PHYSICAL REVIEW A **88**, 022314 (2013)

Implementation of a simple operator-quantum-error-correction scheme

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(Received 6 September 2012; revised manuscript received 20 May 2013; published 12 August 2013)

We provide a simple yet interesting example of operator quantum error correction avoiding fully correlated noise. Our scheme requires no initialization of ancillae, which can thus be in the uniformly mixed state. We demonstrate our scheme experimentally by making use of a three-qubit NMR quantum computer.

DOI: 10.1103/PhysRevA.88.022314 PACS number(s): 03.67.Pp, 03.65.Yz, 03.67.Ac, 76.60.−k

I. INTRODUCTION

A quantum computer is vulnerable to the interaction with the environment, and it must be protected in one way or another. Quantum error correction (QEC) strategies [1], such as *quantum error correcting codes* (QECCs) [2–4], *noiseless subsystems* (NS) [5–11], and *decoherence-free subspaces* (DFSs) [12–15], were devised to protect quantum information. More recently, a unified approach called *operator quantum error correction* (OQEC), encompassing and generalizing the above techniques, was introduced in Refs. [16,17]. We review examples of QECC [2,3] and NS [9–11] with three qubits in terms of OQEC. Note that we use the term NS in a broader sense defined in Refs. [16,17], which does not necessarily entail the decomposition of the error operators into a direct sum of irreducible representations of SU(2). We then provide the simplest example with three qubits, in terms of the number of CNOT gates, that is a NS robust against fully correlated noise and that requires no initialization of ancillae. A somewhat analogous problem is discussed in Refs. [18,19] for a related but different system. Our QEC scheme provides another realization of quantum computing using uniformly mixed state qubits, such as, e.g., *deterministic quantum computation with one pure qubit* (DQC-1) [20,21].

II. OPERATOR QUANTUM ERROR CORRECTION

Let us first review the operator quantum error correction scheme following Refs. [16,17]. Suppose $\mathcal{H}$ is the Hilbert space for some quantum system, and that it is decomposed as $\mathcal{H} = (\mathcal{H}^A \otimes \mathcal{H}^B) \oplus \mathcal{K}$ with $\dim(\mathcal{H}^{\mathrm{A}}) = m$, $\dim(\mathcal{H}^{\mathrm{B}}) = n$ and $\dim(\mathcal{K}) = \dim(\mathcal{H}) - mn$. We assume $\mathcal{H}^A$ and $\mathcal{H}^B$ are spanned by the orthonormal sets $|\alpha_i\rangle$ and $|\beta_j\rangle$, respectively. We define the projection operator

$$P_{\mathfrak{A}} = \sum_i |\alpha_i\rangle\langle\alpha_i| \otimes \mathbb{I}^B \oplus 0_{\dim(\mathcal{K})},$$

so that $P_{\mathfrak{A}}(\mathcal{H}) = \mathcal{H}^A \otimes \mathcal{H}^B$, and a super-operator $\mathcal{P}_{\mathfrak{A}}(\cdot) = P_{\mathfrak{A}}(\cdot)P_{\mathfrak{A}}$. Hereafter, we will discuss the case when $\dim(\mathcal{K}) = 0$. Suppose $\mathcal{E}$ is a quantum (error) operation (or channel) acting on $\mathfrak{B}(\mathcal{H})$. Every such channel admits an operator sum representation

$$\mathcal{E}(\sigma) = \sum_k E_k \sigma E_k^\dagger$$

for any $\sigma \in \mathfrak{B}(\mathcal{H})$. Note that $\mathfrak{B}(\cdot)$ is the set of operators on $\cdot$. For a given decomposition of $\mathcal{H}$, let us define $\mathfrak{A}$ as the operator semigroup in $\mathfrak{B}(\mathcal{H})$ such that

$$\mathfrak{A} = \{\sigma \in \mathfrak{B}(\mathcal{H}) : \sigma = \sigma^A \otimes \sigma^B, \text{ for some } \sigma^A \in \mathfrak{B}(\mathcal{H}^A) \text{ and } \sigma^B \in \mathfrak{B}(\mathcal{H}^B)\}. \quad (1)$$

The B-sector of $\mathfrak{A}$ is said to be $\mathcal{E}$-correctable with respect to the above decomposition if there exists a trace-preserving quantum (recovery) operation $\mathcal{R}$ on $\mathfrak{B}(\mathcal{H})$ such that

$$(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{R} \circ \mathcal{E})(\sigma) = \mathrm{Tr}_A(\sigma) \quad (2)$$

for any $\sigma \in \mathfrak{A}$. Any given quantum error correction scheme is determined by specifying the triple $\{\mathcal{R},\mathcal{E},\mathfrak{A}\}$. When $\mathfrak{A}$ is a subspace and $\mathcal{R} \neq id$ (identity channel), the scheme is identified as QECC. On the other hand, DFS and NS are obtained when $\mathcal{R} = id$ and $\mathfrak{A}$'s are a subspace and an algebra, respectively.

A typical QECC with three qubits is presented in its quantum circuit form (Fig. 1) [2,3]. $\mathfrak{A}$ is implemented by $\mathcal{U}_E$, which is realized with the two controlled NOT gates before $\mathcal{E}$. *Independent* bit-flip errors of qubits are defined by $\{E_k\} = \{\sqrt{p_1}\,\sigma_0 \otimes \sigma_0 \otimes \sigma_0, \sqrt{p_2}\,\sigma_x \otimes \sigma_0 \otimes \sigma_0, \sqrt{p_3}\,\sigma_0 \otimes \sigma_x \otimes \sigma_0, \sqrt{p_4}\,\sigma_0 \otimes \sigma_0 \otimes \sigma_x\}$, where $p_i \geqslant 0$ and $\sum p_i = 1$. σ_0 is a unit matrix of dimension 2, while $\sigma_{x,y,z}$ is the x,y,z component of the Pauli matrices. $\mathcal{R}$ is defined by the rightmost Toffoli gate.

Let us consider a *fully correlated* noise where all the qubits, unlike the error channel in Fig. 1, suffer from the same noise. This may happen when the dimensions of the quantum computer are microscopic compared with the wavelength of external disturbances [9–11]. This situation also takes place when photons are sent one by one through an optical fiber with a fixed imperfection, assuming the scattering of photons by the imperfection is “elastic.” Then the imperfection acts on each photon in the same way, resulting in the collective noise. A NS example with three qubits was discussed in Ref. [9].

The error channel is defined by $\{E_k\} = \sqrt{p_1}\sigma_0^{\otimes 3}, \sqrt{p_2}(e^{i\alpha\sigma_x})^{\otimes 3}, \sqrt{p_3}(e^{i\beta\sigma_y})^{\otimes 3}, \sqrt{p_4}(e^{i\gamma\sigma_z})^{\otimes 3}\}$, where $p_i \geqslant 0$ and $\sum p_i = 1$. Its quantum circuit is shown in Fig. 2, where $G_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$ and $G_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. In this case,

$$\begin{aligned}(\mathcal{U}_E^\dagger \circ \mathcal{E} \circ \mathcal{U}_E)(|0\rangle\langle 0| \otimes |v\rangle\langle v| \otimes |\psi\rangle\langle\psi|) \\ = |0\rangle\langle 0| \otimes \left(\sum_{i=0}^{3} p_i U_i |v\rangle\langle v| U_i^\dagger\right) \otimes |\psi\rangle\langle\psi|,\end{aligned}$$

where $\{U_i\} = \{\sigma_0, e^{i\alpha\sigma_x}, e^{i\beta\sigma_y}, e^{i\gamma\sigma_z}\}$. Note that one of the ancillae required in Fig. 2 is in an arbitrary initial state including the fully mixed state. We used a pure-state notation $|v\rangle$ in Fig. 2 to simplify the expressions. The general case with mixed initial states is obtained by simply mixing the pure-state

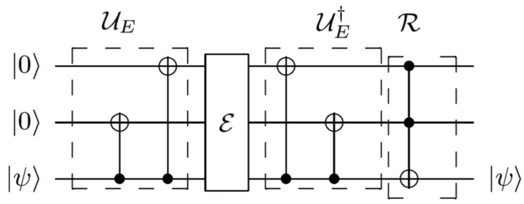


FIG. 1. Traditional QECC against independent bit-flip errors. $\{E_k\} = \{\sqrt{p_1}\,\sigma_0 \otimes \sigma_0 \otimes \sigma_0, \sqrt{p_2}\,\sigma_x \otimes \sigma_0 \otimes \sigma_0, \sqrt{p_3}\,\sigma_0 \otimes \sigma_x \otimes \sigma_0, \sqrt{p_4}\,\sigma_0 \otimes \sigma_0 \otimes \sigma_x\}$, where $p_i \geqslant 0$ and $\sum p_i = 1$.

results using linearity. We will use this notation in the rest of this paper.

We will hereafter discuss a *very simple, yet interesting* error channel. The E_k are $\{E_k\} = \{\sqrt{p_1}\,\sigma_0^{\otimes 3}, \sqrt{p_2}\,\sigma_x^{\otimes 3}, \sqrt{p_3}\,\sigma_y^{\otimes 3}, \sqrt{p_4}\,\sigma_z^{\otimes 3}\}$, where we assume $\sum p_i = 1$. This noise channel is a special case of the one discussed in Fig. 2.

First, we consider the case when pure state ancillae are prepared initially, as shown in Fig. 3. This quantum circuit was discussed in Ref. [11], although not very explicitly. The state $|\psi\rangle\langle\psi|$ to be protected is encoded as $\rho_E = (H|0\rangle\langle 0|H^\dagger) \otimes |0\rangle\langle 0| \otimes |\psi\rangle\langle\psi|$. Therefore, the projection operator $P_{\mathfrak{A}}$ should be the identity operator of dimension 8. After $\mathcal{E}$ is operated on ρ_E, ρ_E is decoded by $\mathcal{U}_E^\dagger$ and then $\mathcal{R}$ recovers $|\psi\rangle\langle\psi|$. In summary, we obtain $(\mathrm{Tr}_A \circ \mathcal{R} \circ \mathcal{E})\rho_E = |\psi\rangle\langle\psi|$. This QEC scheme is regarded as QECC with the simplest encoding and decoding circuits.

For such a restricted noise channel, an encoding scheme in which two qubits are protected with one arbitrary topmost ancilla was reported [10] (see Fig. 4). The circuit in Ref. [10] was not exactly the same as that in Fig. 4, because in the latter the first CNOT gate was removed at the cost of initializing the ancilla to the $|0\rangle$ state.

III. SIMPLEST OQEC WITH MIXED STATE ANCILLAE

We now introduce a new and even simpler NS with three qubits. As we shall later prove, this is the *simplest* possible NS in terms of the number of CNOT gates, although only one qubit is protected with two arbitrary ancillae, as shown in Fig. 5. We can show that

$$\begin{aligned}(\mathcal{U}_E^\dagger \circ \mathcal{E} \circ \mathcal{U}_E)(|v,u\rangle\,\langle v,u| \otimes |\psi\rangle\,\langle\psi|) \\ = \left(\sum_{i=0}^{3} p_i M_i\,|v,u\rangle\,\langle v,u|\,M_i^\dagger\right) \otimes |\psi\rangle\,\langle\psi|,\end{aligned} \quad (3)$$

where $\{M_i\} = \{\sigma_0^{\otimes 2}, \sigma_x^{\otimes 2}, -\sigma_x \otimes \sigma_y, \sigma_0 \otimes \sigma_z\}$.

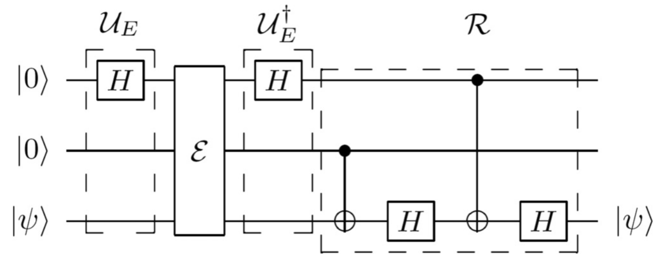


FIG. 3. Simple QECC against restricted fully correlated errors. $\{E_k\} = \{\sqrt{p_1}\,\sigma_0^{\otimes 3}, \sqrt{p_2}\,\sigma_x^{\otimes 3}, \sqrt{p_3}\,\sigma_y^{\otimes 3}, \sqrt{p_4}\,\sigma_z^{\otimes 3}\}$, where $\sum p_i = 1$.

The one-qubit density matrix $|\psi\rangle\langle\psi|$ can be represented by the Bloch vector $\boldsymbol{n} = (n_x, n_y, n_z)$ so that $|\psi\rangle\langle\psi| = \frac{1}{2}(\sigma_0 + \boldsymbol{n}\cdot\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. After encoding the initial state with fully mixed state ancillae, we obtain the encoded state:

$$\rho_E = \mathcal{U}_E(\sigma_0 \otimes \sigma_0 \otimes |\psi\rangle\langle\psi|/4) = \tfrac{1}{8}(\Sigma_0 + \boldsymbol{n}\cdot\boldsymbol{\Sigma}), \quad (4)$$

where $\Sigma_0 = \sigma_0^{\otimes 3}$ and $\boldsymbol{\Sigma} = (\Sigma_x, \Sigma_y, \Sigma_z) = (\sigma_0 \otimes \sigma_x \otimes \sigma_x, \sigma_z \otimes \sigma_x \otimes \sigma_y, \sigma_z \otimes \sigma_0 \otimes \sigma_z)$. $\{\Sigma_i\}$ satisfies the ordinary $\mathfrak{su}(2)$ algebra, and the quantum information $\boldsymbol{n}$ is stored in the space spanned by $\{\Sigma_i\}$. It is easy to obtain general gate operations acting on the physical qubits by $e^{-i\alpha\Sigma_i}$ in order to implement operations acting on a logical qubit. We note that direct operations on logical qubits in DFS and NS were discussed in Ref. [22].

In the present case,

$$\begin{aligned}\mathfrak{A} &= \left\{\frac{1}{8}\Sigma_0, \frac{1}{8}\Sigma_x, \frac{1}{8}\Sigma_y, \frac{1}{8}\Sigma_z\right\},\\ P_{\mathfrak{A}} &= \frac{1}{2}(\sigma_0 \otimes \sigma_0 + \sigma_0 \otimes \sigma_x + \sigma_z \otimes \sigma_x + \sigma_z \otimes \sigma_0) \otimes \sigma_0\\ &= \begin{pmatrix} 1 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \sigma_0.\end{aligned}$$

Then, we obtain $(\mathrm{Tr}_A \circ \mathcal{P}_{\mathfrak{A}} \circ \mathcal{E})(\rho_E) = |\psi\rangle\langle\psi|$.

This NS is obviously the simplest under our noise model when we consider a system of three or more qubits, because at least two CNOT gates to correlate the three qubits are required. We will now prove the nonexistence of a NS with a two-qubit system under our noise model. Suppose there exists $U \in U(4)$, which satisfies $U(\sigma_x \otimes \sigma_x)U^\dagger = \sigma_0 \otimes \sigma_x, U(\sigma_y \otimes \sigma_y)U^\dagger = \sigma_0 \otimes \sigma_y$ and $U(\sigma_z \otimes \sigma_z)U^\dagger = \sigma_0 \otimes \sigma_z$. By multiplying the left-hand sides and the right-hand sides of these relations one

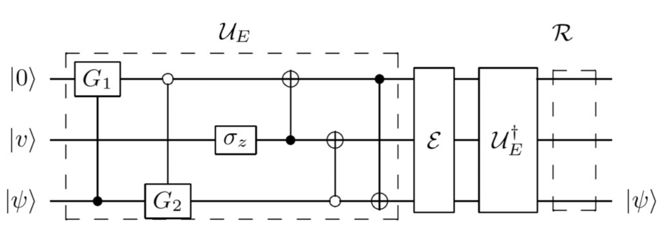


FIG. 2. NS against fully correlated noise. $\{E_k\} = \sqrt{p_1}\sigma_0^{\otimes 3}, \sqrt{p_2}(e^{i\alpha\sigma_x})^{\otimes 3}, \sqrt{p_3}(e^{i\beta\sigma_y})^{\otimes 3}, \sqrt{p_4}(e^{i\gamma\sigma_z})^{\otimes 3}\}$, where $p_i \geqslant 0$ and $\sum p_i = 1$.

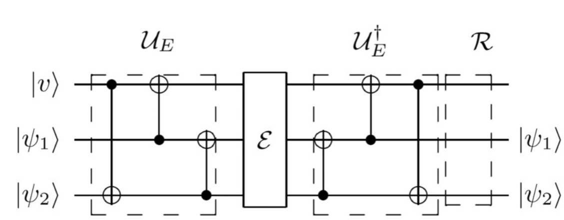


FIG. 4. NS that protects two qubits with one arbitrary ancilla, including a fully mixed state one. The noise model is the same as in Fig. 3.

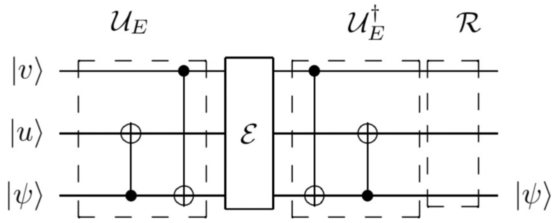


FIG. 5. Simplest NS, in terms of the number of CNOT gates, to avoid restricted fully correlated noise. The noise model is the same as in Fig. 3.

by one, we find $U[(\sigma_x\sigma_y\sigma_z)\otimes(\sigma_x\sigma_y\sigma_z)]U^\dagger = \sigma_0 \otimes \sigma_x\sigma_y\sigma_z$. By using $\sigma_x\sigma_y\sigma_z = i\sigma_0$, we obtain $UU^\dagger = -iI$, which is a contradiction. It was shown that there is no $N = 2$ NS under our noise model.

We will briefly discuss quantum mutual information and quantum discord (hereafter, abbreviated as QD) introduced in Ref. [23] in order to analyze another aspect of our scheme. First, let us observe that the encoded state ρ_E given by Eq. (4) is fully separable. Let us write $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha,\beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$. Then,

$$\rho_E = \tfrac{1}{2}(|0\rangle\langle 0| \otimes \rho' + |1\rangle\langle 1| \otimes \rho''),$$

where ρ' is an equal mixture of $\alpha|00\rangle + \beta|11\rangle$ and $\beta|01\rangle + \alpha|10\rangle$, and ρ'' is an equal mixture of $\beta|00\rangle + \alpha|11\rangle$ and $\alpha|01\rangle + \beta|10\rangle$. It is easy to prove that ρ' and ρ'' are separable, by employing the positive partial transpose (PPT) criterion [24,25]. The state ρ_E is thus fully separable, which entails that, in this case, entanglement is clearly not a relevant resource for error avoidability. It is therefore interesting to investigate nonclassical correlations other than entanglement. The quantum mutual information $\mathcal{I}$ of the system described by the density matrix ρ_{AB} is $\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, where $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ and $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$. Here, we are considering the state given by Eq. (4); hence, System B is the qubit originally storing the information, while System A is the ancillae, and $\mathcal{I}(\rho_{AB}) = 1$ regardless of the state $|\psi\rangle\langle\psi|$.

QD is a measure of nonclassical correlations between two subsystems of a quantum system. We follow the notation described in Refs. [23,26]. We find, after some lengthy calculations, that $\mathcal{D}(B:A)$ vanishes while $\mathcal{D}(A:B)$ has the structure shown in Fig. 6 [27]. They are not necessarily equal to each other [23]. $\mathcal{D}(A:B)$ strongly depends on $|\psi\rangle\langle\psi|$, as shown in Fig. 6, and vanishes for certain points on the Bloch sphere. For those points, our scheme can successfully implement error correction even though $\mathcal{D}(B:A)$ and $\mathcal{D}(A:B)$ vanish simultaneously. (We again note that $\mathcal{I}(\rho_{AB}) = 1$ everywhere.) We therefore hypothesize that the nonclassical correlations quantified by QD may not be a relevant resource for error avoidability in our scheme.

By assuming another, more artificial, noise model we may be able to construct a NS simpler than that in Fig. 5, with just two qubits.

The scheme shown in Fig. 5, however, has an interesting application: it may be employed to allow communication between two parties (Alice and Bob) without a shared reference frame (SRF). Let us denote Alice's basis with $|i\rangle$ and Bob's basis with $|i'\rangle$, and assume that $|0'\rangle = V|0\rangle$ and $|1'\rangle = V|1\rangle$,

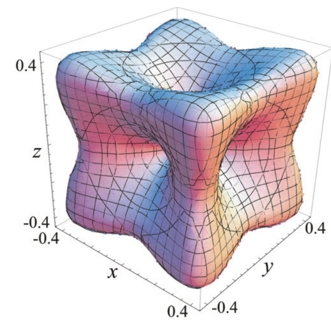


FIG. 6. (Color online) Quantum discord $\mathcal{D}(A:B)$ as a function of the initial state of the data qubit parameterized by the Bloch vector $\boldsymbol{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Coordinates $(r = \mathcal{D}(A:B),\theta,\phi)$ depict QD as a function of θ,ϕ. $\mathcal{D}(A:B)$ vanishes when $\boldsymbol{n} = (\pm 1,0,0),(0,\pm 1,0),(0,0,\pm 1)$. The function $\mathcal{D}(A:B)$ takes the maximum value $(3/4)\log_2 3 - 1/2$ when $\boldsymbol{n} = (\pm 1, \pm 1, \pm 1)/\sqrt{3}$.

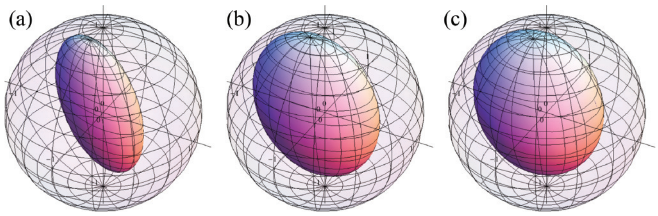
where V is one of σ_i. Alice uses $\mathcal{U}_{\mathrm{E}}$ in Fig. 5 to encode a data qubit $|\psi\rangle$ with two fully mixed ancillae and sends them to Bob. The state that Bob receives is $\mathcal{U}_{\mathrm{E}}(\sigma_0 \otimes \sigma_0 \otimes |\psi\rangle\langle\psi|/4)$ in Alice's basis. He decodes it with $(\mathcal{U}'_{\mathrm{E}})^\dagger$ in his basis, that is, $\mathcal{V}^{\otimes 3} \circ \mathcal{U}_{\mathrm{E}}^\dagger \circ \mathcal{V}^{\dagger\otimes 3}$ in Alice's basis. Since $\mathcal{E} = \mathcal{V}^{\otimes 3}$ in Fig. 5, Bob obtains $\mathcal{V}^{\otimes 3}(\sigma_0 \otimes \sigma_0 \otimes |\psi\rangle\langle\psi|/4) = (\sigma'_0 \otimes \sigma'_0 \otimes |\psi'\rangle\langle\psi'|/4)$. This implies that Alice and Bob can transfer a data qubit without knowing the relation between their respective bases. A more general theory of quantum communication without a SRF was discussed in Ref. [28], wherein the reference frames of Alice and Bob may differ by arbitrary rotations on account of the definition of NS being restricted to its original meaning [5–11]. On the other hand, in the present case the allowed rotations are limited to the σ_i because we are considering a generalized definition of NS [16,17].

IV. EXPERIMENTS

We implement our scheme experimentally with a three-qubit NMR quantum computer; we employ a JEOL ECA-500 NMR spectrometer, whose hydrogen Larmor frequency is approximately 500 MHz. As a three-spin molecule, we employ ^{13}C-labeled L-alanine (98% purity, Cambridge Isotope) solved in D_2O. We simplify the quantum circuit shown in Fig. 5 by taking into account the fact that the phases of states are not independently observed in a NMR quantum computer. Both the encoding and the decoding require only five pulses including refocusing pulses, taking about 25 ms.

TABLE I. The four initial states are transformed to the final states by the map $\mathcal{M}$. The states are represented by using their Bloch vectors.

	Final state		
	(a) No error $\{p_i\} = (1,0,0,0)$	(b) E_1 error $\{p_i\} = (0,1,0,0)$	(c) E_2 error $\{p_i\} = (0,0,1,0)$
$\hat{z}$	(0.05, 0.03, 0.89)	(0.02, 0.02, 0.85)	(0.00, −0.02, 0.78)
$\hat{x}$	(0.50, 0.11, −0.25)	(0.65, 0.05, −0.14)	(0.61, 0.06, −0.16)
$\hat{y}$	(−0.04, 0.37, 0.03)	(−0.03, 0.56, −0.02)	(−0.14, 0.65, 0.01)
$-\hat{z}$	(−0.01, 0.14, −0.74)	(0.00, 0.06, −0.70)	(−0.07, 0.08, −0.69)



Error Operator	no error	E_1	E_2
$F_e(\sigma_0, \mathcal{M})$	0.68	0.73	0.75
$\mathrm{Tr}(\mathcal{M})$	1.03	1.00	1.00

FIG. 7. (Color online) Geometrical illustration of the action of our scheme on the data qubit. For each of the cases (a), (b), and (c), the outer Bloch sphere represents the set of all the pure states, while the inner, deformed sphere represents the output states after the recovery process. The three cases depicted in (a), (b), and (c) correspond to different error operators, no error, E_1, and E_2, respectively. Entanglement fidelities $F_e(\sigma_0,\mathcal{M})$ and traces $\mathrm{Tr}(\mathcal{M})$ are summarized in the table.

The density matrix of the thermal state, ρ_{th}, is well approximated by

$$\rho_{\mathrm{th}} = (\sigma_0/2)^{\otimes 3} + \frac{\epsilon}{8}(\sigma_z \otimes \sigma_0 \otimes \sigma_0 + \sigma_0 \otimes \sigma_z \otimes \sigma_0 + \sigma_0 \otimes \sigma_0 \otimes \sigma_z),$$

where $\epsilon \sim 10^{-6}$. Since $(\sigma_0/2)^{\otimes 3}$ is not visible in NMR, ρ_{th} is effectively regarded as $\sigma_0 \otimes \sigma_0 \otimes \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ when the third spin is "seen." This implies that no initialization process is necessary, and thus the experiments become very simple.

We perform three sets of experiments, as summarized in Table I.

The four initial states, $\hat{z},\hat{x},\hat{y}, -\hat{z}$, are transformed to the final state by the map $\mathcal{M}$, which is determined by the encoding, error, and decoding processes shown in Fig. 5. The states are represented by using their Bloch vectors. We apply quantum process tomography [29], and the results are summarized in Table I and Fig. 7. Although the surfaces in Fig. 7 are distorted, it is clear that our QEC scheme indeed eliminates the effects of the fully correlated noises. One may find that the two initial states, $\hat{z}$ state and $-\hat{z}$ state, are mapped to final states that show a certain discrepancy in the amplitudes of their z components. For example, 0.89 for $\hat{z}$ state and $|-0.74|$ for $-\hat{z}$ state in case (a). Note that the only difference is that we use a π pulse to prepare the $-\hat{z}$ state from the $\hat{z}$ state before encoding. The most of fidelity loss may be due to an inhomogeneous control field (H_1 in NMR terminology) of pulses [30]. For better results, we might have to make use of composite pulses [31].

V. CONCLUSIONS

We demonstrate a simple, yet interesting, quantum error correction scheme avoiding fully correlated errors. The ancillae can be in a uniformly mixed state for this purpose. The analysis of quantum mutual information and quantum discord (QD) may indicate that the nonclassical correlations quantified by QD are not relevant as a resource for error avoidability in our scheme. We anticipate further progress both in the understanding of quantum correlations and the development of QEC schemes.

ACKNOWLEDGMENTS

We are grateful to Hiroyuki Tomita for his valuable inputs, to Akira SaiToh for his critical reading and suggestions, and to an anonymous referee who provided us with various useful comments. We are also grateful to the "Open Research Center" Project for Private Universities, matching fund subsidy from the MEXT (Ministry of Education, Culture, Sports, Science and Technology) for financial support. Y.K. and M.N. would like to thank partial supports of Grants-in-Aid for Scientific Research from the JSPS (Grant Nos. 23540470 and 25400422). C.B. acknowledges the financial support of the MEXT Scholarship for foreign students.

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