Entanglement of movable mirrors in a correlated-emission laser

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(Received 27 March 2013; published 30 August 2013)

We propose an experimental scheme for entangling two macroscopic mechanical resonators (movable mirrors) by their coupling to the two-mode fields of a correlated-emission laser inside a doubly resonant cavity. With this aim we investigate the quantum Langevin equations that describe the interaction of the field-mirror system in conjuction with the master equation of the correlated-emission laser. We show that the steady-state entanglement of two mirrors as well as that of two-mode fields can be obtained in the regime of strong field-mirror coupling when the input lasers are scattered at the anti-Stokes sidebands. Remarkably, the degree of entanglement for both the mirror pair and the field pair can be controlled by an external field driving the gain medium. Our scheme is able to entangle two macroscopic objects with current state-of-the-art experimental apparatus.

DOI: 10.1103/PhysRevA.88.022338 PACS number(s): 03.67.Bg, 03.65.Ud, 42.50.Dv, 37.30.+i

I. INTRODUCTION

Quantum entanglement [1] has received a lot of interest in different areas of physics. Entanglement of microscopic objects such as photons and ions [2] has been demonstrated and it is of fundamental and practical importance whether we can generate entanglement between mesoscopic and even macroscopic objects.

Optomechanical coupling via radiation pressure between a cavity field and a movable mirror [3] is a promising approach to study entanglement of mesoscopic systems. Quantum superposition states of a microscopic cavity field and a massive moving mirror have been studied through optomechanical control [4–7]. Other works have investigated the optomechanical entanglement of a movable mirror and a cavity field [8], of a micromechanical resonator and output fields [9], and even atom-light-mirror tripartite entanglement [10]. Experimentally, strong coupling between a micromechanical resonator and a cavity field has been demonstrated [11], paving the way towards quantum optical control of entanglement in mesoscopic objects.

Recently different schemes have also been proposed to entangle two movable mirrors via optomechanical coupling in a ring cavity [12], in two separate cavities using two entangled output fields from a nondegenerate parametric amplifier [13], in a double-mode cavity driven by squeezed light fields [14], in a classically driven Fabry-Pérot cavity [15], and by injecting broadband squeezed vacuum light and laser light into a cavity [16]. On the other hand, a doubly resonant cavity system has been recently proposed as an entangled-light amplifier using both a three-level cascade [17] and a four-level atomic system [18].

With all these works considered, an interesting question arises as to whether a doubly resonant cavity with a gain medium inside can be used to entangle its two movable mirrors coupled via radiation pressure to the intracavity fields. First, entanglement of movable mirrors manifests Schrödinger's idea [1] on macroscopic entanglement, which is of conceptual importance. Second, from a practical point of view, entanglement between the end mirrors in an active Michelson interferometer is relevant in the detection of gravitational waves [19], as in the Laser Interferometer Gravitational Wave Observatory (LIGO) [20].

In this regard, Zhou *et al*. recently studied a scheme of using injected atomic coherence, that is, the coherence necessary to generate entanglement is supplied in the form of the initial superposition state of atoms [21]. They showed that the two cavity modes are entangled when interacting with cascading three-level atoms with initial coherence and that the entanglement can be transferred to the mechanical mirrors via radiation pressure. Considering the laser operation, however, one may apply an external driving field to a gain medium in order to establish the atomic coherence, e.g., in a correlated-spontaneous-emission laser (CEL) [22], rather than simply preparing an initial state of coherence. In this paper, therefore, we consider the CEL system with an external driving field [17,22] to generate entanglement between two movable mirrors or two-mode fields. We show that in the regime of strong field-mirror coupling, the entanglement between the field modes can be transferred to the movable mirrors when the cavity-driving laser frequencies are both scattered to the anti-Stokes sidebands. Remarkably, in contrast to previous work [14,16,21], our scheme can control the degree of entanglement of two movable mirrors and of two field modes with an external field driving the gain medium.

The paper is structured as follows. In Sec. II we describe our scheme and introduce the Hamiltonian of the system. In Sec. III we derive the linearized quantum Langevin equations for the field-mirror subsystem. In Sec. IV we study the entanglement generation between two mirrors in three different scattering processes. In Sec. V we summarize the main results of this work.

II. THE MODEL AND HAMILTONIAN OF THE SYSTEM

We consider a scheme for using a gain medium inside a doubly resonant cavity with two movable mirrors M_1 and M_2 (see Fig. 1). The gain medium consists of three-level atoms with the natural linewidth γ in a cascade configuration [Fig. 1(b)] pumped to the lowest level $|c\rangle$ at a rate r_a. The atoms have two dipole-allowed transitions $|a\rangle$-$|b\rangle$ and

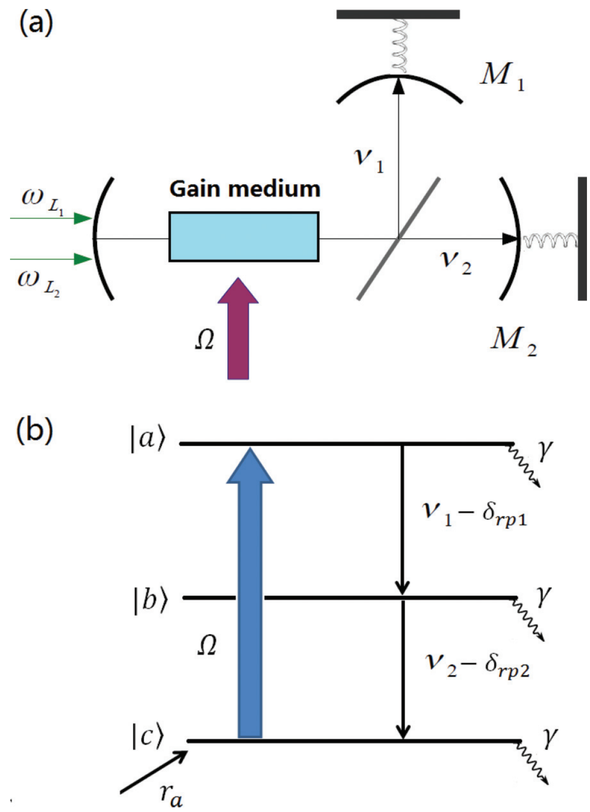

FIG. 1. (Color online) (a) Scheme of entangling movable mirrors in a doubly resonant cavity. The gain medium inside the cavity is driven by an external field and also interacts with two cavity modes. (b) Level diagram of the gain medium.

$|b\rangle$-$|c\rangle$ that are coupled to two nondegenerate cavity modes at frequencies ν_1 and ν_2, respectively. They also have a dipole-forbidden transition $|a\rangle$-$|c\rangle$ that can be induced by a resonant semiclassical laser field with Rabi frequency Ω and phase ϕ. As shown in Ref. [17], the dipole-forbidden transition can be induced, for example, by applying a strong magnetic field for a magnetic-dipole-allowed transition. Each of the mirrors is treated as a quantum-mechanical harmonic oscillator with an effective mass m and frequencies ω_{m_j} $(j = 1,2)$. The annihilation and the creation operators of each vibrational mode are b_j and $b_j^\dagger$ satisfying $[b_j, b_j^\dagger] = 1$.

In the interaction picture, the Hamiltonian of the system under the rotating-wave approximation is given by $H_I = H_I^{af} + H_I^{fm}$, where [8,17]

$$H_I^{af} = \hbar g_1(\sigma_{ab}a_1 + a_1^\dagger \sigma_{ba}) + \hbar g_2(\sigma_{bc}a_2 + a_2^\dagger \sigma_{cb}) - \hbar \frac{\Omega}{2}(\sigma_{ac}e^{-i\phi} + \sigma_{ca}e^{i\phi}), \tag{1}$$

$$H_I^{fm} = \sum_{j=1}^{2} \left(\hbar \delta_{rp_j} a_j^\dagger a_j + \hbar \omega_{m_j} b_j^\dagger b_j\right) + \sum_{j=1}^{2} i\hbar \left(E_j a_j^\dagger e^{i\delta_j t} - E_j^* a_j e^{-i\delta_j t}\right) - \hbar G_1 q_1 a_1^\dagger a_1 - \hbar G_2 q_2 a_2^\dagger a_2. \tag{2}$$

Here H_I^{af} and H_I^{fm} describe the atom-field and the field-mirror interaction, respectively. In Eq. (1), the first two terms represent the interaction between the atomic medium and the two cavity modes with coupling strengthes g_1 and g_2, respectively. σ_{ij} $(i,j = a,b,c)$ is the Pauli pseudospin operator $|i\rangle\langle j|$ of the atoms. a_1, a_2 and $a_1^\dagger, a_2^\dagger$ are the annihilation and the creation operators of each cavity mode. The third term in Eq. (1) generates an atomic coherence between the levels $|a\rangle$ and $|c\rangle$, which contributes to the lasing operation of the CEL.

In Eq. (2), the first line represents the energies of the cavity modes and the movable mirrors. Note that δ_{rp_1} and δ_{rp_2} denote the frequency shift of each cavity mode due to radiation pressure, which will be clarified later. We assume a resonant condition on the shifted cavity modes, i.e., $(\nu_1 - \delta_{rp_1}) - (\omega_a - \omega_b) = 0$ and $(\nu_2 - \delta_{rp_2}) - (\omega_b - \omega_c) = 0$. The second line in Eq. (2) represents two laser fields driving the cavity modes, where $|E_j| = \sqrt{\frac{2P_j\kappa_j}{\hbar\omega_{L_j}}}$ with input power P_j, laser frequencies ω_{L_j}, and decay rates κ_j of each mode. Here $\delta_j \equiv \nu_j - \delta_{rp_j} - \omega_{L_j}$ is the effective detuning of each cavity-driving field. The third line in Eq. (2) corresponds to the coupling via radiation pressure [3] of mirrors and cavity modes with the coupling rates $G_j \equiv \frac{\nu_j}{L_j}\sqrt{\frac{\hbar}{m\omega_{m_j}}}$ and cavity lengths L_j. We also define dimensionless position and momentum operators $q_j = (b_j + b_j^\dagger)/\sqrt{2}$ and $p_j = (b_j - b_j^\dagger)/i\sqrt{2}$ for the mirrors.

III. DYNAMICAL EQUATIONS OF THE ATOM-CAVITY AND MIRROR-CAVITY SUBSYSTEMS

In this section, we introduce the master equation for the reduced density operator of the cavity field modes in the CEL and then derive the quantum Langevin equations for the field-mirror subsystem separately. This approach is justified if the atom-field interaction is much stronger than the field-mirror interaction, and the atomic medium is here treated as a general reservoir to the two cavity modes.

A. Master equation for the two-mode fields

We first introduce the master equation for the two-mode fields in the CEL. We focus on the regime $\gamma \gg \kappa_j$ where the atoms reach their steady state much faster than the cavity fields so that we can eliminate the dynamics of the atoms. The master equation is obtained from the interaction Hamiltonian H_I^{af} using the standard methods of laser theory [17,23] as

$$\begin{aligned}\dot{\rho} = &-\alpha_{11}(a_1 a_1^\dagger \rho + \rho a_1 a_1^\dagger - 2a_1^\dagger \rho a_1)\\ &-\alpha_{22}(a_2^\dagger a_2 \rho + \rho a_2^\dagger a_2 - 2a_2 \rho a_2^\dagger)\\ &-[\alpha_{12}a_1 a_2 \rho + \alpha_{21}\rho a_1 a_2 - (\alpha_{12}+\alpha_{21})a_2 \rho a_1]\\ &-[\alpha_{12}\rho a_1^\dagger a_2^\dagger + \alpha_{21}a_1^\dagger a_2^\dagger \rho - (\alpha_{12}+\alpha_{21})a_1^\dagger \rho a_2^\dagger]\\ &-\kappa_1(a_1^\dagger a_1 \rho + \rho a_1^\dagger a_1 - 2a_1 \rho a_1^\dagger)\\ &-\kappa_2(a_2^\dagger a_2 \rho + \rho a_2^\dagger a_2 - 2a_2 \rho a_2^\dagger).\end{aligned} \tag{3}$$

The first two terms proportional to α_{11} and α_{22} in the above equation describe the emission from level $|a\rangle$ and the absorption from level $|c\rangle$, respectively. The next two terms correspond to the effective coupling of the two cavity modes via atomic coherence generated by the classical field Ω. The last two terms represent the damping of each cavity mode with rate κ_j ($j = 1,2$).

In Eq. (3), the two quantum fields of the cavity are taken into account to second order in the coupling constants g_1 and g_2 while the classical laser field is considered to all orders in the Rabi frequency Ω. The coefficients α_{11}, α_{22}, α_{12}, and α_{21} are then given by

$$\alpha_{11} = \frac{g_1^2 r_a}{4} \frac{3\Omega^2}{(\Omega^2+\gamma^2)(\Omega^2/4+\gamma^2)}, \quad (4)$$

$$\alpha_{22} = g_2^2 r_a \frac{1}{(\Omega^2+\gamma^2)}, \quad (5)$$

$$\alpha_{12} = -g_1 g_2 r_a \frac{\Omega}{\gamma(\Omega^2+\gamma^2)}, \quad (6)$$

$$\alpha_{21} = \frac{g_1 g_2 r_a}{4} \frac{\Omega(\Omega^2-2\gamma^2)}{\gamma(\Omega^2+\gamma^2)(\Omega^2/4+\gamma^2)}, \quad (7)$$

where the phase of the driving field is assumed to be $\phi = -\frac{\pi}{2}$ for simplicity.

With the above master equation, we can address the effective field-field coupling due to the gain medium in the CEL. Furthermore, the diffusion coefficients for the cavity modes are readily derived from the master equation using Einstein's relation [23]. For any operators O_1, O_2 and their noise operators F_{O_1}, F_{O_2}, it follows from Einstein's relation

$$2\langle D_{O_1O_2}\rangle = \frac{d}{dt}\langle O_1 O_2\rangle - \left\langle\left(\frac{dO_1}{dt} - F_{O_1}\right)O_2\right\rangle - \left\langle O_1\left(\frac{dO_2}{dt} - F_{O_2}\right)\right\rangle \quad (8)$$

that the nonzero diffusion coefficients are

$$\begin{aligned} &2\langle D_{a_1^\dagger a_1}\rangle = 2\alpha_{11}, \quad 2\langle D_{a_1 a_1^\dagger}\rangle = 2\kappa_1, \\ &2\langle D_{a_2 a_2^\dagger}\rangle = 2(\kappa_2 + \alpha_{22}), \\ &2\langle D_{a_2 a_1}\rangle = 2\langle D_{a_1^\dagger a_2^\dagger}\rangle = -(\alpha_{12}+\alpha_{21}). \end{aligned} \quad (9)$$

In the following, we use these diffusion coefficients in specifying the correlation functions of the cavity-mode noise operators.

B. The quantum Langevin equations and the steady-state covariance matrix

Now we derive the quantum Langevin equations for the field-mirror subsystem and obtain their covariance matrix in steady state. We consider a general analysis for the field-mirror subsystem including cavity decay, mirror damping, cavity-mode noise, and the Brownian noise of the mirrors. With the help of the master equation (3) and the field-mirror interaction Hamiltonian H_I^{fm}, the nonlinear quantum Langevin equations are obtained as

$$\begin{aligned} \dot{b}_1 &= -i\omega_m b_1 + i\frac{G_1}{\sqrt{2}} a_1^\dagger a_1 - \gamma_m b_1 + \xi_1, \\ \dot{b}_2 &= -i\omega_m b_2 + i\frac{G_2}{\sqrt{2}} a_2^\dagger a_2 - \gamma_m b_2 + \xi_2, \\ \dot{a}_1 &= -(\kappa_1 + i\delta_{rp_1})a_1 + iG_1 a_1 q_1 + E_1 e^{i\delta_1 t} \\ &\quad + \alpha_{11} a_1 + \alpha_{12} a_2^\dagger + F_{a_1}, \\ \dot{a}_2 &= -(\kappa_2 + i\delta_{rp_2})a_2 + iG_2 a_2 q_2 + E_2 e^{i\delta_2 t} \\ &\quad - \alpha_{22} a_2 - \alpha_{21} a_1^\dagger + F_{a_2}. \end{aligned} \quad (10)$$

We here assume that the two mirrors have the same damping rate γ_m and the same oscillation frequency ω_m. In Eq. (10), we have the quantum Brownian noise operators ξ_j and $\xi_j^\dagger$ with their δ-correlated function at temperature T in the limit of large mechanical quality factor $Q = \omega_m/\gamma_m \gg 1$ [24],

$$\langle \xi_j(t)\xi_k^\dagger(t') + \xi_k^\dagger(t')\xi_j(t)\rangle/2 \approx \gamma_m(2n+1)\delta_{jk}\delta(t-t'), \quad (11)$$

where $n = [\exp(\hbar\omega_m/k_B T) - 1]^{-1}$ denotes the average thermal photon number and k_B is the Boltzmann constant. We introduce the nonzero correlation functions of the cavity noise operators F_{a_1}, $F_{a_1^\dagger}$, F_{a_2}, and $F_{a_2^\dagger}$ in the presence of the atomic medium as

$$\langle F_{O_1}(t) F_{O_2}(t')\rangle = 2\langle D_{O_1O_2}\rangle\,\delta(t-t'), \quad (12)$$

where $\langle D_{O_1O_2}\rangle$ are given by Eq. (9).

The nonlinear Langevin equations can be transformed to linearized Langevin equations of zero-mean fluctuation operators around c-number steady values. This is justified if the input power P_j to the cavity modes is very large [25]. That is, we take $b_j = b_{js} + \delta b_j$ and $\tilde{a}_j = \alpha_{js} + \delta\tilde{a}_j$, where $\tilde{a}_j \equiv a_j e^{-i\delta_j t}$ are the slowly varying field-mode operators. The steady values are given by

$$\begin{aligned} &p_{1s} = p_{2s} = 0, \quad q_{js} = \frac{G_j|\alpha_{js}|^2}{\omega_m}, \\ &\alpha_{1s} = \frac{E_1}{s_1}, \quad \alpha_{2s} = \frac{E_2}{s_2}, \end{aligned} \quad (13)$$

with $s_j = i(\nu_j - G_j q_{js} - \omega_{L_j}) + \kappa_j + (-1)^j \alpha_{jj}$ and $p_{js} = (b_{js} - b_{js}^*)/i\sqrt{2}$, $q_{js} = (b_{js} + b_{js}^*)/\sqrt{2}$. The term $\nu_j - G_j q_{js} - \omega_{L_j}$ represents the effective detuning of each cavity-mode frequency. Thus it follows that the mean frequency shift due to radiation pressure, which is introduced in Eq. (2), is given by $\delta_{rp_j} \equiv G_j q_{js}$.

We introduce the slowing varying fluctuation operators $\delta\tilde{b}_j(t) \equiv \delta b_j(t) e^{i\omega_{m_j} t}$ and $\delta a_j(t) \equiv \delta\tilde{a}_j(t) e^{i\delta_j t}$ and write the linear quantum Langvin equations for them as

$$\begin{aligned} \delta\dot{\tilde{b}}_1 &= -\gamma_m \delta\tilde{b}_1 + \sqrt{2\gamma_m} b_{1\text{in}} \\ &\quad + i\frac{G_1\alpha_{1s}^*}{\sqrt{2}}\delta a_1 e^{i(\omega_m-\delta_1)t} + i\frac{G_1\alpha_{1s}}{\sqrt{2}}\delta a_1^\dagger e^{i(\omega_m+\delta_1)t}, \\ \delta\dot{\tilde{b}}_2 &= -\gamma_m \delta\tilde{b}_2 + \sqrt{2\gamma_m} b_{2\text{in}} \\ &\quad + i\frac{G_2\alpha_{2s}^*}{\sqrt{2}}\delta a_2 e^{i(\omega_m-\delta_2)t} + i\frac{G_2\alpha_{2s}}{\sqrt{2}}\delta a_2^\dagger e^{i(\omega_m+\delta_2)t}, \\ \delta\dot{a}_1 &= -\kappa_{11}\delta a_1 + \alpha_{12}\delta a_2^\dagger + F_{a_1} \\ &\quad + i\frac{G_1\alpha_{1s}}{\sqrt{2}}(\delta\tilde{b}_1 e^{-i(\omega_m-\delta_1)t} + \delta\tilde{b}_1^\dagger e^{i(\omega_m+\delta_1)t}), \\ \delta\dot{a}_2 &= -\kappa_{22}\delta a_2 - \alpha_{21}\delta a_1^\dagger + F_{a_2} \\ &\quad + i\frac{G_2\alpha_{2s}}{\sqrt{2}}(\delta\tilde{b}_2 e^{-i(\omega_m-\delta_2)t} + \delta\tilde{b}_2^\dagger e^{i(\omega_m+\delta_2)t}), \end{aligned} \quad (14)$$

where $\kappa_1 = \kappa_2 = \kappa$ for simplicity and $\kappa_{11} \equiv \kappa - \alpha_{11}$ and $\kappa_{22} \equiv \kappa + \alpha_{22}$. We have introduced the noise operators

$b_{\rm jin} \equiv \xi_j e^{i\omega_m t}/\sqrt{\gamma_m}$ for the mirrors' vibrational modes ($j = 1,2$), which satisfy the correlation relations [26]

$$\begin{aligned}\langle b_{\rm jin}^\dagger(t) b_{\rm kin}(t')\rangle &= n\delta_{jk}\delta(t-t'),\\ \langle b_{\rm jin}(t) b_{\rm kin}^\dagger(t')\rangle &= (n+1)\delta_{jk}\delta(t-t').\end{aligned} \tag{15}$$

It has been shown in [9,10] that the optomechanical interaction and consequently the field-mirror entanglement are enhanced when the cavity-driving light is scattered by the vibrating cavity boundary at the first Stokes ($\omega_{L_j} - \omega_m$) and anti-Stokes ($\omega_{L_j} + \omega_m$) sidebands. Therefore there are two choices of interest to us for the detuning of each cavity-driving field, i.e., $\delta_1 = \pm\omega_m$ and $\delta_2 = \pm\omega_m$. It is seen from Eqs. (14) that for the anti-Stokes sidebands $\delta_j = +\omega_m$, the field fluctuation operators δa_j are coupled to the movable-mirror fluctuation operator $\delta\tilde{b}_j$ effectively in a beam-splitter-like (BSL) process. On the other hand, for the first Stokes sidebands $\delta_j = -\omega_m$, each pair of operators δa_j and $\delta\tilde{b}_j$ is coupled effectively in a parametric down-conversion (PDC) process. Due to the symmetric configuration of field-field and mirror-mirror in our setup, we may deal with three different situations, i.e., $\delta_1 = \delta_2 = +\omega_m$, $\delta_1 = \delta_2 = -\omega_m$, and $\delta_1 = -\delta_2 = \pm\omega_m$.

We define the dimensionless position and momentum fluctuation operators $\delta q_j = (\delta\tilde{b}_j + \delta\tilde{b}_j^\dagger)/\sqrt{2}$, $\delta p_j = (\delta\tilde{b}_j - \delta\tilde{b}_j^\dagger)/i\sqrt{2}$, $\delta x_j = (\delta a_j + \delta a_j^\dagger)/\sqrt{2}$, and $\delta y_j = (\delta a_j - \delta a_j^\dagger)/i\sqrt{2}$ and their corresponding noise operators $q_{\rm jin}$, $p_{\rm jin}$, F_{x_j}, and F_{y_j} ($j = 1,2$). We also define $u = (\delta q_1,\delta p_1,\delta q_2,\delta p_2,\delta x_1,\delta y_1,\delta x_2,\delta y_2)^T$. Then, the linear Langevin equations in the rotating-wave approximation ($\omega_m \gg \kappa, G$) can be written in a compact form as

$$\dot{u}(t) = Au(t) + B(t), \tag{16}$$

where

$$A = \begin{pmatrix} -\gamma_m & 0 & 0 & 0 & 0 & -(\frac{\delta_1}{\omega_m})G & 0 & 0\\ 0 & -\gamma_m & 0 & 0 & G & 0 & 0 & 0\\ 0 & 0 & -\gamma_m & 0 & 0 & 0 & 0 & -(\frac{\delta_2}{\omega_m})G\\ 0 & 0 & 0 & -\gamma_m & 0 & 0 & G & 0\\ 0 & -(\frac{\delta_1}{\omega_m})G & 0 & 0 & -\kappa_{11} & 0 & \alpha_{12} & 0\\ G & 0 & 0 & 0 & 0 & -\kappa_{11} & 0 & -\alpha_{12}\\ 0 & 0 & 0 & -(\frac{\delta_2}{\omega_m})G & -\alpha_{21} & 0 & -\kappa_{22} & 0\\ 0 & 0 & G & 0 & 0 & \alpha_{21} & 0 & -\kappa_{22} \end{pmatrix}, \tag{17}$$

and $B(t) = (\sqrt{\gamma_m}q_{1\rm in},\sqrt{\gamma_m}p_{1\rm in},\sqrt{\gamma_m}q_{2\rm in},\sqrt{2\gamma_m}p_{2\rm in},F_{x_1},F_{y_1},F_{x_2},F_{y_2})$. For simplicity the parameters of the two field-mirror pairs are chosen such that $G_1\alpha_{1s} = G_2\alpha_{2s} = |G_j\alpha_{js}|$. The effective coupling rate is thus defined as $G \equiv G_j\alpha_{js}/\sqrt{2}$, which is controlled by the cavity-driving input power $P \equiv P_1 = P_2$.

We study the quantum fluctuations of the operators when the system reaches a steady state. Since the quantum noises $q_{\rm jin}, p_{\rm jin}$ and F_{x_j}, F_{y_j} are all zero-mean Gaussian noises and the dynamics has been linearized, the steady state of the system becomes a zero-mean multipartite Gaussian state. We define the covariance matrix (CM) of the system whose elements are $V_{ij} = [\langle u_i(\infty)u_j(\infty) + u_j(\infty)u_i(\infty)\rangle]/2$. The system can reach a stable steady state when all real parts of the eigenvalues of the drift matrix A are negative. The eigenvalues s are given by the roots of the equation $s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$ with $a_0 = (\kappa_{11}\gamma_m + \frac{\delta_1}{\omega_m}G^2)(\kappa_{22}\gamma_m + \frac{\delta_2}{\omega_m}G^2) + \alpha_{12}\alpha_{21}\gamma_m^2$, $a_1 = \gamma_m^2(\kappa_{11} + \kappa_{22}) + 2\gamma_m(\kappa_{11}\kappa_{22} + \alpha_{12}\alpha_{21}) + G^2(\frac{\delta_1}{\omega_m}\kappa_{22} + \frac{\delta_2}{\omega_m}\kappa_{11}) + \gamma_m(\frac{\delta_1}{\omega_m} + \frac{\delta_2}{\omega_m})$, $a_2 = 2\gamma_m(\kappa_{11} + \kappa_{22}) + (\kappa_{11}\kappa_{22} + \gamma_m^2 + \alpha_{12}\alpha_{21}) + G^2(\frac{\delta_1}{\omega_m} + \frac{\delta_2}{\omega_m})$, and $a_3 = (\kappa_{11} + \kappa_{22} + 2\gamma_m)(\delta_j = \pm\omega_m,\ j = 1,2)$. We obtain from the Routh-Hurwitz stability criterion [27] the following stability conditions:

$$a_i > 0 \quad (i = 0,1,2,3), \quad a_3a_2a_1 > a_1^2 + a_3^2 a_0. \tag{18}$$

Now we simply assume that the parameters satisfy the stationarity conditions; then the CM in the steady state satisfies the Lyapunov equation [15]

$$AV + VA^T = -D, \tag{19}$$

where

$$D = \begin{pmatrix} D_m & 0\\ 0 & D_f \end{pmatrix}, \tag{20}$$

with $D_m = {\rm diag}[\gamma_m(2n+1),\gamma_m(2n+1),\gamma_m(2n+1),\gamma_m(2n+1)]$ and

$$D_f = \begin{pmatrix} \kappa + \alpha_{11} & 0 & -\frac{\alpha_{12}+\alpha_{21}}{2} & 0\\ 0 & \kappa + \alpha_{11} & 0 & \frac{\alpha_{12}+\alpha_{21}}{2}\\ -\frac{\alpha_{12}+\alpha_{21}}{2} & 0 & \kappa + \alpha_{22} & 0\\ 0 & \frac{\alpha_{12}+\alpha_{21}}{2} & 0 & \kappa + \alpha_{22} \end{pmatrix}. \tag{21}$$

The exact solution of the CM can be obtained from Eq. (19) with nontrivial elements under three different situations of interest to us. We pick up relevant elements in each case to obtain a two-mode covariance matrix V^s in the steady state with $u = (\delta Q_1,\delta P_1,\delta Q_2,\delta P_2)^T$ where $Q_j = q_j,x_j$ and $P_j = p_j,y_j$ ($j = 1,2$) and test the entanglement conditions for three different pairs of macroscopic objects, namely, field-field, field-mirror, and mirror-mirror.

IV. QUANTITATIVE MEASURE OF ENTANGLEMENT FOR A BIPARTITE GAUSSIAN STATE

In this section, we investigate the degree of entanglement for each bipartite Gaussian state of the field-mirror system

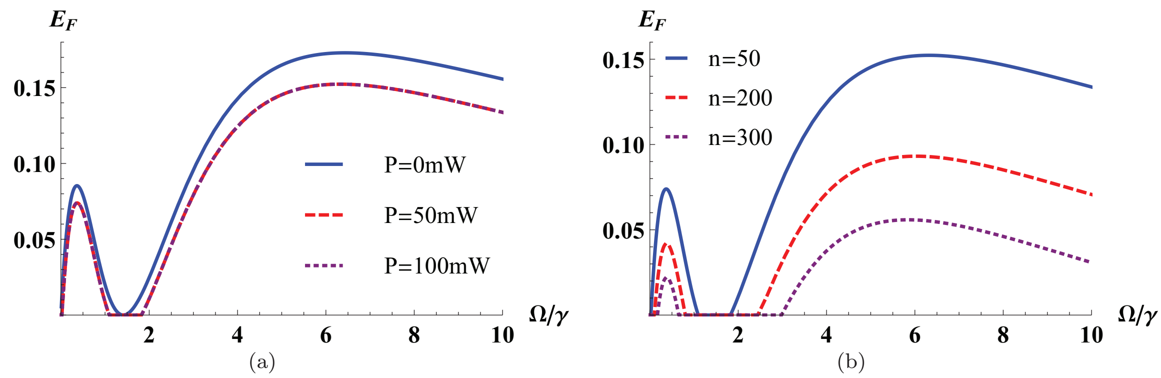

FIG. 2. (Color online) Logarithmic negativity $E_{\mathcal{F}}$ of the two-mode fields as a function of Ω/γ. (a) Effect of the cavity-driving laser power P on $E_{\mathcal{F}}$ at $n = 50$ for $P = 0$ mW (blue solid curve), $P = 50$ mW (red dashed curve), and $P = 100$ mW (purple dotted curve). (b) Effect of thermal noise of the mirrors on $E_{\mathcal{F}}$ at $P = 100$ mW for $n = 50$ (blue solid curve), $n = 200$ (red dashed curve), and $n = 300$ (purple dotted curve). The optomechanical doubly resonant cavity parameters are taken as $\gamma_m = 2\pi \times 50$ Hz, $\kappa = 2\pi \times 215$ kHz, $\omega_m = 2\pi \times 3$ MHz, $m = 5$ ng, $L_1 = 1.064$ mm, and $L_2 = 0.810$ mm according to recent experiments [11,35]. The input laser wavelengths are $\lambda_1 = 810$ nm and $\lambda_2 = 1064$ nm and the input power P varies from 0 to 100 mW so that $0 \leqslant G/\sqrt{\kappa\gamma_m} \leqslant 140$. The atom-cavity coupling strength is $g_1 = g_2 = g = 2\pi \times 2.5$ MHz, the injection rate $r_a = 1.6$ MHz, and the atomic decay rate $\gamma = 1.7$ MHz.

under three different detuning conditions leading effectively to a BSL or PDC process for the mirror-field coupling. By definition, a quantum state $\hat{\rho}$ of a bipartite system is said to be separable if and only if $\hat{\rho}$ can be written as a convex combination of product states, i.e.,

$$\hat{\rho} = \sum_j p_j \hat{\rho}_{jA} \otimes \hat{\rho}_{jB}, \tag{22}$$

where $\hat{\rho}_{jA}$ and $\hat{\rho}_{jB}$ are the density operators of mode A and mode B respectively with the probability sum $\sum_j p_j = 1$ ($0 \leqslant p_j \leqslant 1$). Many criteria have been proposed to test the separability for a continuous-variable (CV) bipartite system [28–34].

In this paper we employ a quantitative measure of entanglement, i.e., the logarithmic negativity $E_{\mathcal{N}}$ [32] that is based on the negative eigenvalues of the density matrix under partial transposition. In the case of a two-mode Gaussian state, the logarithmic negativity is given by

$$E_{\mathcal{N}} = \max\{0, -\ln 2\eta^-\} + \max\{0, -\ln 2\eta^+\}, \tag{23}$$

where $\eta^\pm$ are the two positive roots of the characteristic function of the covariance matrix,

$$\eta^4 - (\det V_A + \det V_B - 2\det V_C)\eta^2 + \det V^s = 0. \tag{24}$$

In the above we assume a block-matrix form of the covariance matrix as

$$V^s = \begin{pmatrix} V_A & V_C \\ V_C^T & V_B \end{pmatrix}. \tag{25}$$

In the following we investigate the effects of various parameters such as the input power P, the driving field Ω, and the temperature T of the mirrors on the degree of output entanglement. In turn this analysis shows that the generated entanglement can be controlled by adjusting experimental conditions, particularly the external driving field Ω. We consider three different detuning conditions, i.e., $\delta_j = +\omega_m$, $\delta_j = -\omega_m$, and $\delta_1 = -\delta_2 = \omega_m$, to find an experimental configuration relevant to output entanglement.

A. Both field-mirror pairs coupled in a BSL process

We first discuss the case of $\delta_j = +\omega_m$ ($j = 1,2$) so that the field-mirror effective coupling is a BSL process for both pairs. As has been shown in Refs. [9,35,36], the effective BSL process for the field-mirror coupling is very stable since this process followed by the cavity photon decay leads to the cooling of the movable mirrors [36]. We find from the stability conditions (18) that the cross-coupling strengths α_{12} and α_{21} cannot be too large since $\alpha_{12}\alpha_{21}$ becomes negative for $\Omega > \sqrt{2}\gamma_m$. On the other hand, the cavity loss κ and field-mirror coupling strength G are preferably large from Eq. (18). We study our scheme in the strong radiation-pressure coupling regime, i.e., $G^2/\gamma_m \gg \kappa$.

1. Field-field pair

Let us first look at the entanglement of two-mode fields. In Figs. 2(a) and 2(b), we plot the logarithmic negativity of the two-mode fields as a function of the driving field strength Ω for different cavity-driving laser powers P and thermal noises n of the mirrors. Except for the case of $\Omega = \sqrt{2}\gamma$, we see that there generally arises entanglement between the two-mode fields caused by the external driving field, which couples the coherence of the gain medium to the cavity modes.

To understand the behavior of the logarithmic negativity with the driving field Ω, we also plot in Fig. 3 the effective field-field coupling strength $\alpha \equiv \alpha_{21}^2/(\kappa_{11}\kappa_{22} + \alpha_{12}\alpha_{21})$. By comparing Figs. 2 and 3, we see that the effective coupling α and the degree of entanglement behave very similarly as a function of Ω. That is, the driving field Ω controls the effective field-field coupling which in turn determines the shape of the degree of entanglement in Fig. 2. There are two peaks of entanglement $E_{\mathcal{F}}$ in Fig. 2 at $\Omega \approx 0.5\gamma$ and $\Omega \approx 6\gamma$, which correspond to two maxima of the effective field-field coupling strength in Fig. 3.

When the cavity-driving power P changes from 0 to a nonzero value [Fig. 2(a)], we see that the degree of entanglement is slightly reduced. This is due to the coupling of the cavity-field mode to a mirror. The effective coupling G of

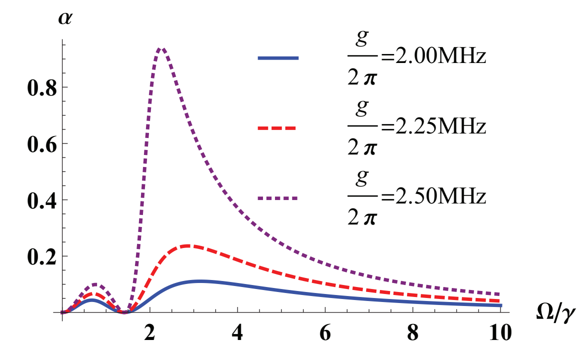

FIG. 3. (Color online) Influence of atom-cavity coupling on the effective two-mode field coupling strength α; α as a function of Ω/γ for the coupling $g = 2\pi \times 2.00$, $2\pi \times 2.25$, and $2\pi \times 2.50$ MHz. The other parameters are the same as in Fig. 2.

the field-mirror pair in Eq. (17) is defined as $G \equiv G_j \alpha_{js}/\sqrt{2}$ so that it increases with the input power P in view of Eq. (13). Therefore, noting that there is no direct interaction between the two mirrors in our setup, the effective coupling G indirectly transfers the entanglement of two fields to that of two mirrors (Fig. 4). We also observe in Fig. 2(b) that the degree of entanglement decreases with the thermal noise n of the mirrors. Except for two small regions around $\Omega = 0$ and $\Omega = \sqrt{2}\gamma$ where $\alpha_{21} = 0$, we generally obtain steady-state entanglement for the two-mode fields. The range of those two regions with no entanglement increases as the thermal noise n increases.

2. *Field-mirror pair*

On the other hand, the field-mirror pairs are coupled in a BSL process and it has been proved [37,38] that a nonclassical input state is required to have an entangled output state in a BSL process. Under this theorem, we readily see that there arises no entanglement between field and mirror due to the classical input states in our scheme.

3. *Mirror-mirror pair*

Although there is no entanglement between the cavity fields and the movable mirrors, we show that the entanglement between the two-mode fields can be transferred to entanglement of the mirrors. To be more elaborate, the effective coupling between the two mirrors can be obtained from Eq. (14) by eliminating adiabatically the dynamics of the field modes δa_j under the condition $\kappa \gg \gamma_m(2n+1)$ and substituting into the mirrors' vibrational modes [13]. It can be shown in the corresponding equations that the two vibrational modes are coupled to each other in a PDC process so that it is possible to generate entanglement between the two movable mirrors.

In Fig. 4, the degree of entanglement of the two movable mirrors is plotted against the external field Ω for different cavity-driving laser powers P and thermal noises n. We observe in the figures that the degree of entanglement for the movable mirrors has a similar curve to that of the two-mode fields for a large input power P and a small thermal noise n. In our scheme there is no direct interaction between the mirrors so one may say that the entanglement of the two-mode fields is transferred to entanglement of the mirrors due to the field-mirror coupling (radiation pressure). In Fig. 4(a), we observe that at a large cavity-driving power P, the value of $E_{\mathcal{M}}$ is comparable to that of $E_{\mathcal{F}}$ in the strong-coupling regime ($G/\sqrt{\kappa\gamma_m} = 140$). In Fig. 4(b), we also see that the degree of entanglement for the mirrors is reduced with increasing temperature. From the figure, we see that a macroscopic bipartite entanglement can be realized despite the condition $k_B T > \hbar\omega_m$.

To see the effect of the atom-cavity coupling strength on the output entanglement, we plot $E_{\mathcal{M}}$ versus T for different values of g with $\Omega = 6\gamma$ in Fig. 5. We observe that the degree of entanglement $E_{\mathcal{M}}$ increases with the atom-cavity coupling g. This is consistent with the increasing field-field coupling strength α so that a higher entanglement from the fields can be transferred to the mirrors. However, g has an upper bound restricted by both the cavity loss κ and the field-mirror coupling strength G such that the stability condition Eq. (18) is satisfied.

In Fig. 6, we plot $E_{\mathcal{M}}$ versus T for different values of κ with $\Omega = 6\gamma$ to see the effect of cavity loss on entanglement. This plot shows that the slope of $E_{\mathcal{M}}$ versus T decreases with increasing cavity loss; however, it is desirable to have a

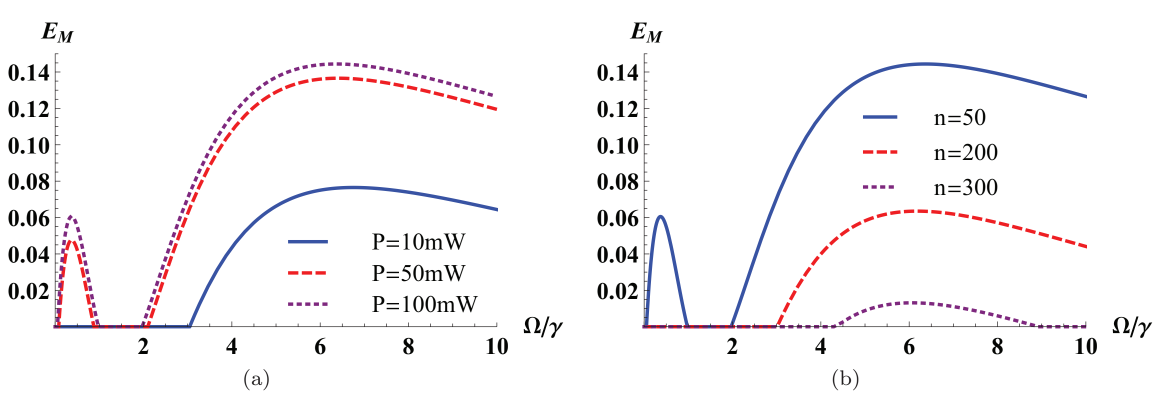

FIG. 4. (Color online) Logarithmic negativity $E_{\mathcal{M}}$ of the movable mirrors as a function of Ω/γ. (a) Effect of the cavity-driving laser power p on entanglement for $n = 50$ with $P = 10$ mW (blue solid curve), $P = 50$ mW (red dashed curve), and $P = 100$ mW (purple dotted curve). (b) Effect of thermal noise on entanglement at $P = 100$ mW for $n = 50$ (blue solid curve), $n = 200$ (red dashed curve), and $n = 300$ (purple dotted curve). The other parameters are the same as in Fig. 2.

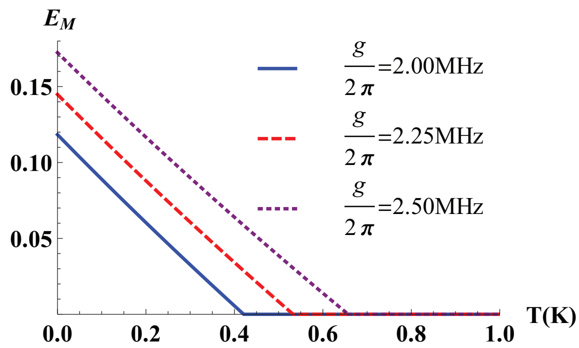

FIG. 5. (Color online) Effect of atom-cavity coupling on the two mirrors' entanglement $E_\mathcal{M}$; $E_\mathcal{M}$ as a function of the mirrors' environmental temperature T at $\Omega = 6\gamma$ for the couplings $g = 2\pi \times 2.00$, $2\pi \times 2.25$, and $2\pi \times 2.50$ MHz. The other parameters are the same as in Fig. 2.

smaller κ in order to obtain a higher entanglement. A larger cavity loss results in a smaller field-field coupling strength α so that the degree of entanglement for both the field-field pair and the mirror-mirror pair is reduced with otherwise the same parameters. We also see that the critical temperature above which the entanglement $E_\mathcal{M}$ disappears increases with decreasing cavity loss.

Therefore, we have demonstrated that macroscopic bipartite entanglement of both the field-field and the mirror-mirror pair can be obtained with the degree of entanglement controllable by an external driving field that implements a correlated-emission laser under experimentally realizable conditions.

B. Field-mirror pairs coupled in different processes

We now study the case of the two field-mirror pairs coupled in two different processes, e.g., field 1 (F1) and mirror 1 (M1) are coupled in a BSL process and field 2 (F1) and mirror 2 (M2) are coupled in a PDC process. We obtain from Eq. (18) one necessary condition needed for the steady-state solution in this case as

$$\left(\kappa_{11} + \frac{G^2}{\gamma_m}\right)\left(\kappa_{22} - \frac{G^2}{\gamma_m}\right) + \alpha_{12}\alpha_{21} > 0. \tag{26}$$

Now the parametric region is restricted to the regime of weak field-mirror coupling as one of the field-mirror pairs is coupled in the PDC process which is hindered by the stability condition [10]. This stability region is approximately $\frac{G^2}{\gamma_m} < \alpha_{11} + \alpha_{22}$.

In this case we find there are only three bipartite pairs, i.e., F1F2, F2M2, and F1M2, that are coupled effectively in a PDC process to generate an entangled stationary state. All the other bipartite states are Gaussian states coupled effectively in a BSL process which cannot be entangled states due to the classical inputs. Due to the weak radiation-pressure coupling, the field-mirror pair coupled in a PDC process is poorly entangled. Therefore, in this case we obtain only the entangled two-mode fields of the correlated-emission laser significantly and its degree of entanglement is slightly reduced by the weak radiation-pressure coupling.

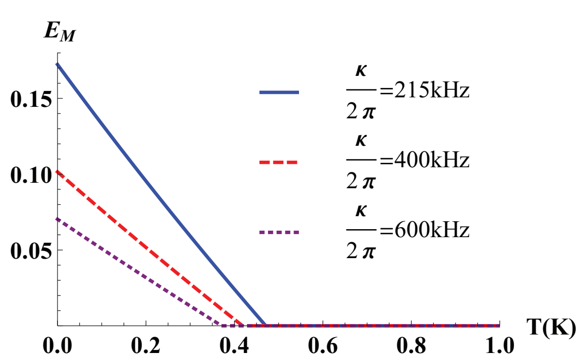

FIG. 6. (Color online) Influence of cavity loss on the two mirrors' entanglement $E_\mathcal{M}$; $E_\mathcal{M}$ as a function of environmental temperature T at $\Omega = 6\gamma$ for cavity losses $g = 2\pi \times 215$, $2\pi \times 400$, and $2\pi \times 600$ kHz. The other parameters are the same as in Fig. 2.

C. Both field-mirror pairs coupled in a PDC process

In this section, the field-mirror coupling is considered in a PDC process for both pairs; therefore the systems of field-mirror pairs are very unstable. To see this, we derive from Eq. (18) two necessary conditions by setting $\delta_j = -\omega_m$:

$$\begin{aligned}&\left(\kappa_{11} - \frac{G^2}{\gamma_m}\right)\left(\kappa_{22} - \frac{G^2}{\gamma_m}\right) + \alpha_{12}\alpha_{21} > 0,\\ &\gamma_m(2\alpha_{12}\alpha_{21} + 2\kappa_{11}\kappa_{22} + \kappa_{11}\gamma_m + \kappa_{22}\gamma_m)\\ &\qquad - G^2(\kappa_{11} + \kappa_{22} + 2\gamma_m) > 0.\end{aligned} \tag{27}$$

This condition is more stringent than Eq. (26) leaving us very little to play with. Although all the bipartite states (F1M1, F2M2, F1F2, F1M2, F2M1, and M1M2) in this case are Gaussian states coupled effectively in a PDC process, there is no significantly entangled bipartite state except the two-mode fields due to the weak radiation-pressure coupling restricted by the stability condition.

Therefore, we conclude that the seemingly achievable entanglement of movable mirrors coupled effectively in a PDC process is not practicable due to the stability condition Eq. (27). In this case, we obtain only the entangled bipartite Gaussian states of the two-mode fields in the steady-state solution.

V. CONCLUSION

We have studied the gain medium of cascading three-level atoms placed inside a doubly resonant cavity as a scheme for entangling two-mode fields whose entanglement can be transferred to two movable mirrors through radiation pressure. We first studied the master equations of the atom-cavity subsystem and the quantum Langevin equations of the mirror-cavity subsystem in order to derive the dynamical coupling equations among the two cavity fields and the two mirrors. We considered three different cases of tuning the two cavity-driving laser frequencies such that $\delta_j = +\omega_m$, $\delta_1 = -\delta_2 = +\omega_m$, and $\delta_j = -\omega_m$ and generalized the three cases to the Lyapunov equation (19) for the stationary covariance matrix V with a generalized drift matrix A in the rotating-wave approximation. In each case, we investigated three types bipartite Gaussian states in steady state and the entanglement conditions quantitatively with the logarithmic negativity $E_\mathcal{N}$ as well as the stability conditions obtained from the drift matrix.

Remarkably, the generated entanglement can be controlled by adjusting the external driving field Ω. We have shown that in the regime of strong field-mirror coupling ($G^2/\gamma_m \gg \kappa$) the entanglement of two-mode fields can be transferred to the two movable mirrors. Among the three cases considered, we obtain the macroscopic entanglement of movable mirrors only in the case of both field-mirror pairs interacting in a BSL process. The two mirrors are coupled effectively in a PDC process in this case and the steady state of the field-mirror system is stable in the regime of strong field-mirror coupling. The degree of entanglement $E_{\mathcal{M}}$ is significant for a high atom-cavity coupling g and a low cavity loss κ. With the stability condition Eq. (18) and experimentally accessible parameters [11,35], macroscopic entanglement for two movable mirrors can be realized with the current state-of-the-art experimental apparatus.

ACKNOWLEDGMENTS

This research is supported by a grant from King Abdulaziz City for Science and Technology (KACST) and an NPRP Grant (No. 5-102-1-026) from Qatar National Research Fund.

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