a long-standing problem. This scattering mechanism induces long-range interactions in the system, which are responsible for the occurrence of plasma-like phenomena. In the present paper, we investigate the importance of the long-range character of the mediated atom-atom interaction in the equilibrium and dynamical features of a magneto-optical trap. Making use of a hydrodynamical formulation, we derive a generalized Lane-Emden equation modeling the polytropic equilibrium of a magneto-optical trap, allowing us to describe the crossover between the two limiting cases: temperature-dominated and multiple-scattering-dominated traps. The normal collective modes of the system are also computed.

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I. INTRODUCTION

Since the first realizations of cold atomic gases [1], both theoretical and experimental investigations reveal that magneto-optical traps (MOT) pave the way for very exciting and complex physical phenomena [2]. The interest in studying the basic properties of MOTs has, however, considerably decreased after the production of Bose-Einstein condensates [3,4], as they started to be used mainly as a riding horse to achieve quantum degeneracy. However, the study of the dynamical properties of MOTs has received much attention recently, which revives the investigation of the basic properties of laser-cooled gases. Examples of such a growing interest can be found in the work realized by Kim *et al.* [5], where a parametric instability is excited by an intensity-modulated laser beam, and in the works of di Stefano [6] and coworkers [7,8], where the feedback of retroreflected laser beams can induce stochastic or deterministic chaos for a large optical thickness of the MOT.

A route for the most intriguing complex behavior in magneto-optical traps relies exactly on the multiple scattering of light, a mechanism which has been described since the early stages of MOTs as the principal limitation for the compressibility of the cloud [9,10]. Under these circumstances, the atoms experience a mediated long-range interaction potential similar to a Coulomb potential ($\sim 1/r$) [11], and the system can therefore be regarded as a one-component trapped plasma. In a series of previous works, we have given evidence of important consequences of such a plasma description of cold atomic traps [12], whereas the application of plasma physics have produced very interesting predictions. In particular, driven mechanical instabilities [13] or even more exciting instability phenomena, such as photon bubbles [14], phonon lasing [15], and the appearance of a roton minimum in the classical regime [16], form a broad set of theoretical results.

In this work, we investigate the hydrodynamic equilibrium and normal modes of cold atomic traps. For that purpose, we combine the effects of multiple scattering (described by the Coulomb-like potential) and the thermal fluctuations inside the system, which can be cast in the form of a polytropic equation of state. We derive a generalized Lane-Emden equation for the equilibrium density profiles and calculate the corresponding solutions. It is shown that the long-range interactions significantly change the Maxwell-Boltzmann equilibrium of a thermal gas. We linearize the equations to calculate the normal modes of the system for both small and large clouds, dominated by thermal and multiple-scattering effects, respectively. We describe the collective modes of the system in both temperature-limited and multiple-scattering regimes of the gas.

II. POLYTROPIC HYDRODYNAMICS AND THE GENERALIZED LANE-EMDEN EQUATION

A simple description of a cold gas in the presence of long-range interactions can be done using a set of hydrodynamic equations considered in our previous paper [12,17]:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla P}{Mn} + \frac{\mathbf{F}_t}{M} + \frac{\mathbf{F}_c}{M}, \tag{2}$$

$$\nabla \cdot \mathbf{F_c} = Qn, \tag{3}$$

where n and $\mathbf{v}$ represent the gas density and velocity, respectively, and M is the atomic mass. Here, $\mathbf{F}_c$ is the collective force, and $Q = (\sigma_R - \sigma_L)\sigma_L I_0/c$ represents the square of the effective *electric charge* of the atoms [11,12], with c being the speed of light and I_0 being the total intensity of the six laser beams. σ_R and σ_L represent the emission and absorption cross sections, respectively [2]. The term $\mathbf{F}_t = -\nabla U$ stands for the trapping force. The hydrodynamical model in Eqs. (1) and (2) provides the best description so far of the dynamics of the MOT in the presence of multiple scattering of light, a collective mechanism that prevents cooling below the recoil limit, which greatly hinders the task of increasing phase-space degeneracy [9]. The trapping potential is assumed to be harmonic,

$$U(\mathbf{r}) = \tfrac{1}{2}\kappa r^2, \tag{4}$$

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and κ represents the spring constant (in the low-saturation Doppler limit, the spring constant is approximately given by $\kappa = \alpha\mu_B \nabla B/\hbar k$; for the sake generality, we shall consider a generic value of κ in the remainder of the paper). Here, μ_B represents the Bohr magneton, $\kappa = \kappa(\delta, I_0/I_s)$ is the spring constant, $\alpha = \alpha(\delta, I_0/I_s)$ is the friction coefficient, δ is the laser detuning, I_s is the atomic saturation intensity, and $\nabla B = |\nabla B|$ represents the magnetic field gradient [12]. The harmonic trapping potential (4) is valid for a MOT of maximum radius $R_{\rm max} = |\delta|/(\gamma_M \nabla B)$, where $\gamma_M = g_J\mu_B/\hbar$ and g_J is the Land factor. Under typical experimental conditions, $\nabla B \simeq 10\ {\rm G\,cm}^{-1}$, $\delta = -2\Gamma_0$ (with Γ_0 standing for the atomic transition lifetime), we find $R_{\rm max} \sim 0.5$ cm. Above that limit, self-sustained mechanical instabilities take place [18], so we should avoid this situation in the present work.

In the absence of a microscopic theory of the ultracold gas, we assume a polytropic equation of state for the MOT of the form

$$P = C_\gamma n^\gamma, \tag{5}$$

where γ is the polytropic exponent, $C_\gamma = P(0)/n(0)^\gamma$ is a constant depending on conditions of the thermodynamic transformation, and $P(0)$ and $n(0)$ represent the values of pressure and density at the center of the cloud. By introducing Eq. (5), we are considering a wide class of thermodynamic transformations which are not necessarily adiabatic. Therefore, "polytropes" refer to the largest family of fluids that undergo polytropic transformations for which a relation between pressure and density is possible. The hydrodynamic equilibrium condition applied to Eqs. (1), (2), and (3) simply yields

$$\nabla\cdot\frac{\nabla P}{n} = 3M\omega_0^2 - Qn, \tag{6}$$

where $\omega_0 = \sqrt{\kappa/M}$ is the trapping frequency. Assuming radial symmetry, the density is given by $n = n(0)\theta(r)$, where $n(0)$ represents the peak density. Putting Eqs. (5) and (6) together, one easily obtains

$$\gamma\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\theta^{\gamma-2}\frac{d\theta}{d\xi}\right) - \Gamma\theta + 1 = 0, \tag{7}$$

where $\Gamma = Qn(0)/3M\omega_0^2 C_\gamma$ represents the ratio of interaction to kinetic energy (coupling parameter). The distance $\xi = r/a_\gamma$ is given in units of a generalized Wigner-Seitz radius

$$a_\gamma = \sqrt{\frac{3M\omega_0^2}{C_\gamma}}\, n(0)^{-(\gamma-1)/2}. \tag{8}$$

Equation (7) corresponds to a generalization to the Lane-Emden equation derived to study astrophysical fluids [19]. The important modifications in our model include both the trapping and the long-range interaction induced by multiple scattering. The present model allows us to generalize the work of Walker *et al.* [2] by relating γ to experimentally accessible density profiles. The case of rotating clamps of atoms, however, is not included in this paper and will be considered in a separate publication. In what follows, we examine the analytical solutions of Eq. (7) for some limiting cases and compare them with numerical solutions.

III. MULTIPLE-SCATTERING REGIME

In the multiple-scattering regime, typically achieved for a number of particles above $N \sim 10^4$–10^5, the MOT is essentially dominated by the collective forces [2]. In that case, $\Gamma \to 1$, and one can safely neglect the effects of the pressure. By setting $\gamma = 0$, Eq. (7) simply yields the so-called *water-bag* equilibrium profile

$$\theta(\xi) = \Gamma^{-1}\Theta(\xi - \xi_1), \tag{9}$$

where $\xi_1 = 3/(4\pi\Gamma)^{1/3}$ is the Lane-Emden radius of the MOT. In physical units, it corresponds to a MOT of radius

$$R = \left(\frac{3N}{4\pi n_0}\right)^{1/3} \approx 0.92\left(\frac{Q}{M\omega_0^2}\right)^{1/3} N^{1/3}. \tag{10}$$

This exactly corresponds to the scaling law observed in the experiments of Ref. [20]. For a typical ^{85}Rb MOT, $\delta = -\Gamma_0/2$ and $\nabla B = 10$ G cm^{-1}, we get a density $n_0 = 3\times 10^9$ cm^{-3}. This density would be reached for $N = 1500$ atoms already. This means that in typical MOTs, the large density limit very easily holds. In this case, the restoring frequency and the effective plasma frequency $\omega_P = \sqrt{Qn_0/M}$ are related as

$$\omega_0 = \frac{\omega_P}{\sqrt{3}}. \tag{11}$$

We will later see that this is related to the Mie mode, which is a natural consequence of treating the system as a trapped one-component plasma. Figure 1 shows the numerical solution to Eq. (7) for different polytropes. We observe that the Gaussian profile is modified when Γ is increased towards a water-bag profile. For traps containing a small number of particles ($N \lesssim 10^4$), the effects of the multiple scattering can be neglected. In that case, we set $\Gamma \to 0$ and obtain the following polytropic equilibrium:

$$\theta(\xi) = \left(1 - \frac{\gamma-1}{6\gamma}\xi^2\right)^{1/(\gamma-1)}. \tag{12}$$

For the interesting case of an isothermal gas, $\gamma = 1$ and $C_1 = k_BT$, this simply leads to the Maxwell-Boltzmann

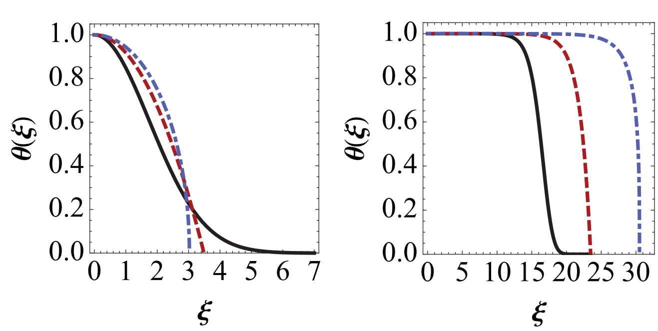

FIG. 1. (Color online) Effect of the long-range interaction on the density profile for different polytropic exponents. The left panel depicts the case where thermal effects dominate ($\Gamma = 0$), while the right panel illustrates the case where multiple scattering dominates ($\Gamma = 0.99$). The black thick line depicts the normalized density profile for the isothermal case, $\gamma = 1$. The red dashed and blue dot-dashed lines are obtained for $\gamma = 2$ and $\gamma = 4$, respectively.

equilibrium

$$n_1(r) = n(0)e^{-V_t/k_BT} = n(0)e^{-r^2/6a_1^2}, \quad (13)$$

where $a_1 = \sqrt{3m\omega_0^2/k_BT}$.

IV. NORMAL MODES

We now discuss the case of the localized oscillations, or *normal modes*, in trapped gases. By linearizing the set of fluid equations (1), (2), and (3) we get

$$-\omega^2\delta n - \frac{\gamma C_\gamma}{m}\nabla\cdot\left(n_0^{\gamma-1}\nabla\delta n\right) = \frac{1}{m}\nabla\cdot[n_0(r)\nabla\delta\phi_c],$$
$$\nabla^2\delta\phi_c = -Q\delta n, \quad (14)$$

where we have used $\nabla\delta P \simeq \gamma C_\gamma n_0^{\gamma-1}\nabla\delta n$. Here, we have assumed that the collective force can be derived from a potential, i.e., $\mathbf{F}_c = -\nabla\phi_c$. We now define the auxiliary quantity η, defined as $\delta n = (1/4\pi r^2)d\eta/dr$, which together with Eq. (14) implies

$$\frac{d}{dr}\delta\phi_c = -\frac{Q}{4\pi r^2}\eta. \quad (15)$$

Making proper substitutions, the linearized equations can be put together and result in a single expression:

$$-\frac{\gamma C_\gamma n(0)^{\gamma-1}}{m}\frac{1}{r^2}\frac{d}{dr}\left[r^2 n_0(r)^{\gamma-1}\frac{d}{dr}\left(\frac{1}{r^2}\frac{d\eta}{dr}\right)\right]$$
$$-\omega^2\frac{1}{r^2}\frac{d\eta}{dr} + \omega_p^2\frac{1}{r^2}\frac{d}{dr}[n_0(r)\eta] = 0, \quad (16)$$

where we have used $\omega_p^2 = Qn(0)/m$. In a dimensionless form, one obtains

$$-\omega^2\frac{1}{\xi^2}\frac{d\delta\theta}{d\xi} - \frac{1}{2}\gamma\omega_0^2\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\theta\frac{d}{d\xi}\left(\frac{1}{\xi^2}\frac{d\delta\theta}{d\xi}\right)\right]$$
$$+\omega_p^2\frac{1}{\xi^2}\frac{d}{d\xi}(\theta^{1/(\gamma-1)}\delta\theta) = 0. \quad (17)$$

Solutions to the latter eigenvalue problem depend on the details of the equilibrium $\theta(\xi)$ considered. The general case involves a numerical solution, for which we hereby provide analytical solutions for some limiting cases. For small clouds, lying in the temperature-limited regime ($N < 10^4$) [21], the effects of multiple scattering may be neglected, and therefore, one can set $\omega_p = 0$ in Eq. (17). Using the equilibrium profile in Eq. (12), the eigenvalue problem yields

$$-\omega^2\delta\theta - \frac{1}{2}\gamma\omega_0^2\frac{1}{\zeta^2}\frac{d}{d\zeta}[(1-\zeta^2)\zeta^2\delta\theta] = 0, \quad (18)$$

where we have performed a change of variables, $\zeta = \sqrt{6\gamma/(\gamma-1)}\xi$. The solution can be given in terms of the ansatz [22]

$$\delta\theta = \sum_{\nu,\ell} a_{\nu\ell}\zeta^{2\nu+\ell}. \quad (19)$$

Replacing this in Eq. (18), one easily obtains a recurrence relation for the coefficients $a_{\nu\ell}$, which is found to converge provided that

$$\omega^2 = \omega_0^2\{\ell + 2\nu[(\gamma-1)(\nu+\ell+1/2)+1]\}. \quad (20)$$

This result resembles the solution given by Stringari for the oscillations of a Bose-Einstein condensate ($\gamma = 2$) in a spherical harmonic trap [23]. For comparison, we recall the result known for the free Bose-Einstein condensate in the collisionless regime, $\omega = \omega_0(2\nu + \ell)$. The case $\gamma < 1$ is naturally unstable. Pure surface modes, which may eventually be more easily detectable experimentally, are obtained for $\nu = 0$ and have the following frequencies:

$$\omega_S = \omega_0\sqrt{\ell}. \quad (21)$$

Such modes correspond to the absence of nodes in the perturbation δn for $r < R$. On the other hand, breathing modes corresponding to zero-surface fluctuations at the boundary $r = R$ (thus obtained for $\ell = 0$) are also theoretically possible in small traps, with frequencies given by

$$\omega_B = \omega_0\sqrt{2\nu + (\gamma-1)(\nu+1/2)}. \quad (22)$$

Because experimental techniques usually make the measurement of frequencies in a very precise manner possible, these results can be very useful, as they relate the polytropic exponent γ to the mode frequencies.

In the deep multiple-scattering regime, we can model the equilibrium by a water-bag solution in Eq. (9). In that case, one readily obtains $(3\omega_0^2 - \omega^2)\delta n = 0$, which corresponds to a *breathing mode* in a system with long-range interactions

$$\omega_B = \omega_p = \sqrt{3}\omega_0. \quad (23)$$

The latter result is very well known in plasma physics and corresponds to an uncompressional monopole oscillation of the system at the classical plasma frequency ω_p. However, this solution is not unique. From Eqs. (1) and (2), we can obtain the following equation in the multiple-scattering regime ($\nabla P = 0$): $-\omega^2\delta n + \nabla\cdot[n(0)M^{-1}\nabla\delta\phi_c] = 0$. Combining with Eq. (11) and inserting in the linearized version of Eq. (3), $\nabla^2\delta\phi_c = Q\delta n$, we obtain

$$\nabla\cdot\left[\epsilon(\omega)\nabla\delta\phi_c^{\text{in}}\right] = 0, \quad \epsilon(\omega) = 1 - 3\frac{\omega_0^2}{\omega^2}, \quad (24)$$

which holds in the interior region of the MOT, $r < R$. Outside the MOT ($r > R$), the collective force should not vary, so we have $\nabla^2\delta\phi_c^{\text{out}} = 0$. The general solution to Eq. (24) is therefore given by

$$\delta\phi_c^{\text{in}}(r) = \sum_{\ell,m} a_{\ell,m}r^\ell Y_\ell^m(\theta,\varphi),$$
$$\delta\phi_c^{\text{out}}(r) = \sum_{\ell,m} b_{\ell,m}r^{-(\ell+1)}Y_\ell^m(\theta,\varphi). \quad (25)$$

Imposing regular continuity conditions at the surface, $r = R$, $\phi_c^{\text{in}}(R) = \phi_c^{\text{out}}(R)$, and $\frac{d}{dr}\phi_c^{\text{in}}(r)|_{r=R} = \frac{d}{dr}\phi_c^{\text{out}}(r)|_{r=R}$, one obtains the frequencies corresponding to incompressible *surface* modes:

$$\omega_S = \omega_0\sqrt{\frac{3\ell}{2\ell+1}}. \quad (26)$$

A similar result can be obtained from the Mie theory for scattering in the context of surface plasmon-polaritons (see Ref. [24] for a review). We remark here that the frequency of surface modes is bounded between the Mie and the surface plasmon resonances, $\omega_p/\sqrt{3} < \omega < \omega_p/\sqrt{2}$, totally differing from the Tonks-Dattner resonances described in our previous work [12]. We expect that this feature can be very easily observed in fluorescence measurements, for which a plateau in a frequency-resolved measurement would appear. Notice that the modes in Eq. (26) do not depend on the size of the cloud R. This is a result of neglecting the dispersion, introduced by thermal fluctuations.

V. RETARDED SURFACE MODES IN LARGE TRAPS

The previous calculation of the frequencies of surface oscillations, considered in the multiple-scattering regime, completely neglects the effects of temperature. In reality, the thermodynamic pressure is always present, and therefore, the effects of dispersion must be considered. In the same spirit of the derivation of Eq. (24), we manipulate Eqs. (1) and (2) for an isothermal gas ($P = k_B T n$) to obtain $(-\omega^2 - u_s^2\nabla^2)\delta n + \nabla \cdot [n(0)M^{-1}\nabla\delta\phi_c] = 0$, where $u_s = \sqrt{k_B T/M}$ is the thermal sound speed. Combining with the linearized Eq. (3), $\nabla^2\delta\phi_c = Q\delta n$, and after dividing everything by ω_p^2, we finally obtain

$$\left(\nabla^2 + k_{\text{in}}^2\right)\delta\phi_c^{\text{in}} = 0, \quad \nabla^2\delta\phi_c^{\text{out}} = 0, \tag{27}$$

where $k_{\text{in}} = \sqrt{\epsilon(\omega)}\omega/u_s$ and $\epsilon(\omega)$ is the frequency response given in (24). The general solution to Eq. (27) reads, after ruling out exponential growth at the origin,

$$\begin{aligned} \phi_c^{\text{in}}(r) &= \sum_{\ell,m} a_{\ell,m} j_\ell(k_{\text{in}} r) Y_\ell^m(\theta,\varphi), \\ \phi_c^{\text{out}}(r) &= \sum_{\ell,m} b_{\ell,m} r^{-(\ell+1)} Y_\ell^m(\theta,\varphi), \end{aligned} \tag{28}$$

where $j_\ell(z)$ represents the spherical Bessel functions of the first kind. Imposing the same regular boundaries as in (26), one finally gets the eigenvalues through the following condition:

$$\frac{j_\ell(\sqrt{\epsilon(\omega)}\omega x)}{\sqrt{\epsilon(\omega)}\, j_\ell'(\sqrt{\epsilon(\omega)}\omega x)} = -\frac{\ell+1}{x}, \tag{29}$$

where $x = R/\lambda_D$, with $\lambda_D = u_s/\omega_p$ standing for the effective Debye length [12,17]. The numerical solution to the latter is plotted in Fig. 2. Notice that the frequencies tend to decrease their values as R increases. Such a feature is often referred to as *retardation* in the context of surface plasmon-polaritons [24]. In real-life experiment, the effect of retardation can be tested by taking the spectrum of the Fourier transform density profile. By increasing the number of atoms N in the cloud, a broadening of the plateau $\omega_p/\sqrt{3} < \omega < \omega_p/\sqrt{2}$ should be observed, as the difference between two consecutive surface modes increases with R (see Fig. 2).

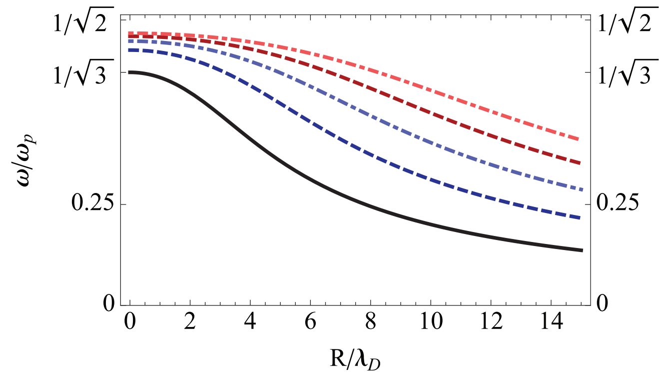

FIG. 2. (Color online) Effect of the retardation on the surface modes. From bottom to top: $\ell = 1,2,3,4$, and 5.

VI. CONCLUSIONS

We have used a hydrodynamic polytropic description of the magneto-optical trap and derived the equilibrium condition. The equilibrium density is given as the solution of a generalized Lane-Emden equation. We have also computed the spectrum of the normal modes in the two relevant limits of the system (dominated by temperature or dominated by multiple scattering). We have established a distinction between radial and surface modes as a function of the polytropic exponent γ and showed that a retardation (decrease of frequency) of the surface modes is associated with the thermal effects. The importance of the present results is twofold: first, a polytropic equation of state, which phenomenologically models a large class of hydrodynamic problems, can be easily confirmed experimentally, both by measuring the density profiles and by determining the spectrum of the normal modes; second, our problem may establish an important bridge to investigate astrophysical systems in the laboratory, especially concerning the stability and dynamics of the Lane-Emden equation. Although we have used the parameters of ^{85}Rb, our results are completely general. Indeed, for the case of Na or Cs atoms, the changes in the mass, for example, will slightly modify the nominal value of the plasma frequency by a factor of $\omega_p/\omega_p^{\text{Rb}} = \sqrt{M^{\text{Rb}}/M}$. Although this fact could have an important impact in experiments (a considerable variation in the parameters would necessarily influence the measurement techniques and even the devices involved in the measurement), the mass difference is not significant, and the typical length and frequency scales ($\lambda_D \sim 10$–$100\ \mu$m and $\omega_p \sim 50$–100 Hz [12]) remain the same.

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[1] S. Chu, Rev. Mod. Phys. **70**, 685 (1998); C. Cohen-Tannoudji, *ibid.* **70**, 707 (1998); W. D. Philips, *ibid.* **70**, 721 (1998).
[2] T. Walker, D. Sesko, and C. Wieman, Phys. Rev. Lett. **64**, 408 (1990).
[3] A. J. Legget, Rev. Mod. Phys. **73**, 307 (2001).
[4] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999).
[5] K. Kim, H.-R. Nohm, and W. Jhe, Opt. Commun. **236**, 349 (2004).

[6] A. di Stefano, M. Fauquembergue, P. Verkerk, and D. Hennequin, Phys. Rev. A **67**, 033404 (2003); A. di Stefano, P. Verkerk, and D. Hennequin, Eur. Phys. J. D **30**, 243 (2004).
[7] D. Wilkowski, J. Ringot, D. Hennequin, and J. C. Garreau, Phys. Rev. Lett. **85**, 1839 (2000).
[8] D. Hennequin, Eur. Phys. J. D **28**, 135 (2004).
[9] J. Dalibard, Opt. Commun. **68**, 203 (1988).
[10] D. W. Sesko, T. G. Walker, and C. E. Wieman, J. Opt. Soc. Am. B **8**, 946 (1991).
[11] L. Pruvost, I. Serre, H. T. Duong, and J. Jortner, Phys. Rev. A **61**, 053408 (2000).
[12] J. T. Mendonça, R. Kaiser, H. Terças, and J. Loureiro, Phys. Rev. A **78**, 013408 (2008); J. T. Mendonca, *ibid.* **81**, 023421 (2010).
[13] H. Terças, J. T. Mendonça, and R. Kaiser, Europhys. Lett. **89**, 53001 (2010).
[14] J. T. Mendonça and R. Kaiser, Phys. Rev. Lett. **108**, 033001 (2012).
[15] J. T. Mendonça, H. Terças, G. Brodin, and M. Marklund, Eur. Phys. Lett. **91**, 33001 (2010).
[16] H. Terças, J. T. Mendonça, and V. Guerra, Phys. Rev. A **86**, 053630 (2012).
[17] J. T. Mendonça and H. Terças, *Physics of Ultra-Cold Matter*, Spring Series on Atomic, Optical and Plasma Physics Vol. 70 (Springer, Berlin, 2012).
[18] G. Labeyrie, F. Michaud, and R. Kaiser, Phys. Rev. Lett. **96**, 023003 (2006).
[19] C. G. Bömer and T. Harko, J. Cosmol. Astropart. Phys. **6**, 25 (2007).
[20] G. L. Gattobigio, T. Pohl, G. Labeyrie, and R. Kaiser, Phys. Scr. **81**, 025301 (2010).
[21] C. G. Townsend, N. H. Edwards, C. J. Cooper, K. P. Zetie, C. J. Foot, A. M. Steane, P. Szriftgiser, H. Perrin, and J. Dalibard, Phys. Rev. A **52**, 1423 (1995).
[22] More formally, the solution can be found directly by inspection, where the departure from equilibrium is found to be $\delta n \propto r^{\ell}(1-\zeta^2)^{1/(\gamma-1)}F[-\nu,\nu+\ell+1/(\gamma-1),\ell+3/2;\zeta^2]$, where $F[a,b,c;z]$ represents the hypergeometric function.
[23] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[24] J. M. Pitarke, V. M. Silkin, E. V. Chulkov, and P. M. Echenique, Rep. Prog. Phys. **70**, 1 (2007).