Classification of entanglement via rigged string configurations

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We investigate the structure of bipartite entanglement in one-dimensional spin systems. The standard approach is based on a reduced two-qubit density matrix and uses concurrence as the measure of entanglement. We show that for a spin system consisting of four qubits the structure of bipartite entanglement can be classified by rigged string configurations in a strictly combinatorial manner.

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I. INTRODUCTION

A key ingredient in quantum-information theory [1] is entanglement between subsystems of a given system. In recent years, there has been an ongoing effort to understand this phenomenon and, for bipartite systems, one can say that this problem has essentially been solved. How to characterize entanglement both quantitatively and qualitatively has been well established, in particular the necessary and sufficient condition for inseparability has been derived [2,3] and measures of bipartite entanglement have been introduced [4–6].

However, since a generalization of the notion of entanglement for multipartite systems remains vague, the method of entangled graphs [7–10] has been introduced. The method allows one to study the structure of multipartite entanglement in terms of bipartite reduced density matrices, and introduces concurrence as a measure of entanglement.

To have a clear understanding of the concept of entanglement let us consider two classes of well-known states: N-party Greenberger-Horne-Zeilinger (GHZ) states of the form

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0^{\otimes N}\rangle + |1^{\otimes N}\rangle), \tag{1}$$

representing maximally entangled states of N qubits, and so-called W states

$$|W\rangle = \frac{1}{\sqrt{N}}(|0\ldots01\rangle + |0\ldots10\rangle + \cdots + |1\ldots00\rangle), \tag{2}$$

where all qubits are maximally bipartite entangled.

The above picture suggests that if we prepare a system in a given state then we create in the system the specific structure of entanglement by creating some nonlocal (in terms of nonlocal quantum mechanics) quantum channels between the qubits. In other words the state imposes on the system a network of quantum channels (an entangled graph) which are the subject of further investigation. Deep understanding of this approach gives us a possibility of building quantum algorithms with a wide range of potential implementations.

On the other hand, according to classical information theory, a basic memory unit is a register and every calculation comes down to a change of state of one of these. This implies a question about all the possible states of this register, at any instant of time, in order to control our calculations. If we restrict ourselves to the register being a one-dimensional Heisenberg magnet consisting of four nodes with a one-node spin $s = 1/2$ and a Heisenberg Hamiltonian responsible for the dynamics of the system, then all eigenstates can be labeled by combinatorial objects known as rigged string configurations [11–13]. Here we want to show that such combinatorial objects (quantum numbers) contain also full information about the quantum structure of bipartite entanglement in the system.

The paper is organized as follows: Section II starts with a brief description of a one-dimensional Heisenberg model which forms a quantum register of a basic memory unit of quantum computer. Next, in Sec. III we introduce the notion of rigged string configuration and classification of the one-dimensional Heisenberg magnetic ring solutions. In Sec. IV we present the standard way of calculation of quantum structure of bipartite entanglement in terms of the reduced density matrix and concurrence as a measure of entanglement. Section V is devoted to bipartite aspects of multipartite entanglement leading to the introduction of the notion of entangled molecules and resulting in an exemplary classification of entangled molecules by rigged string configurations for a Heisenberg magnet with four nodes of spin 1/2. Conclusions, along with a brief summary of our results, can be found in Sec. VI.

II. MODEL

Let us consider a one-dimensional Heisenberg model (spin system) which reveals the symmetry under collective unitary *rotations* and some permutations of subsystems (Fig. 1). The vector space of this model $\mathrm{SU}(n)^{\otimes N}$ can be interpreted either as the state-space of a quantum system or as a representation of the symmetric and unitary groups. Here, the single-node spaces h_j, $j = 1,\ldots,N$ are spanned by single-node spin projections $|i\rangle$, $i = 1,\ldots,n$, i.e., by the set $\tilde{n} = \{i \mid i = 1,2,\ldots,n\}$ and can be modeled mathematically by the fundamental irreducible representation (irrep) $D^{(1)}$ of the unitary group $\mathrm{SU}(n)$ with transformation properties $D^{(1)}(u)|i\rangle = \sum_{i'\in\tilde{n}} u_{i'i}|i'\rangle$. The group Σ_N with elements $\sigma_i \in \Sigma_N$ is the permutation group of nodes, and the set $\tilde{N} = \{j \mid j = 1,2,\ldots,N\}$ labels the nodes of the system.

In the language of quantum computation this model can be interpreted as the quantum register consisting of N qudits (qubits for $n = 2$), each with the local (computational) basis $|i\rangle, i \in \tilde{n}$. A natural state of this system is represented by the

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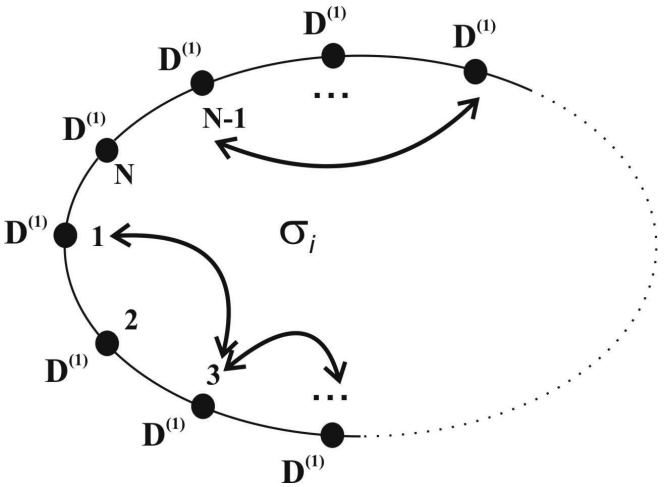


FIG. 1. One-dimensional Heisenberg model with a symmetry under collective unitary rotations and some permutations of subsystems.

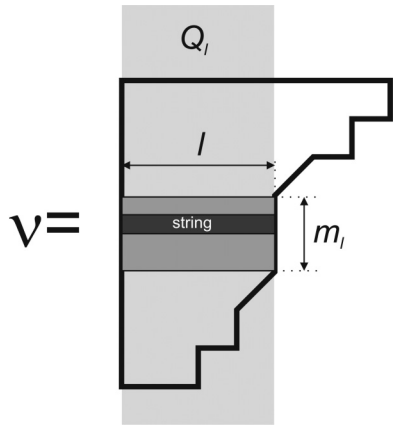


FIG. 2. Here, $\nu = (m_1, m_2, \ldots)$ denotes the Young diagram of the *string configuration* with m_l being the number of strings of length $l = 1,2,\ldots$. Each part of the partition ν is referred to as a *string*, and its size (the number of boxes in a row of the Young diagram of ν) - the *length* of the string is the number of boxes in the first l *columns* of the Young diagram of the string configuration ν.

configuration $|f\rangle = |i_1, i_2, \ldots, i_N\rangle$, $i_j \in \tilde{n}$, $j \in \tilde{N}$, which is a product of single-node states in the space $\mathrm{SU}(n)^{\otimes N}$. In the remainder of this work we intend to call this state *the configuration*.

The set of all configurations forms a basis $b = \{|f\rangle, |f \in \tilde{n}^{\tilde{N}}\}$ which spans the Hilbert space $\mathcal{H} = lc_{\mathbb{C}}\, b \cong h_j^{\otimes N}$ of the model. The space $\mathcal{H}$ of all quantum states, with $\dim \mathcal{H} = 2^N$, can be decomposed as

$$\mathcal{H} = \sum_{r=0}^{N} \oplus \mathcal{H}^r, \quad \dim \mathcal{H}^r = \binom{N}{r} \tag{3}$$

into subspaces $\mathcal{H}^r$, with fixed number r of *Bethe pseudoparticles* (spin deviations), where $\mathcal{H}^r$ is given by

$$\mathcal{H}^r = \sum_{S=N/2-r}^{N/2} \oplus \mathcal{H}^{rS} \quad \text{where} \quad S = N/2 - r', \quad 0 \leqslant r' \leqslant r. \tag{4}$$

The space $\mathcal{H}^{rS}$ carries the irreducible representations Δ^λ of the symmetric group Σ_N, with $\lambda = \{N - r', r'\}$.

The dynamics of the Heisenberg magnet is governed by the Hamiltonian

$$\hat{H} = J \sum_{j \in \tilde{N}} (\hat{\mathbf{s}}_j \cdot \hat{\mathbf{s}}_{j+1} - 1/4), \quad \text{where} \quad \hat{\mathbf{s}}_j = \left(\hat{s}_j^x, \hat{s}_j^y, \hat{s}_j^z\right), \tag{5}$$

and $J = 1$ is the coupling constant for a one-dimensional model. Using this particular one-dimensional spin model with the above properties we are also required to introduce a way of classification of the eigenstate and to choose an appropriate measure of entanglement in order to derive a set of states capable of creating any entangled graph of interest.

III. CLASSIFICATION OF EIGENSTATES BY RIGGED STRING CONFIGURATIONS

It is known that the exact solutions of such a model are given by the famous Bethe ansatz [11] (BA), and can be classified in terms of combinatorial objects: rigged string configurations [12,13]. A string configuration ν (Fig. 2) is, by definition, a partition of the integer r', $\nu \vdash r'$, and $\sum_l l m_l = r'$. Each string (lv), $v \in \tilde{m}_l = \{1,2,\ldots,m_l\}$, is equipped with a nonnegative integer L, called its *rigging*. The rigging L varies within the range $0 \leqslant L \leqslant P_l$, where $P_l = N - 2Q_l$. The integer P_l is referred to as the number of *holes* for strings of length l. The pair $\nu\mathcal{L}$, where

$$\mathcal{L} = \{L_{lv} | l = 1,2,\ldots; \quad v \in \tilde{m}_l\}, \tag{6}$$

is the set of all riggings of ν, referred to as a *rigged string configuration*. All m_l strings of length l are distinguished by their riggings only, so that they can be arranged in $\nu\mathcal{L}$ in nondecreasing order (from the top to the bottom of the $l \times m_l$ - rectangle incorporated in the Young diagram $\nu \vdash r'$), that is,

$$0 \leqslant L_{lv} \leqslant L_{lv'} \leqslant P_l \quad \text{for} \quad 1 \leqslant v < v' \leqslant m_l, \quad l = 1,2,\ldots. \tag{7}$$

Such a counting implies that the set $z(\nu) = \{\nu\mathcal{L}\}$ of all rigged string configurations corresponding to a given $\nu \vdash r'$ has the cardinality

$$|z(\nu)| = \prod_l \binom{P_l + m_l}{m_l}, \quad \text{and} \quad \dim \Delta^\lambda = \sum_{\nu \vdash r'} |z(\nu)|. \tag{8}$$

Each rigged string configuration $\nu\mathcal{L}$ classifies an exact eigenstate $|\Psi\rangle$ of the Heisenberg Hamiltonian (5) in each space $\mathcal{H}^r$, $r' \leqslant r \leqslant N/2$. In another words string configurations $\nu\mathcal{L}$ can be seen as multiquantum numbers which label the eigenstates of the considered system.

IV. REDUCED DENSITY MATRIX, CONCURRENCE

Now we are interested in the quantum structure of the bipartite entanglement. To find such a structure in the standard way let us consider an eigenstate of the Hamiltonian (5) of the form

$$|\Psi\rangle = \sum a(i_1 \ldots i_N) |i_1 \ldots i_N\rangle \tag{9}$$

with a density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix

$$R_{jk} = \mathrm{Tr}_{\text{rest of the qubits}}\,\rho \tag{10}$$

for a pair of qubits j,k can be constructed by performing a partial trace over the rest of the qubit to be eliminated

$$(R_{jk})^{i'_j i'_k}_{i_j i_k} = \sum_{i_1 \dots i_N} a(i_1 \dots i_j \dots i_k \dots i_N) \times a^*(i_1 \dots i'_j \dots i'_k \dots i_N), \tag{11}$$

where the sum runs over all permutations of nodes $1 \dots N$ except for nodes j and k. In general the reduced density matrix for the nodes (j,k) represents mixed states.

To determine how these qubits are entangled among themselves, the concurrence measure $C = \max\{\sqrt{r_1} - \sqrt{r_2} - \sqrt{r_3} - \sqrt{r_4},0\}$ can be used, where $\{r_i : i \in \tilde{4}\}$ are the eigenvalues in decreasing order of the matrix $R = R_{jk}\tilde{R}_{jk}$, and $\tilde{R}_{jk}$ is the time-reversed matrix $\tilde{R}_{jk} = (\sigma_y \otimes \sigma_y)[R_{jk}]^T(\sigma_y \otimes \sigma_y)$. We take the concurrence as a quantitative measure of entanglement since the amount of entanglement required for an assembly of a state monotonically increases with the increase of concurrence, so it proves to be a good measure of the strength of the quantum "bindings."

Here we concentrate on the bipartite aspects of multipartite entanglement by considering the entanglement between a pair (j,k) in a particular exact BA eigenstate $|\nu\mathcal{L}\rangle$, cf. (9). We use the notion of entangled molecules as a generalization of Wootters idea of an "Entangled chain" [14]. In this work, we evaluate concurrence C on each pair (j,k) of the chain for all eigenstates of the system. Thus, for each eigenstate $|\nu\mathcal{L}\rangle$, we have $N(N-1)/2$ different bipartite reduced density operators

$$\rho_{jk},\ j < k, \quad j,k \in \tilde{N},$$

with matrix elements given by Eq. (11). These operators describe whether nodes j and k are entangled or not, and how strong this quantum correlation is. This approach allows us to build entangled molecules, which characterize the quantum structure of the system in a given eigenstate $|\nu\mathcal{L}\rangle$.

V. ENTANGLEMENT MOLECULES VIA RIGGED STRING CONFIGURATIONS

From a different point of view, if we take into account the irreducible basis of the Schur-Weyl duality [15–20] for N spins, then $1/2$ is characterized by the chain of intermediate angular momenta

$$\tfrac{1}{2} = S_1, S_{12}, S_{123}, \dots, S_{1\dots j}, \dots, S_{1\dots N} = S, \tag{12}$$

where $S_{1\dots j}$ is the angular momentum of the first j nodes, and thus S is (a particular value of) the total angular momentum of the magnetic ring. The neighbor angular momenta in this chain differ by ± 1, that is,

$$S_{1\dots j} = S_{1\dots j-1} \pm 1. \tag{13}$$

The chain (12) corresponds to the chain of Young diagrams [21,22] $\lambda^{(j)} = (\lambda_1^{(j)},\lambda_2^{(j)})$, differing by a single box. The Robinson-Schensted-Knuth [23–25] algorithm associates thus the chain (12) with the Young tableau $y = \{y_{\alpha\beta}\}$ of the shape

$$\lambda = \{\lambda_1,\lambda_2\}, \quad \lambda_1 = (N+2S)/2, \quad \lambda_2 = (N-2S)/2, \tag{14}$$

filled in by letters $j \in \tilde{N}$ by the following prescription

$$S_{1\dots j} - S_{1\dots j-1} = \begin{cases} 1 \ \Rightarrow & j = y_{1\beta}, \\ -1 \ \Rightarrow & j = y_{2\beta}, \end{cases} \tag{15}$$

for some β. In other words, if the intermediate angular momentum at the step j increases (decreases), then the letter $j \in \tilde{N}$ is placed in the first (second) row of the Young diagram λ. This process of consecutive coupling of angular momenta can be equivalently coded in the *tableau word*

$$w(y) = i_1 i_2 \dots i_N, \tag{16}$$

defined by the formula

$$i_j = \begin{cases} 1 & \text{if} \quad j = y_{1\beta}, \\ 2 & \text{if} \quad j = y_{2\beta}. \end{cases} \tag{17}$$

The word $w(y)$ is thus a magnetic configuration which represents the chain (12) of all intermediate angular momenta $S_{1\dots j}$, $j \in \tilde{N}$. Clearly, combinatorial data y and $w(y)$ are equivalent.

The chain (12), or, equivalently, its corresponding Young tableau y given by Eqs. (14) and (15), defines N points, which are $(1,1)$, $(2,2S_{12}), \dots, (j,2S_{1\dots j}), \dots, (N,2S)$ in the plane $(j,2S)$. Joining the nearest neighbor points and, moreover, joining the first point $(1,1)$ with the origin $(0,0)$, one gets a *path* which lies in the first quarter of the $(j,2S)$ plane, and is constrained from the top by the line $j = 2S$, and from the right by N. Such a path has the form of a zigzag, with each step j being either ascent or descent by one, according to the position of j in y. These paths prove to be a surprisingly powerful combinatoric tool for a complete classification of solutions of the system of Bethe equations in terms of rigged string configurations. This correspondence can be confirmed using the KKR (Kerov-Kirillov-Reshetikhin) algorithm [12,26].

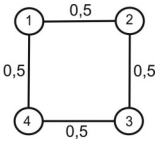
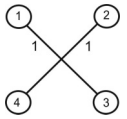
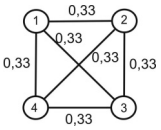
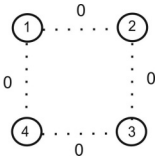
Results for the Heisenberg magnet with $N = 4$ nodes and $r = 2$ spin deviations are presented in Table I. Here, the second column presents quantum numbers E,k connected with energy and quasimomentum, the third presents the corresponding exact eigenstates, while the fourth and fifth give rigged string configurations and entangled molecules, respectively.

VI. CONCLUSIONS

We have shown in Sec. II that all eigenstates of the Heisenberg magnet with spin $s = 1/2$ are classified by rigged string configurations. On the other hand, if we calculate an entanglement structure (using tools from Sec. III) between all pairs of nodes of the magnet, which is prepared in a certain state, we obtain an entangled molecule (as described in Sec. IV). It implies connections between rigged string configurations and entangled molecules.

In Table I we present results for the Heisenberg magnet with $N = 4, r = 2$. Here, if the system is in the state given by quantum numbers $E = -6, k = 0$ (first row of the table) then only the nearest neighbor nodes are entangled, which is in good agreement with the physical interpretation of the string configuration $\begin{smallmatrix}\boxed{0}\\ \boxed{0}\end{smallmatrix}0$, which corresponds to two magnons

TABLE I. Classification of entangled molecules by rigged string configurations for Heisenberg magnet with $N = 4$ nodes and $r = 2$ spin deviations. An entanglement between two nodes of the magnet is presented on the picture of the molecule by the solid line, and the number near this line is equal to concurrence.

No.	Quantum numbers	Eigenstate	Rigged string configuration	Entangled molecule
1	$E = -6, k = 0$	$\|\Psi\rangle = -\frac{\sqrt{3}}{6}(\|12\rangle + \|23\rangle + \|34\rangle + \|14\rangle) + \frac{\sqrt{3}}{3}(\|13\rangle + \|24\rangle)$	$\begin{matrix}\boxed{0}\\ \boxed{0}\end{matrix}\,0$	
2	$E = -2, k = 2$	$\|\Psi\rangle = \frac{\sqrt{1}}{2}(-\|12\rangle + \|23\rangle - \|34\rangle + \|14\rangle)$	$\boxed{0}\boxed{}\,0$	
3	$E = 0, k = 0$	$\|\Psi\rangle = \frac{\sqrt{2}}{2}(\|12\rangle + \|23\rangle + \|34\rangle + \|14\rangle + \|13\rangle + \|24\rangle)$	$\emptyset$	
4	$E = -4, k = 2$	$\|\Psi\rangle = \frac{\sqrt{2}}{2}(-\|13\rangle + \|24\rangle)$	$\boxed{0}\,2$	
5	$E = -2, k = -1$	$\|\Psi\rangle = \frac{1}{2}(-I\|12\rangle - \|23\rangle + I\|34\rangle + \|14\rangle)$	$\boxed{1}\,2$	
6	$E = -2, k = 1$	$\|\Psi\rangle = \frac{1}{2}(I\|12\rangle - \|23\rangle - I\|34\rangle + \|14\rangle))$	$\boxed{2}\,2$	

moving independently across the magnet. The eigenstate in the second row is labeled by string configuration $\boxed{0}\boxed{}0$ which corresponds to two magnons bound onto a string. This system of two bound magnons can exist only in two space configurations: either node 1 is bound to 3, or node 2 to 4. As we see on the entangled molecule there are only two possible positions of this string. In the case of the second descendant of the vacuum state (third row), we have obtained that the concurrence is the same for each pair of nodes. In the last case (fourth, fifth, and sixth rows), corresponding to the first descendants of the single magnon states, the concurrence is zero everywhere, in accordance with physical interpretation of a 1-string.

This article had shown how rigged string configurations can be used for classification purposes based on an example of the Heisenberg magnet with $N = 4$ nodes and spin $s = 1/2$. Further extension of this work for more advanced systems is currently being carried out which should lead us to a general picture and understanding of physics of the magnon. The main purpose of this work is presentation of the effect along with revealing the possibility of its helping to solve the cases for bigger sets.

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