Discord and quantum computational resources

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Discordant states appear in a large number of quantum phenomena and seem to be a good indicator of divergence from classicality. While there is evidence that they are essential for a quantum algorithm to have an advantage over a classical one, their precise role is unclear. We examine the role of discord in quantum algorithms using the paradigmatic framework of *restricted distributed quantum gates* and show that manipulating discordant states using local operations has an associated cost in terms of entanglement and communication resources. Changing discord reduces the total correlations and reversible operations on discordant states usually require nonlocal resources. Discord alone is, however, not enough to determine the need for entanglement. A more general type of similar quantities, which we call *K* discord, is introduced as a further constraint on the kinds of operations that can be performed without entanglement resources.

I. INTRODUCTION

In recent years quantum discord and similar quantities have become an actively studied topic. There is mounting evidence that discordant states play an essential role in a wide variety of quantum phenomena (see [1] and references therein). In many cases discord seems to be a more natural measure of quantum correlations than entanglement [1,2] and has sometimes been hailed as an essential resource for mixed state quantum computation [3]. The term *resource* is, however, debatable, mainly due to the fact that very few limitations have been found on the creation and manipulation of discord. Most notably, unlike entanglement, discord can be created and increased using local operations [1].

Nevertheless, there is strong evidence that discordant states play a role in mixed state protocols and discord is in many ways an indication of the divergence from classicality [1,3–5]. In the context of quantum algorithms there is interest in identifying "the elusive source of quantum speedup" [6]. Recent results indicate that discordant states play an essential role in quantum algorithms that display an advantage over classical ones [1,2,7].

Discord shares many properties with pure state entanglement measures [1]. These similarities and the inability to extend some results regarding entanglement in pure state quantum computation to mixed states [8–10] make discord one possible candidate for the source of computational speedup [11]. There are, however, obvious differences between discord in mixed states and entanglement in pure states. The most significant is that discord is not known to be monotonic under a relevant class of operations.

Another approach for identifying the source of the quantum advantage is to attribute it to quantum gates [12,13]. One way to reconcile the two pictures (gates vs states) is to look at the relation between the sets of input and output states. It is also possible to bring entanglement back into the picture by examining a *local operations and classical communication* (LOCC) implementation of the relevant quantum gates. Indeed

under reasonable assumptions an LOCC protocol that does not require ancillary entanglement can be simulated efficiently on a classical computer [14].

In [15] we introduced *restricted, distributed gates* to study the question *when is a reversible operation effectively nonclassical?* in terms of entanglement resources. The result was a relation between discord in the set of input and output states and entanglement. It was, however, limited to cases where only rank-1 measurements are applicable. The original definition of discord naturally selects rank-1 measurements since they maximize information gain [1,16–18]. However, when discussing LOCC protocols it is often useful to consider measurements that reveal less information and are consequently less disturbing. In this regard one should define a more general version of discord. Similar issues were recently discussed in [19,20], where orthogonal projective measurements of different ranks were considered.

Here we extend the results of [15] and tackle a number of issues described above. First we show that changing discord using local operations has an associated cost in terms of mutual information. Next we define a discordlike quantity that takes into account more general measurements. We use this quantity to extend the results of [15] to more general types of states including those used in NMR mixed state quantum computation. The results indicate that in the context of reversible operations discord acts more like an obstacle than a resource. Nonlocal resources are required to *change* discord rather than just increase it. These results, as well as the tools used to obtain them, are useful for a conceptual understanding of the quantum advantage and can be extended to answer more general questions.

A. Notation

The state of a quantum system shared by two parties, Alice and Bob, is denoted by ρ_{AB} with reduced states $\rho_A = \mathrm{tr}_B[\rho_{AB}]$ and $\rho_B = \mathrm{tr}_A[\rho_{AB}]$. Operations on Alice's side are described by their Kraus operators $\{M_a\}$. In principle, any operation can be called a measurement with POVM elements $\{E_a = M_a^\dagger M_a\}$ and a "classical" measurement outcome corresponding to each term. We use the term *measurement* whenever an operation

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has at least one POVM element which is not proportional to the identity $\mathbb{1}$. A measurement is rank-1 if and only if all the POVM elements $\{E_a\}$ are rank-1.

The probabilities for the classical outcomes are given by $p_a = \mathrm{tr}\,(E_a \rho_A)$. The resulting (conditional) state is $\rho_{AB|a} = M_a \rho_{AB} M_a^\dagger$. The conditional state on Bob's side is $\rho_{B|a} = \mathrm{tr}_A(\rho_{AB|a}) = \mathrm{tr}_A(E_a \rho_{AB})$. If we discard the classical outcomes, the resulting (average) state is given by $\rho'_{AB} = \sum_a M_a \rho_{AB} M_a^\dagger$. This final state can include any ancillary systems used on Alice's side; thus $d_A = \dim(\rho_A)$ is not necessarily the same as $d'_A = \dim(\rho'_A)$. In principle, it is possible to encode the results in orthogonal states, for example, $\rho'_{AB} = \sum_a |a\rangle\langle a| \otimes \rho_{B|a}$ or $\rho'_{AB} = \sum_a |a\rangle\langle a|_{A_L} \otimes \rho_{A_R B|a}$, where the Hilbert space $\mathcal{H}_A = \mathcal{H}_{A_L} \otimes \mathcal{H}_{A_R}$ and $\langle a|b\rangle = \delta_{ab}$. For simplicity of notation we sometimes use Λ_A to denote an operation on Alice's side such that $\Lambda_A(\cdot) = \sum_a M_a \cdot M_a$. Λ without a subscript represents a local operation on both sides.

II. DISCORD AND LOCAL OPERATIONS

A. Quantum correlations

The total correlation in a bipartite quantum system is given by the mutual information $I(A:B) = S(\rho_{AB}||\rho_A \otimes \rho_B)$, where $S(\rho||\tau) = \mathrm{tr}\,[\rho\,\log_2\rho - \rho\,\log_2\tau]$ is the quantum relative entropy.

Discord was originally defined via the probabilities for the outcomes of a measurement [16,17]:

$$D(B|A) = I(A:B) - \max_{\{E_a\}}\left[S(B) - \sum_a p_a S(\rho_{B|a})\right].\quad (1)$$

The maximization depends only on the POVM elements $\{E_a\}$ and naturally selects rank-1 measurements [1,18]. Interpreting the last term as the classical part of the correlations, $J(B|A) = \max_{\{E_a\}}[S(B) - \sum_a p_a S(\rho_{B|a})]$, gives

$$I(B|A) = J(A:B) + D(B|A).\quad (2)$$

Discord is then the quantum part of the correlations [16,17,21].

We can also define quantum discord as the minimal change in mutual information after a rank-1 measurement [22]:

$$D(B|A) = \min_{\{\bar{M}_a\}}[I(AB) - I(A'B')].\quad (3)$$

The bar indicates that the corresponding POVM elements $\{\bar{E}_a = \bar{M}_a^\dagger \bar{M}_a\}$ are rank-1. The two definitions, Eqs. (1) and (3), are equivalent.

Zero discord states are called *classical* [1]. A state is classical if an only if it has the form

$$\rho_{AB} = \sum_a p_a \Pi_a \otimes \rho_{B|a},\quad (4)$$

where $\{\Pi_a\}$ is a set of orthogonal projectors [16]. These states have only classical correlations. For classical states it is possible to find a local rank-1 measurement that does not induce any loss of information. The POVM elements of this measurement are $\{\Pi_a\}$.

Discord, classical correlations, and mutual information are invariant under local unitary operations. Mutual information and classical correlations are also nonincreasing under local operations [17]. Discord, on the other hand, can both increase

and decrease under local operations [1]. Both discord and classical correlations are not symmetric under the interchange of the subsystems. Here we always consider the discord under a measurement on Alice's subsystem.

B. Changing discord with local operations

To show that changing discord implies the loss of correlations we will use a special case of Petz's theorem [23,24] regarding the reversibility of completely positive trace preserving (CPTP) operations.

Lemma 1. Given two states ρ_{AB} and $\tau_{AB} = \tau_A \otimes \tau_B$, where $S(\rho_{AB}||\tau_{AB}) < \infty$, and a local (CPTP) operation Λ, the equality $S[\rho_{AB}||\tau_{AB}] = S[\Lambda_A(\rho_{AB})||\Lambda(\tau_{AB})]$ holds if and only if there exists a local operation Λ^* such that $\Lambda^*[\Lambda_A(\tau_{AB})] = \tau_{AB}$ and $\Lambda^*[\Lambda(\rho_{AB})] = \rho_{AB}$ [25].

Taking $\tau_{AB} = \rho_A \otimes \rho_B$ we get a corollary: $I[\rho_{AB}] = I[\Lambda(\rho_{AB})]$ if and only if Λ can be reversed locally. We now present our first result.

Theorem 1. If a local operation $\Lambda(\rho_{AB}) = \rho'_{AB}$ changes discord, $D(B|A) \neq D(B'|A')$, it also decreases mutual information, $I(\rho_{AB}) > I(\rho'_{AB})$.

Alternatively, using the fact that local operations cannot increase mutual information, the theorem reads as follows.

For a local operation, Λ, we have $I[\rho_{AB}] = I[\Lambda(\rho_{AB})] \Rightarrow D(B|A) = D(B'|A')$.

Proof. First we show that discord cannot be decreased without affecting mutual information. Neither mutual information, $I(A:B)$, nor classical correlation, $J(B|A)$, can increase under local operations [17]. Using Eq. (2) we see that decreasing discord reduces mutual information by at least the same amount.

To show that discord cannot be increased we use lemma 1. The operation Λ must be reversible if mutual information does not change. If Λ increases discord without changing mutual information then its reverse can decrease discord without changing mutual information, violating the first part of the proof. ∎

Note that this proof is valid for local operations on both Alice and Bob's side.

Theorem 1 implies that a local operation that *changes* discord results in some loss of information. In the case where the initial state is unknown this loss of information will usually affect reversibility even when we allow classical communication. In what follows we will show that reversible operations taking sets of initial states to final states require nonlocal resources if discord changes. This is a consequence of the effects of measurements on discord and mutual information.

III. DISCORD AND ENTANGLEMENT RESOURCES

A. Restricted distributed gates

To illustrate some of the implications of theorem 1 we extend the *restricted distributed gates* paradigm first introduced in [15]: Alice and Bob may use arbitrary LOCC operations to implement a reversible quantum gate $\mathcal{G}$ on a restricted set of input states, $\mathcal{L}$. This corresponds to realistic situations where the input states are very seldom arbitrary. In particular it corresponds to the case where at each stage of the computation the system is separable as in some mixed state algorithms [1].

Restricted distributed gates may prove to be useful for some quantum computing tasks. However, here we use this paradigm as a theoretical tool for studying "standard" quantum information processing scenarios, in particular the quantum circuit model. With this in mind we restrict ourselves to reversible operations.

Definition 1. A quantum CPTP operation, $\mathcal{G}$, is called reversible on the set of states $\mathcal{L}$ if and only if there exists an inverse CPTP operation $\mathcal{G}^{-1}$ such that $\mathcal{G}^{-1}[\mathcal{G}(\rho_{AB}^i)] = \rho_{AB}^i$ for all $\rho_{AB}^i \in \mathcal{L}$.

A unitary gate is reversible for any input set. More generally irreversibility is a consequence of information loss and reversibility can be related to error correction.

Definition 2. Given a reversible operation, $\mathcal{G}$, on a set of bipartite states, $\mathcal{L}$, we define the distributed, restricted gate, $\mathcal{G}_{\text{res}}$ on $\mathcal{L} = \{\rho_{AB}^i\}$ as a CPTP, LOCC operation with

$$\mathcal{G}_{\text{res}}(\rho_{AB}^i) = \mathcal{G}(\rho_{AB}^i) \quad (5)$$

for all states $\rho_{AB}^i \in \mathcal{L}$.

In general $\mathcal{G}_{\text{res}}(\rho_{AB}^x) \neq \mathcal{G}(\rho_{AB}^x)$ when $\rho_{AB}^x \notin \mathcal{L}$. Since the operations are linear we can assume $\mathcal{L}$ is convex without losing generality.

When there are no entangled ancillary systems we can, without losing generality, describe the implementation of $\mathcal{G}_{\text{res}}$ as a set of local CPTP operations Λ_μ at each stage followed by classical communication from Alice to Bob $C_{A \to B}$ or Bob to Alice $C_{B \to A}$. The classical information is then encoded as part of the local quantum state before the next step. It is retained throughout the operation and discarded (traced out) at the end,

$$\mathcal{G}_{\text{res}}(\rho_{AB}) = \text{tr}_{\text{cl}} \Lambda_{A_n} C_{B \to A} \cdots C_{A \to B} \Lambda_{A_1}(\rho_{AB}). \quad (6)$$

Without loss of generality we always assume Alice goes first.

Reversibility of $\mathcal{G}_{\text{res}}$ requires $S(\mathcal{G}[\rho_{AB}^i]||\mathcal{G}[\rho_{AB}^j]) = S(\rho_{AB}^i||\rho_{AB}^j)$ for all $\rho_{AB}^i, \rho_{AB}^j \in \mathcal{L}$. Using the above structure this requires that at any stage the relative entropy remains constant, in particular $S(\Lambda_{A_1}[\rho_{AB}^i]||\Lambda_{A_1}[\rho_{AB}^j]) = S(\rho_{AB}^i||\rho_{AB}^j)$. Note that G^{-1} is not restricted to LOCC.

If any communication is necessary during the protocol then the first step, Λ_{A_1}, should involve a measurement whereby Alice can gather some information about Bob's (conditional) state. A measurement that reveals maximal information [and maximizes the last term in Eq. (1)] would in general change discord and in most cases decrease the relative entropy between some states in the set of initial states. To overcome this problem Alice can choose her first measurement to reveal less information. In principle Alice and Bob may have some physical restriction on their local measurements. We denote the set of allowed measurements $\mathcal{S}^K$. In the most general case Alice and Bob can make any measurement. We denote the set of all measurements $\mathcal{S}^2$ for a reason that will be apparent below.

We now ask the following. Given a set of measurements $\mathcal{S}^K$ can Alice and Bob implement the gate $\mathcal{G}_{\text{res}}$ without ancillary entanglement? If the answer is yes, we say that $\mathcal{G}_{\text{res}}$ is pseudoclassical for $\mathcal{S}^K$; otherwise, we say it is nonlocal for $\mathcal{S}^K$. $\mathcal{G}_{\text{res}}$ is fully nonlocal when it is nonlocal for $\mathcal{S}^2$. It is fully local if it is pseudoclassical for the empty set. In the fully local case Alice and Bob do not need to communicate in order to implement the gate.

1. Examples for $\mathcal{S}^K$

The set of measurements $\mathcal{S}^K$ will generally depend on the physical scenario. One set that has been extensively studied in the past is the set of all rank-1 orthogonal projective measurements or fully dephasing channels $\mathcal{S}^\Pi$. The corresponding set of Π-classical states (see below for a definition of K-classical states) is the same as the set of zero discord (or classical) states in Eq. (4).

More general are the sets of all rank r orthogonal projections, $\mathcal{S}^{R=r}$. Classicality under these sets of measurements was studied in [20]. While these sets have a simple mathematical structure it is not clear what is the physical scenario where such restrictions apply. In [19] these sets were studied from the perspective of local unitary operations with degenerate eigenvalues.

Other sets of interest are the sets of measurements with at least N outcomes (or linearly independent POVM elements) $\mathcal{S}^N$. The physical motivation here is a bit more clear since the number of outcomes is a property of the measurement device. A special case which has a clear physical significance is $N = 2$. This is the set of all possible measurements since a measurement must have at least two outcomes. Some of our main results below will be derived using this set of all measurements, $\mathcal{S}^2$.

B. *K*-classical states

We now define a notion of classical states with respect to a set of measurements $\mathcal{S}^K$. The relation between this definition and the classicality of distributed gates will become apparent in the next subsection.

Definition 3. A state is called K classical if $I[\rho_{AB}] = I[\Lambda_A(\rho_{AB})]$ for some measurement $\Lambda_A \in \mathcal{S}^K$. A state is K discordant if it is not K classical.

K-classical states can have nonzero discord. An example is a 3×3 discordant state which is a mixture of a maximally entangled state, $1/\sqrt{2}[|00\rangle + |11\rangle]$, and a product state, $|22\rangle$, such a state is two classical (classical with respect to the set of all measurements) and entangled.

Classicality under $\mathcal{S}^N$—the set of measurements with at least N outcomes—is related to reversibility under $\mathcal{S}^N$.

Proposition 1. A state ρ_{AB} is N classical if and only if there exists $\Lambda_A \in \mathcal{S}^N$ such that $\Lambda_A(\rho_{AB}) = \rho_{AB}$.

Proof follows directly from lemma 1.

We use the following necessary and sufficient condition for K classicality to show a relation between K-discordant states, the change in discord, and entanglement resources.

Lemma 2. A state ρ_{AB} is K classical if and only if there is a local measurement $\Lambda_A \in S^K$ and a product state $\tau_{AB} = \tau_A \otimes \tau_B$, with $S(\rho_{AB}||\tau_{AB}) < \infty$, such that $S[\rho_{AB}||\tau_{AB}] = S[\Lambda_A(\rho_{AB})||\Lambda_A(\tau_{AB})]$.

Proof. From lemma 1 the equality above holds if and only if there is a local reverse operation Λ_A^*. This reversibility condition is also necessary and sufficient for K classicality since a local operation which reverses Λ_A on ρ_{AB} will also reverse it on ρ_A so Λ_A preserves mutual information. ∎

C. Operations that require entanglement

Let us now examine the following general scenario: Alice and Bob are limited to local operations where all measurements

are in the set $\mathcal{S}^K$. They implement a restricted gate $\mathcal{G}_{\mathrm{res}}$ over a set of states $\mathcal{L}$ that includes a product state $\tau_{AB} = \tau_A \otimes \tau_B$ and a K-discordant state ρ_{AB} with $S(\rho_{AB}||\tau_{AB}) < \infty$. What can we say about the nonlocal resources required for this gate?

From lemma 2 any measurement on Alice's side $\Lambda_{A_1} \in S^K$, will decrease the relative entropy and will conflict with the reversibility condition. This leaves two options: (a) the implementation of $\mathcal{G}_{\mathrm{res}}$ does not require any measurements in which case it is fully local or (b) some ancillary entanglement is necessary for the implementation, so $\mathcal{G}_{\mathrm{res}}$ is nonlocal for $\mathcal{S}^K$.

In the special case where there is no restriction on the local measurements, $\mathcal{S}^K = \mathcal{S}^2$, the restricted gate is either fully local (can be implemented without communication or entanglement) or fully nonlocal (requires nonlocal resources). In the fully local case we can appreciate theorem 1. Since the operation is fully local any change in discord would reduce mutual information. Consequently the operation would not be reversible locally and the relative entropy between ρ^{AB} and τ^{AB} would decrease (lemma 2), violating the (nonlocal) reversibility condition. The above proves the following.

Theorem 2. A restricted, distributed gate on an input set $\mathcal{L}$ that includes a product state $\tau_{AB} = \tau_A \otimes \tau_B$ cannot be implemented without entanglement resources if it changes discord for any 2-discordant state $\rho_{AB} \in \mathcal{L}$, with $S(\rho_{AB}||\tau_{AB}) < \infty$.

The condition $S(\rho_{AB}||\tau_{AB}) < \infty$ is always satisfied when τ_{AB} is the completely mixed state. In this case the convex set $\mathcal{L}$ also includes noisy states of the form $\rho_{AB} = (1 - N)\rho^0_{AB} + N \mathbb{1}_{AB}/d_{AB}$. These are the standard states for NMR quantum information processing [11].

Corollary 1. Take the noisy family of states $\rho_{AB} = (1 - N)\rho^0_{AB} + N \mathbb{1}_{AB}/d_{AB}$ where $0 \leqslant N \leqslant 1$ is a free parameter and ρ^0_{AB} is two discordant. A restricted gate on this family cannot be implemented without entanglement resources if it changes the discord of ρ^0_{AB}.

1. Example

Consider a distributed quantum simulation of an NMR protocol that includes only a single factorized input state $(1 - N)\rho_A \otimes \rho_B + N \mathbb{1}_{AB}/d_{AB}$ but varied amounts of noise N. Assume that at some point during the computation the state becomes two discordant across the relevant bi-partition. The next quantum gate that changes discord will require some ancillary entanglement to simulate on a distributed quantum computer. Simulate here means producing the exact output state. The relation to classical simulation is not immediate, but in the case where a fully local implementation exists there is an efficient classical algorithm for obtaining an arbitrarily good estimate of the possible measurement probabilities at the end of the quantum algorithm [14].

IV. DISCUSSION AND OUTLOOK

The results presented here are useful for tackling a number of questions regarding the role of discord and discordant states in various processes. Manipulating discordant states has an associated resource cost. Any change in discord using local operations results in a loss of mutual information.

With respect to quantum computational resources, it is possible to regard entanglement as a resource by examining a distributed implementation of the relevant algorithm. In this picture there is a cost associated with manipulating discord. This cost is related to a more general sense of classicality than the one defined by zero discord. Classicality under the set of all local measurements, which we call 2-classicality, is the most sensitive indicator of the requirement for entanglement resources. Changing discord in a noisy 2-discordant set requires nonlocal resources when the operation is reversible. In light of theorem 1 this is also a statement about decreasing mutual information. Either way discord should not be viewed as a *resource* in this context.

2-discordant states provide an obstacle that can only be overcome using nonlocal resources. More generally K-discordant states are an obstacle that can be overcome by either nonlocal resources or less disturbing measurements than those in $\mathcal{S}^K$. Each of these limitations has implications regarding quantum resources. One can also try to quantify the set of allowed measurements as a resource.

This notion of discord as an obstacle was used to discuss the verification of nonlocal gates in the cases of one way communication [5] and qubits [26]. The definition of K discord allows a more general approach to this problem. Surprisingly variable depolarizing noise can help the verifier (corollary 1).

A. *K* discord

The results are qualitative: we showed that changing discord has an associated resource cost but we did not calculate this cost explicitly. A quantitative relation between discord in a set of initial states and the entanglement and communication resources required to manipulate the states in this set is still missing. Another missing parameter is a quantitative measure of K discord defined in an operationally meaningful way. Apart from vanishing for K-classical states it should be directly related to the problem at hand. One candidate is the expression $\inf_{\Lambda_A \in \mathcal{S}^K} \{I[\rho_{AB}] - I[\Lambda_A(\rho_{AB})]\}$. In the case of $\mathcal{S}^{R=1}$ this is the standard quantum discord, Eq. (1). For $R > 1$ this expression was analyzed in [20]. In many other cases and in particular for $\mathcal{S}^N$ this quantity may vanish for all states [27] and some meaningful normalization should be used. Alternatively the distance from the set of K-classical states $\mathcal{C}^K$ can be used, e.g., $\min_{\varrho_{AB} \in \mathcal{C}^K} S(\rho_{AB}||\varrho_{AB})$, in a similar way to other operationally meaningful quantities [1,28,29].

K discord can also be related to the thermal discord (see [1,22,30]) using the distance to the completely mixed state, e.g., by taking the difference: $S[\rho_{AB}||\mathbb{1}_{AB}/d_{AB}] - S[\Lambda_A(\rho)||\Lambda_A(\mathbb{1}_{AB}/d_{AB})]$; such a quantity may play a role in finding the number of rounds (times the parties exchange information) required for extracting maximal work from a heat engine. It is also related to an open question regarding the difference between the one-way and two-way quantum deficit [1,31].

A quantitative relation between discord and entanglement resources in restricted gates may also answer practical questions regarding the use of restricted gates in place of standard reversible gates. It would be interesting to find a useful algorithm for restricted gates that are simple to implement

compared to their nonrestricted counterparts. Moreover, such a quantitative relation will provide an operational approach to a long sought-after resource theory for discord. Here discord indicates the need for other resources rather then playing the role of a resource *per se*.

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