Hybrid entanglement purification for quantum repeaters

Yu-Bo Sheng,[1,2,*] Lan Zhou,[2,3] and Gui-Lu Long[4,5]

[1]*Institute of Signal Processing Transmission, Nanjing University of Posts and Telecommunications, 210003, Nanjing, China*
[2]*Key Laboratory of Broadband Wireless Communication and Sensor Network Technology, Nanjing University of Posts and Telecommunications, Ministry of Education, Nanjing, 210003, China*
[3]*College of Mathematics and Physics, Nanjing University of Posts and Telecommunications, Nanjing, 210003, China*
[4]*Tsinghua National Laboratory for Information Science and Technology, Beijing 100084, China*
[5]*State Key Laboratory of Low-dimensional Quantum Physics and Department of Physics, Tsinghua University, Beijing 100084, China*
(Received 24 April 2013; published 1 August 2013)

We present an entanglement purification protocol (EPP) to reconstruct some maximally hybrid entangled states (HESs) from nonmaximally mixed HESs. We use simple linear optical elements such as a polarization beam splitter (PBS) and beam splitter (BS) to achieve this task. Meanwhile, it is shown that the parity-check gates acted by PBS and BS are enough to complete the task, and the controlled-NOT (CNOT) gates or similar logic operations are not needed. Unlike the current EPPs, this protocol can purify not only the conventional bit-flip error and phase-flip error but also the dissipation error coming from the photon loss of the coherent state. It can also be extended to achieve the purification for multiphoton and multicoherent state HES. This protocol may be useful in current hybrid quantum repeaters.

DOI: 10.1103/PhysRevA.88.022302 PACS number(s): 03.67.Dd, 03.67.Hk, 03.65.Ud

I. INTRODUCTION

Long-distance quantum communication, which has reached a very interesting stage, is used to deal with the transmission and exchange of quantum information between distant nodes of a network. Now many remarkable applications have been reported, such as quantum key distribution (QKD) [1,2], teleportation [3], quantum dense coding [4], and some other important quantum cryptographic schemes [5–13]. Such quantum communication processes above all need to share the entangled states to set up the quantum entanglement channel between distant locations. Unfortunately, the optical absorption and the other channel noise will make the photon loss the main obstacle during the entanglement distribution.

To overcome the difficulty associated with the exponential fidelity decay, the concept of quantum repeaters has been proposed [14,15]. Currently, quantum repeater protocols are usually implemented with atomic ensembles and linear optics [16–26]. Recently, another type of entanglement, hybrid entanglement [27–30], can also be used to implement the quantum repeaters [31–42]. The hybrid entangled state (HES) means that the entanglement is between different degrees of freedom of a particle pair. In 2006, van Loock *et al*. proposed a hybrid quantum repeater protocol using bright coherent light [31]. In 2008, Munro *et al*. developed this idea and proposed a high-bandwidth hybrid quantum repeater protocol [32]. They also improved their protocol to implement the high-performance quantum networking with quantum multiplexing [38]. Azuma *et al*. proposed an efficient hybrid quantum repeater with two coherent states [39] and they discussed the tight bound on coherent-state-based entanglement generation over lossy channels [40]. Recently, a hybrid long-distance entanglement repeater protocol has been proposed [42]. In above hybrid quantum repeater protocols, the solid memory qubit can be converted into the optical qubit in an individual Λ-type atom [16–18], a trapped ion [26], and so on. Recently, the hybrid entangled state of the form $\frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle + |V\rangle|-\beta\rangle)$ can be generated in principle by performing a weak cross-Kerr nonlinear interaction between a single photon and a strong coherent state with the help of a displacement operation [43,44], as pointed out by Lee and Jeong. $|H\rangle$ and $|V\rangle$ denote horizontal and vertical polarizations of the single photon, respectively. The $|\pm\beta\rangle$ are the coherent states. In their paper [43], they proposed a scheme to realize deterministic quantum teleportation using linear optics and hybrid qubits, and such HESs are the necessary resources for universal gate operations. Moreover, the HESs encoded in other degrees of freedom in optical systems have been widely discussed, such as the experiments of QKD and testing nonlocality encoded in time-bin qubit and polarization qubit [45,46], the remote preparation of single-photon hybird entangled state [47], experimental violation of a Bell inequality with the path (linear momentum) of one photon and the polarization of the other photon [28], the demonstration of spin-orbit hybrid entanglement of photons [48], and so on [49,50].

However, similar to the conventional quantum repeaters, one cannot avoid the channel noises. It will cause the probe beam loss and the single qubit flip. For example, the initial state,

$$|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle + |V\rangle|-\beta\rangle), \tag{1}$$

will become $|\Phi^{-}\rangle$ or $|\Psi^{\pm}\rangle$. If the state $|\Phi^{+}\rangle$ becomes $|\Phi^{-}\rangle$, a phase-flip error occurs. If $|\Phi^{+}\rangle$ becomes $|\Psi^{+}\rangle$, it is a bit-flip error, and if $|\Phi^{+}\rangle$ becomes $|\Psi^{-}\rangle$, both the bit-flip error and phase-flip error occur simultaneously. Here

$$|\Phi^{-}\rangle = \frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle - |V\rangle|-\beta\rangle),$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|V\rangle|\beta\rangle \pm |H\rangle|-\beta\rangle). \tag{2}$$

*shengyb@njupt.edu.cn

If a bit-flip error occurs, the initial state may become a mixed state of the form

$$\rho_{ab} = F|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F)|\Psi^+\rangle_{ab}\langle\Psi^+|. \tag{3}$$

In this way, if the entangled states are polluted, we should convert them into the maximally entangled states.

Entanglement purification is essentially to recover an ensemble of nonmaximally entangled states into a subensemble of high quality of maximally entangled states [51–70]. The concept of entanglement purification was proposed by Bennett *et al.* in 1996 [51]. In 2001, Pan *et al.* proposed an entanglement purification protocol (EPP) with linear optics [54]. We call it the PBS protocol. Later, some other EPPs were proposed, including the purification for the decoherent coherent state, the entangled coherent states, and so on [56–70]. However, existing EPPs are all focused on the entanglement state with only one degree of freedom. Most of them are focused on the entanglement being in polarization degree of freedom [54–61]. They cannot deal with the decoherence of HES.

In this paper, we present an efficient hybrid EPP for hybrid quantum repeaters. We show that this EPP can purify not only the bit-flip and phase-flip errors but also the decoherence caused by the photon loss in the coherent state. Interestingly, after performing this EPP, not only is the fidelity of the mixed state improved, but also the amplitude of the coherent state is increased, which is quite different from the conventional EPPs. It makes this EPP more useful in hybrid quantum repeaters.

This paper is organized as follows: In Sec. II, we explain this EPP for the hybrid entangled mixed state. In Secs. II A and II B, we describe this EPP for conventional bit-flip error and phase-flip error caused by the single qubit. In Sec. II C, we describe this EPP for purifying the photon loss of the coherent state during practical transmission. It is shown that the photon loss can equal the bit-flip error in other bases, which can be purified with the above method. In Sec. III, we extend this EPP to deal with the HES with multiphoton and multicoherent states. Finally, in Sec. IV, we present a discussion and summary.

II. EPP FOR HES WITH SINGLE PHOTON AND SINGLE COHERENT STATE

A. EPP for a bit-flip error

Now we start to explain this protocol with a simple example. The HES encoding in the optical system makes this EPP rather simple. From Fig. 1, two identical mixed states ρ_{a1b1} and ρ_{a2b2} of the form of Eq. (3) are sent to Alice and Bob from S1 and S2, respectively. Alice receives two photons and Bob receives two coherent states. Then Alice and Bob operate their photons simultaneously. The polarization beam splitter (PBS) in Alice's location is to transmit the $|H\rangle$ polarization photon and reflect the $|V\rangle$ polarization. The 50:50 beam splitter (BS) in Bob's location makes

$$|\beta\rangle_{b1}|\beta\rangle_{b2} \longrightarrow |\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}, \tag{4}$$

$$|\beta\rangle_{b1}|-\beta\rangle_{b2} \longrightarrow |0\rangle_{d1}|\sqrt{2}\beta\rangle_{d2}, \tag{5}$$

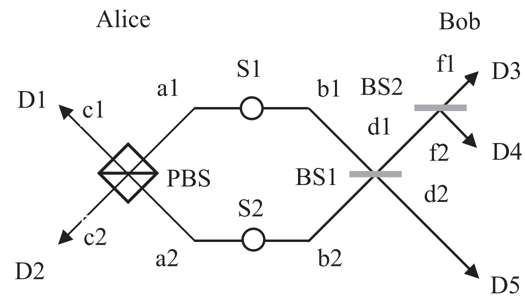

FIG. 1. Schematic diagram of the proposed HES purification. Two pairs of identical mixed HESs are sent to Alice and Bob from source S1 and source S2. Alice receives the single photon, and Bob receives the coherent state. Only two BSs and one PBS are used here. The PBS and BS1 are used to make parity check.

$$|-\beta\rangle_{b1}|\beta\rangle_{b2} \longrightarrow |0\rangle_{d1}|-\sqrt{2}\beta\rangle_{d2}, \tag{6}$$

$$|-\beta\rangle_{b1}|-\beta\rangle_{b2} \longrightarrow |-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}. \tag{7}$$

Here $|0\rangle$ is the vacuum state. We call $|\beta\rangle_{b1}|\beta\rangle_{b2}$ and $|-\beta\rangle_{b1}|-\beta\rangle_{b2}$ even-parity states. $|\beta\rangle_{b1}|-\beta\rangle_{b2}$ and $|-\beta\rangle_{b1}|\beta\rangle_{b2}$ are called odd-parity states. We denote the spatial modes using $c1$ and $d1$ for the upper modes and $c2$ and $d2$ for the lower modes. From Eqs. (4) to (7), we find that the even-parity states will leave the output mode $d2$ (lower mode) no photons, but odd-parity states will leave the output mode $d1$ (upper mode) no photons.

From Eq. (3), one can see that the initial state $\rho_{a1b1}\rho_{a2b2}$ can be seen as a probabilistic mixture of four pure states. With a probability of F^2, states are $|\Phi^+\rangle_{a1b1}|\Phi^+\rangle_{a2b2}$; with an equal probabilities $F(1-F)$, they are in the states $|\Phi^+\rangle_{a1b1}|\Psi^+\rangle_{a2b2}$ and $|\Psi^+\rangle_{a1b1}|\Phi^+\rangle_{a2b2}$; and with a probability of $(1-F)^2$ they are in $|\Psi^+\rangle_{a1b1}|\Psi^+\rangle_{a2b2}$.

We first discuss the item $|\Phi^+\rangle_{a1b1}|\Phi^+\rangle_{a2b2}$. After PBS and BS1, it can be written as

$$\begin{aligned}
&|\Phi^+\rangle_{a1b1}|\Phi^+\rangle_{a2b2}\\
&= \frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle + |V\rangle|-\beta\rangle)_{a1b1}\frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle\\
&\quad + |V\rangle|-\beta\rangle)_{a2b2}\\
&= \frac{1}{2}(|H\rangle_{a1}|H\rangle_{a2}|\beta\rangle_{b1}|\beta\rangle_{b2} + |V\rangle_{a1}|V\rangle_{a2}|-\beta\rangle_{b1}|-\beta\rangle_{b2}\\
&\quad + |H\rangle_{a1}|V\rangle_{a2}|\beta\rangle_{b1}|-\beta\rangle_{b2} + |V\rangle_{a1}|H\rangle_{a2}|-\beta\rangle_{b1}|\beta\rangle_{b2})\\
&\to \frac{1}{2}(|H\rangle_{c1}|H\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}\\
&\quad + |V\rangle_{c1}|V\rangle_{c2}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2} + |H\rangle_{c2}|V\rangle_{c2}|0\rangle_{d1}|\sqrt{2}\beta\rangle_{d2}\\
&\quad + |V\rangle_{c1}|H\rangle_{c1}|0\rangle_{d1}|-\sqrt{2}\beta\rangle_{d2}).
\end{aligned} \tag{8}$$

The essential step in this protocol is to select the "three-mode case." Three-mode case means that there is exactly one photon in each spatial output mode for Alice, and exactly no photon in lower mode $d2$ for Bob. From Eq. (8), items $|H\rangle_{c2}|V\rangle_{c2}|0\rangle_{d1}|\sqrt{2}\beta\rangle_{d2}$ and $|V\rangle_{c1}|H\rangle_{c1}|0\rangle_{d1}|-\sqrt{2}\beta\rangle_{d2}$ never lead the three-mode case, for the $|H\rangle_{c2}|V\rangle_{c2}$ and $|V\rangle_{c1}|H\rangle_{c1}$ are always in the same output mode. After PBS and BS1, by selecting the three-mode case, one can find that only items $|H\rangle_{c1}|H\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}$ and $|V\rangle_{c1}|V\rangle_{c2}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}$ remain.

The two cross-combinations $|\Phi^+\rangle_{a1b1}|\Psi^+\rangle_{a2b2}$ and $|\Psi^+\rangle_{a1b1}|\Phi^+\rangle_{a2b2}$ never lead to the three-mode case. For example, the item $|\Phi^+\rangle_{a1b1}|\Psi^+\rangle_{a2b2}$ evolves as

$$\begin{aligned}
&|\Phi^+\rangle_{a1b1}|\Psi^+\rangle_{a2b2}\\
&= \frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle + |V\rangle|-\beta\rangle)_{a1b1}\frac{1}{\sqrt{2}}(|V\rangle|\beta\rangle\\
&\quad + |H\rangle|-\beta\rangle)_{a2b2}\\
&= \frac{1}{2}(|H\rangle_{a1}|V\rangle_{a2}|\beta\rangle_{b1}|\beta\rangle_{b2} + |V\rangle_{a1}|H\rangle_{a2}|-\beta\rangle_{b1}|-\beta\rangle_{b2}\\
&\quad + |H\rangle_{a1}|H\rangle_{a2}|\beta\rangle_{b1}|-\beta\rangle_{b2} + |V\rangle_{a1}|V\rangle_{a2}|-\beta\rangle_{b1}|\beta\rangle_{b2})\\
&\to \frac{1}{2}(|H\rangle_{c2}|V\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}\\
&\quad + |V\rangle_{c1}|H\rangle_{c1}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2} + |H\rangle_{c1}|H\rangle_{c2}|0\rangle_{d1}\\
&\quad \times|\sqrt{2}\beta\rangle_{d2} + |V\rangle_{c1}|V\rangle_{c2}|0\rangle_{d1}|-\sqrt{2}\beta\rangle_{d2}). \qquad (9)
\end{aligned}$$

From Eq. (9), the items $|H\rangle_{c2}|V\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}$ and $|V\rangle_{c1}|H\rangle_{c1}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}$ will lead the two photons in the same output modes in Alice's location. The items $|H\rangle_{c1}|H\rangle_{c2}|0\rangle_{d1}|\sqrt{2}\beta\rangle_{d2}$ and $|V\rangle_{c1}|V\rangle_{c2}|0\rangle_{d1}|-\sqrt{2}\beta\rangle_{d2}$ will lead to the coherent state in output mode $d2$ in Bob's location. Therefore, the two combinations can be easily eliminated by choosing the three-mode case.

Finally, by choosing the three-mode case, the remaining states are, with the probability of F^2, in

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|H\rangle_{c1}|H\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2} + |V\rangle_{c1}|V\rangle_{c2}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}), \qquad (10)$$

and with the probability of $(1-F)^2$, they are in

$$|\phi'\rangle = \frac{1}{\sqrt{2}}(|V\rangle_{c1}|V\rangle_{c2}|\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2} + |H\rangle_{c1}|H\rangle_{c2}|-\sqrt{2}\beta\rangle_{d1}|0\rangle_{d2}). \qquad (11)$$

Actually, we should point out that the state like $|H\rangle_{a1}|H\rangle_{a2}|\beta\rangle_{b1}|-\beta\rangle_{b2}$ can also lead the three-mode case, for the coherent state has some probability in the state $|0\rangle$. The probability for $|\beta\rangle$ being in $|0\rangle$ is $|\langle 0|\beta\rangle|^2$. In hybrid quantum repeater protocol, we need $|\beta\rangle \gg 1$. In this way, $|\langle 0|\beta\rangle|^2 \to 0$ and can be neglected here.

Alice then generates the maximally entangled state by performing polarization measurement in the basis $+/-$, where $|+\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$, $|-\rangle = \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle)$. If the measurement is $|+\rangle$, one can get the state $\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle + |V\rangle|-\sqrt{2}\beta\rangle)$, with the fidelity F'. Here

$$F' = \frac{F^2}{F^2 + (1-F)^2}. \qquad (12)$$

Otherwise, one can get $\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle - |V\rangle|-\sqrt{2}\beta\rangle)$ with the same fidelity. One can perform a phase-flip operation on the single polarized photon to convert the $\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle - |V\rangle|-\sqrt{2}\beta\rangle)$ to $\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle + |V\rangle|-\sqrt{2}\beta\rangle)$.

So far, they have created a mixed state of the form

$$\rho'_{ab} = F'|\phi_1\rangle\langle\phi_1| + (1-F')|\phi_2\rangle\langle\phi_2|. \qquad (13)$$

Here $|\phi_1\rangle = \frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle + |V\rangle|-\sqrt{2}\beta\rangle)$ and $|\phi_2\rangle = \frac{1}{\sqrt{2}}(|V\rangle|\sqrt{2}\beta\rangle + |H\rangle|-\sqrt{2}\beta\rangle)$. Certainly, if the fidelity of the initial state $F > \frac{1}{2}$, from Eq. (12), one can obtain $F' > F$. In the traditional EPP, the aim of the purification is to increase the fidelity of the desired state [51–54], such as the $|\Phi^+\rangle$ in Eq. (3) in this EPP. However, from Eq. (13), compared with Eq. (1), the purified state $|\phi_1\rangle$ is not the original one. It is $\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle \pm |V\rangle|-\sqrt{2}\beta\rangle)$, and the coherent amplitude has been increased. In order to get the same form of the original state

$$\rho''_{ab} = F'|\Phi^+\rangle\langle\Phi^+| + (1-F')|\Psi^+\rangle\langle\Psi^+|, \qquad (14)$$

they should convert the $|\phi_1\rangle$ to $|\Phi^+\rangle$ by decreasing the coherent amplitude. From Fig. 1, after BS2, they can get

$$\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2}\beta\rangle + |V\rangle|-\sqrt{2}\beta\rangle) \to \frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle_{f1}|\beta\rangle_{f2} + |V\rangle|-\beta\rangle_{f1}|-\beta\rangle_{f2}). \qquad (15)$$

If they perform the photon number detection $|n\rangle\langle n|$ on the second probe beam in $f2$, they can make the $|\pm\beta\rangle$ undistinguishable. To realize the projection $|n\rangle\langle n|$ deterministically, one should use quantum nondemolition detection (QND) [71,72]. After the measurement $|n\rangle\langle n|$ on $f2$, one can finally get the desired state $\frac{1}{\sqrt{2}}(|H\rangle|\beta\rangle + |V\rangle|-\beta\rangle)$ with the fidelity of F'. Here we call the process which makes $|\pm\sqrt{2}\beta\rangle$ become $|\pm\beta\rangle$ with BS2 and QND an attenuation process. The increased amplitude of coherent state seems to make the whole EPP difficult because they have to resort to the QND. Interestingly, the increased coherent state actually can be turned into an advantage in hybrid quantum repeaters if we consider the photon loss of the coherent state, which is discussed later.

B. EPP for a phase-flip error

So far, we have discussed the bit-flip error purification for the HES. The phase-error flip can be converted into the bit-flip error with some operations. For example, if Alice performs the operation U, with

$$|H\rangle \to \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle), \qquad (16)$$

$$|V\rangle \to \frac{1}{\sqrt{2}}(|V\rangle - i|H\rangle). \qquad (17)$$

Meanwhile, Bob performs the operation U' with

$$|\beta\rangle \to \frac{e^{-i\pi/4}}{\sqrt{2}}(|\beta\rangle + i|-\beta\rangle), \qquad (18)$$

$$|-\beta\rangle \to \frac{e^{-i\pi/4}}{\sqrt{2}}(i|\beta\rangle + |-\beta\rangle). \qquad (19)$$

If both the operations are performed, they will make

$$|\Phi^+\rangle \to |\Phi^+\rangle, \qquad (20)$$

$$|\Psi^+\rangle \to |\Psi^+\rangle, \qquad (21)$$

$$|\Phi^-\rangle \rightarrow |\Psi^-\rangle, \tag{22}$$

$$|\Psi^-\rangle \rightarrow |\Phi^-\rangle. \tag{23}$$

In this way, the phase-flip error has been converted into bit-flip error. Here the operation U and U' are essentially the operation B_y, which are corresponds to $\pi/2$ rotations around y axes. The operation U can be achieved with the half-wave plate. From Ref. [73], the B_z rotation can be realized by displace operator $D(\delta) = \exp(\delta\alpha^\dagger - \delta^*\alpha)$ for coherent state. The displacement operators $D(\alpha)$ and $D(\delta)$ do not commute. Interestingly, the product $D(\alpha)D(\delta)$ is simply $D(\alpha+\delta)$ multiplied by a phase factor, $\exp[\frac{1}{2}(\alpha\delta^* - \alpha^*\delta)]$, which plays an important role to rotate the logical qubit. Therefore, the displacement operator $D(i\epsilon)$, where $\epsilon \ll 1$ is real, on an arbitrary qubit $|\varphi\rangle = a|\beta\rangle + b|-\beta\rangle$ can be regarded as a z rotation of the qubit by $U_z(\theta/2 = 2\alpha\epsilon)$. The rotation B_y can be performed by B_z and B_x with $B_y = -\sigma_z B_x B_z B_x$, where σ_z is a π rotation around the z axis. The operations B_x for Bob can be achieved by using the nonlinear Kerr medium [74]. In a word, from Eqs. (20) to (23), these operations performed by Alice and Bob will have the same purification effect with the protocol proposed in Ref. [52]. Once the bit-flip error can be corrected successfully, the phase-flip error can be corrected in a subsequent purification step. In this way, one can implement an EPP for general mixed HESs.

C. EPP for the photon loss in the coherent state

We have briefly explained the EPP with a simple example. In a practical quantum repeater protocol, the noise acts on both a single qubit and coherent state. For coherent-state transmission, the main problem is the photon loss in the channel. It will make the original HES $|\Phi^+\rangle$ become

$$|\Phi'^+\rangle = \frac{1}{\sqrt{2}}(|H\rangle|\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_E + |V\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_E). \tag{24}$$

The photon loss in the channel can be described as the reflection with a BS. The transmission probability is η, and the reflection probability is $1-\eta$, which means that $1-\eta$ photons are dispelled into an environment mode [75,76].

If a bit-flip error occurs, the mixed state in Eq. (3) can be rewritten as

$$\rho'''_{ab} = F|\Phi'^+\rangle_{ab}\langle\Phi'^+| + (1-F)|\Psi'^+\rangle_{ab}\langle\Psi'^+|, \tag{25}$$

with

$$|\Psi'^+\rangle = \frac{1}{\sqrt{2}}(|V\rangle|\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_E + |H\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_E). \tag{26}$$

Similar to the above step to purify the mixed state described by Eq. (25), by choosing two copies of the mixed states ρ'''_{ab}, with the probability of F^2, states are $|\Phi'^+\rangle|\Phi'^+\rangle$; with the equality probability $F(1-F)$, states are $|\Phi'^+\rangle|\Psi'^+\rangle$ and $|\Psi'^+\rangle|\Phi'^+\rangle$; and with the probability of $(1-F)^2$, states are $|\Psi'^+\rangle|\Psi'^+\rangle$. Only the $|\Phi'^+\rangle|\Phi'^+\rangle$ and $|\Psi'^+\rangle|\Psi'^+\rangle$ can lead the three-mode case. For example,

$$\begin{aligned}
&|\Phi'^+\rangle|\Phi'^+\rangle \\
&= \left[\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_{E_1} + |V\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_{E_1})\right] \\
&\quad\times\left[\frac{1}{\sqrt{2}}(|H\rangle|\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_{E_2} + |V\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_{E_2})\right] \\
&= \frac{1}{2}|H\rangle|H\rangle|\sqrt{\eta}\beta\rangle|\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_{E_1}|\sqrt{1-\eta}\beta\rangle_{E_2} \\
&\quad+\frac{1}{2}|V\rangle|V\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_{E_1}|-\sqrt{1-\eta}\beta\rangle_{E_2} \\
&\quad+\frac{1}{2}|H\rangle|V\rangle|\sqrt{\eta}\beta\rangle|-\sqrt{\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_{E_1}|-\sqrt{1-\eta}\beta\rangle_{E_2} + \frac{1}{2}|V\rangle|H\rangle|-\sqrt{\eta}\beta\rangle|\sqrt{\eta}\beta\rangle| \\
&\quad-\sqrt{1-\eta}\beta\rangle_{E_1}|\sqrt{1-\eta}\beta\rangle_{E_2}.
\end{aligned} \tag{27}$$

Here $|\pm\sqrt{1-\eta}\beta\rangle_{E_1}$ and $|\pm\sqrt{1-\eta}\beta\rangle_{E_2}$ are the photon losses to the separate environmental modes of $|\Phi'^+\rangle$ and $|\Phi'^+\rangle$, respectively. From the above description, as shown in Fig. 1, after the PBS and BS1, by selecting the three-mode case and measuring the single photon in the basis $+/-$, one can ultimately get the state

$$\begin{aligned}
|\Phi''^+\rangle &= \frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2\eta}\beta\rangle|\sqrt{1-\eta}\beta\rangle_{E_1}|\sqrt{1-\eta}\beta\rangle_{E_2} \\
&\quad+|V\rangle|-\sqrt{2\eta}\beta\rangle|-\sqrt{1-\eta}\beta\rangle_{E_1}|-\sqrt{1-\eta}\beta\rangle_{E_2}). \\
&\rightarrow \frac{1}{\sqrt{2}}(|H\rangle|\sqrt{2\eta}\beta\rangle|\sqrt{2(1-\eta)}\beta\rangle_E \\
&\quad+|V\rangle|-\sqrt{2\eta}\beta\rangle|-\sqrt{2(1-\eta)}\beta\rangle_E),
\end{aligned} \tag{28}$$

with the same fidelity of F'. For the whole system, the photon loss $|\pm\sqrt{1-\eta}\beta\rangle_{E_1}$ and $|\pm\sqrt{1-\eta}\beta\rangle_{E_2}$ from two coherent states can be regarded as the large environment mode as $|\pm\sqrt{2(1-\eta)}\beta\rangle_E$. Compared with Eq. (24), the amplitude of the coherent state has been increased. If we denote $|\gamma\rangle \equiv |\sqrt{2}\beta\rangle$, $|\Phi''^+\rangle$ can be rewritten as

$$|\Phi''^+\rangle = \frac{1}{\sqrt{2}}(|H\rangle|\sqrt{\eta}\gamma\rangle|\sqrt{(1-\eta)}\gamma\rangle_E + |V\rangle|-\sqrt{\eta}\gamma\rangle|-\sqrt{(1-\eta)}\gamma\rangle_E). \tag{29}$$

Now we rewrite the above state in an orthogonal two dimensional basis $\{|u\rangle,|v\rangle\}$,

$$\begin{aligned}
|\sqrt{\eta}\gamma\rangle &= \mu|u\rangle + \nu|v\rangle, \\
|-\sqrt{\eta}\gamma\rangle &= \mu|u\rangle - \nu|v\rangle, \\
|\sqrt{1-\eta}\gamma\rangle_E &= \mu_E|u\rangle_E + \nu_E|v\rangle_E, \\
|-\sqrt{1-\eta}\gamma\rangle_E &= \mu_E|u\rangle_E - \nu_E|v\rangle_E,
\end{aligned} \tag{30}$$

and

$$\mu = \sqrt{\frac{1+e^{-2\eta\gamma^2}}{2}}, \quad \mu_E = \sqrt{\frac{1+e^{-2(1-\eta)\gamma^2}}{2}}, \tag{31}$$

where $\nu = \sqrt{1-\mu^2}$ and $\nu_E = \sqrt{1-\mu_E^2}$.

Using the method described in Ref. [35], let us trace over the loss mode. Equation (29) becomes

$$\mu_E^2|\Phi_\mu^+\rangle\langle\Phi_\mu^+| + (1-\mu_E^2)|\Psi_\mu^+\rangle\langle\Psi_\mu^+|, \tag{32}$$

with

$$|\Phi_\mu^+\rangle = \mu|H\rangle|u\rangle + \sqrt{1-\mu^2}|V\rangle|v\rangle, \tag{33}$$

$$|\Psi_\mu^+\rangle = \mu|V\rangle|u\rangle + \sqrt{1-\mu^2}|H\rangle|v\rangle. \tag{34}$$

The basis $|u\rangle, |v\rangle$ are

$$|u\rangle = \frac{1}{2\mu}(|\sqrt{\eta}\gamma\rangle + |-\sqrt{\eta}\gamma\rangle),$$
$$|v\rangle = \frac{1}{2\sqrt{1-\mu^2}}(|\sqrt{\eta}\gamma\rangle - |-\sqrt{\eta}\gamma\rangle). \tag{35}$$

From Eq. (32), the photon-loss errors equal to the bit-flip error in the other basis. Therefore, one can purify this kind of error with the same method in a subsequent purification step. In this way, the photon losses in the coherent-state transmission can be induced into the framework of the mixed state with bit-flip error described above. Therefore, in the subsequent process, we do not need to discuss the photon losses specifically.

III. PURIFICATION FOR HES WITH MULTIPHOTON AND MULTICOHERENT STATES

The above purification method can be extended to the HES with multiphoton and multicoherent states. The HES with multiphoton and multi-coherent states can be described as

$$|\Phi_N^+\rangle = \frac{1}{\sqrt{2}}(|HH\ldots H\rangle_N|\beta\beta\ldots\beta\rangle_N + |VV\ldots V\rangle_N|-\beta-\beta\ldots-\beta\rangle_N). \tag{36}$$

Here N is both the number of the single photon and the number of the coherent state. The state $|\Phi_N^+\rangle$ can be generated from the multiphoton Greenberg-Horne-Zeilinger (GHZ) state $\frac{1}{\sqrt{2}}(|HH\ldots H\rangle_N + |VV\ldots V\rangle_N)$ and the coherent state $|\beta\rangle|\beta\rangle\ldots|\beta\rangle_N$, with the help of the weak cross-Kerr nonlinear interaction [43,44]. In Ref. [43], the authors also discussed the hybrid entangled state $|\alpha\rangle|\alpha\rangle|H\rangle + |-\alpha\rangle|-\alpha\rangle|V\rangle$. Since the hybrid entanglement in Eq. (1) can be used in current quantum repeaters, the state $|\Phi_N^+\rangle$ can also be used to connect the entanglement for multiparties in distant locations. Therefore, it may be applied in the future distributed quantum communication network.

Unfortunately, the noise may also degrade the state. A bit-flip error may occur on the single qubit and make the state in Eq. (36) become

$$|\Psi_N'^+\rangle = \frac{1}{\sqrt{2}}(|HH\ldots V\rangle_N|\beta\beta\ldots\beta\rangle_N + |VV\ldots H\rangle_N|-\beta-\beta\ldots-\beta\rangle_N) \tag{37}$$

or may occur on the coherent state and make it become

$$|\Psi_N''^+\rangle = \frac{1}{\sqrt{2}}(|HH\ldots H\rangle_N|\beta\beta\ldots-\beta\rangle_N + |VV\ldots V\rangle_N|-\beta-\beta\ldots\beta\rangle_N). \tag{38}$$

Now we start to explain the purification for multiphoton and multicoherent HESs. Compared with the purification for HES with single-photon and single-coherent HES like Eq. (1), it is more complicated to purify such a state. We cannot treat multiphoton and multicoherent HESs as straightforward extensions of the single-photon and single-coherent state. Fortunately, they can also be divided as the bit-flip error purification and phase-flip error purification. Here we take $N=2$, for example, to show the principle for this kind of purification.

The original state $|\Phi_2^+\rangle$ can be written as

$$|\Phi_2^+\rangle_{acbd} = \frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_c|\beta\rangle_b|\beta\rangle_d + |V\rangle_a|V\rangle_c|-\beta\rangle_b|-\beta\rangle_d). \tag{39}$$

The photons and the coherent states are sent to Alice, Bob, Charlie, and Davis, respectively. Alice receives the single photons from the spatial modes $a1$ and $a2$. Bob receives the coherent states from the spatial modes $b1$ and $b2$. Charlie receives the single photons from $c1$ and $c2$ and Davis receives the coherent states from $d1$ and $d2$. The subscripts a,c,b, and d denote the states for Alice, Charlie, Bob, and Davis. Suppose a bit-flip error occur with the probability of $1-F$, and it makes $|\Phi_2^+\rangle$ become $|\Psi_2^+\rangle$, with

$$|\Psi_2^+\rangle_{acbd} = \frac{1}{\sqrt{2}}(|H\rangle_a|V\rangle_c|\beta\rangle_b|\beta\rangle_d + |V\rangle_a|H\rangle_c|-\beta\rangle_b|-\beta\rangle_d). \tag{40}$$

The whole mixed state can be described as

$$\rho_{acbd}^B = F|\Phi_2^+\rangle_{acbd}\langle\Phi_2^+| + (1-F)|\Psi_2^+\rangle_{acbd}\langle\Psi_2^+|. \tag{41}$$

The superscript B means the mixed state which contains the bit-flip error. The purification principle is shown in Fig. 2. Similar to the purification for HESs with single-photon and single-coherent state, in each step, two identical copies of the form Eq. (41) are sent to Alice, Bob, Charlie, and Davis. Then $\rho_{acbd}^B\rho_{acbd}^{B'}$ can also be seen as a probabilistic mixture of four pure states. With a probability of F^2, states are $|\Phi^+\rangle_{acbd}|\Phi^+\rangle'_{acbd}$, with equal probabilities $F(1-F)$, states are $|\Phi^+\rangle_{acbd}|\Psi^+\rangle'_{acbd}$ and $|\Psi^+\rangle_{acbd}|\Phi^+\rangle'_{acbd}$, and with a probability of $(1-F)^2$ states are $|\Psi^+\rangle_{acbd}|\Psi^+\rangle'_{acbd}$.

For $N=2$, we choose the "six-mode case" to achieve the purification. Six-mode case means that each output mode after the PBSs will exactly register one photon, and the lower mode (analogy with $d2$) after each BS will register no photon. The two PBSs will register four photons and the lower modes after two BSs will register no coherent state. From Fig. 2, the cross-combination items $|\Phi^+\rangle_{acbd}|\Psi^+\rangle'_{acbd}$ and $|\Psi^+\rangle_{acbd}|\Phi^+\rangle'_{acbd}$ will never cause the six-mode case and can be eliminated automatically.

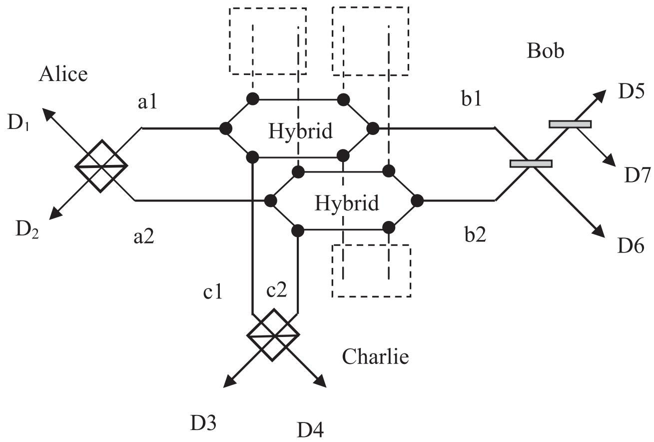

FIG. 2. Schematic diagram of the purification protocol for HES with multiphoton and multicoherent state. The photons and coherent states are sent to Alice, Bob, Charlie, etc. Each party will receive two photons or two coherent states. "Hybrid" means the HES with multiphoton and multicoherent state.

Thus, after selecting the six-mode case, they will get

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|HH\rangle_a|HH\rangle_c|\sqrt{2}\beta\rangle_b|0\rangle|\sqrt{2}\beta\rangle_d|0\rangle + |VV\rangle_a|VV\rangle_c|-\sqrt{2}\beta\rangle_b|0\rangle|-\sqrt{2}\beta\rangle_d|0\rangle), \quad (42)$$

with the probability of F^2. Here the item $|HH\rangle_a$ denotes the two photons belonging to Alice and detectors D1 and D2 both register only one photon. Item $|\sqrt{2}\beta\rangle_b|0\rangle$ denotes that the coherent state is in the upper mode, with lower mode being $|0\rangle$. They will also get

$$|\phi_2\rangle' = \frac{1}{\sqrt{2}}(|HH\rangle_a|VV\rangle_c|\sqrt{2}\beta\rangle_b|0\rangle|\sqrt{2}\beta\rangle_d|0\rangle + |VV\rangle_a|HH\rangle_c|-\sqrt{2}\beta\rangle_b|0\rangle|-\sqrt{2}\beta\rangle_d|0\rangle), \quad (43)$$

with the probability of $(1-F)^2$.

Similar to the Eqs. (10) and (11), Alice and Charlie can generate the maximally entangled state by performing polarization measurement in a $+/-$ basis. Bob and Davis decrease the amplitude of $|\pm\sqrt{2}\beta\rangle$ to $|\pm\beta\rangle|\pm\beta\rangle$ with BS. Finally, if the measurement is $|++\rangle$ or $|--\rangle$, they will obtain the $|\Phi_2^+\rangle_{acbd}$ with the fidelity $F' = \frac{F^2}{F^2+(1-F)^2}$. Otherwise, if the measurement is $|+-\rangle$ or $|-+\rangle$, they will obtain the state $|\Phi_2^-\rangle_{acbd}$, with the same fidelity. Here

$$|\Phi_2^-\rangle_{acbd} = \frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_c|\beta\rangle_b|\beta\rangle_d - |V\rangle_a|V\rangle_c|-\beta\rangle_b|-\beta\rangle_d). \quad (44)$$

One can convert the $|\Phi_2^-\rangle_{acbd}$ to $|\Phi_2^+\rangle_{acbd}$ by performing a phase-flip operation. In this way, they have purified the bit-flip error. Meanwhile, if the bit-flip error occurs on the coherent state like Eq. (38), it can also be corrected using the method described above.

The phase-flip error can also be purified in this way. If a phase-flip error occurs, the original state can be written as

$$\rho_{acbd}^P = F|\Phi_2^+\rangle_{acbd}\langle\Phi_2^+| + (1-F)|\Phi_2^-\rangle_{acbd}\langle\Phi_2^-|. \quad (45)$$

The superscript P means the mixed state which contains the phase-flip error. Phase-flip error can be transformed to bit-flip error with the operations described by Eqs. (16)–(19). However, unlike Eqs. (20)–(23), we cannot get a simple correspondence. After transformation, $|\Phi_2^+\rangle_{acbd}$ becomes

$$|\Phi_2'^+\rangle_{acbd} = \frac{1}{2}[(|HH\rangle - |VV\rangle)_{ac}(|\beta\rangle|\beta\rangle - |-\beta\rangle|-\beta\rangle)_{bd} + (|HV\rangle + |VH\rangle)_{ac}(|\beta\rangle|-\beta\rangle + |-\beta\rangle|\beta\rangle)_{bd}], \quad (46)$$

and $|\Phi_2^-\rangle_{acbd}$ becomes

$$|\Phi_2'^-\rangle_{acbd} = \frac{1}{2}[(|HH\rangle - |VV\rangle)_{ac}(|\beta\rangle|-\beta\rangle + |-\beta\rangle|\beta\rangle)_{bd} + (|HV\rangle + |VH\rangle)_{ac}(|\beta\rangle|\beta\rangle - |-\beta\rangle|-\beta\rangle)_{bd}]. \quad (47)$$

Then the whole system can be rewritten as

$$\rho_{acbd}^{P'} = F|\Phi_2'^+\rangle_{acbd}\langle\Phi_2'^+| + (1-F)|\Phi_2'^-\rangle_{acbd}\langle\Phi_2'^-|. \quad (48)$$

As shown in Fig. 2, they let two copies of the mixed states of the form of Eq. (48) pass through the PBSs and BSs, respectively. With the probability of F^2, states are $|\Phi_2'^+\rangle_{acbd}|\Phi_2'^+\rangle_{acbd}$; with the equal probability $F(1-F)$, states are $|\Phi_2'^+\rangle_{acbd}|\Phi_2'^-\rangle_{acbd}$ and $|\Phi_2'^-\rangle_{acbd}|\Phi_2'^+\rangle_{acbd}$; and with the probability $(1-F)^2$, states are $|\Phi_2'^-\rangle_{acbd}|\Phi_2'^-\rangle_{acbd}$. The cross-combinations $|\Phi_2'^+\rangle_{acbd}|\Phi_2'^-\rangle_{acbd}$ and $|\Phi_2'^-\rangle_{acbd}|\Phi_2'^+\rangle_{acbd}$ cannot lead the six-mode case. By selecting the six-mode case, they will get

$$|\varphi_2\rangle = \frac{1}{2}[(|HHHH\rangle - |VVVV\rangle)(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle - |-\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle) + (|HHVV\rangle + |VVHH\rangle)(|\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle + |-\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle)], \quad (49)$$

with the probability of F^2, and get

$$|\varphi_2'\rangle = \frac{1}{2}[(|HHHH\rangle - |VVVV\rangle)(|\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle + |-\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle) + (|HHVV\rangle + |VVHH\rangle)(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle - |-\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle)], \quad (50)$$

with the probability of $(1-F)^2$.

Equations (49) and (50) can be transformed to Eqs. (46) and (47) by measuring one of photons in each PBS in the $+/-$ basis and decreasing the amplitude of the coherent states. After measurement, if the two photons are $|++\rangle$ or $|--\rangle$, then the state Eq. (49) will become Eq. (46) and Eq. (50) will become Eq. (47). With operations U and U' described by Eqs. (16)–(19), Eq. (46) can be ultimately transformed to $|\Phi_2^+\rangle$, but Eq. (47) will become

$$|\Phi_2^-\rangle_{acbd}^{\perp} = \frac{1}{\sqrt{2}}(|V\rangle_a|V\rangle_c|\beta\rangle_b|\beta\rangle_d - |H\rangle_a|H\rangle_c|-\beta\rangle_b|-\beta\rangle_d). \quad (51)$$

The purified mixed state can be rewritten as

$$\rho^{P''}_{acbd} = F'|\Phi_2^+\rangle_{acbd}\langle\Phi_2^+| + (1-F')|\Phi_2^-\rangle^{\perp}_{acbd}\langle\Phi_2^-|, \quad (52)$$

with $F' = \frac{F^2}{F^2+(1-F)^2}$. If $F > 1/2$, $F' > F$. Therefore, the phase-flip error can also be purified after converting it to the bit-flip error.

For correcting the bit-flip errors and phase-flip errors in multiphoton and multicoherent quantum systems, we can follow the same steps as in the case of $N = 2$. For instance, the N-photon and N-coherent quantum system can be described as Eq. (36). If a bit-flip error occurs on the first photon, which is described as Eq. (37), we choose the $3N$-mode case and the whole system will become

$$|\phi_{2N}\rangle = \frac{1}{\sqrt{2}}[|HH\dots H\rangle_{2N}(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle\dots|\sqrt{2}\beta\rangle \times|0\rangle)_N + |VV\dots V\rangle_{2N}(|-\sqrt{2}\beta\rangle|0\rangle \times|-\sqrt{2}\beta\rangle|0\rangle\cdots|-\sqrt{2}\beta\rangle|0\rangle)_N], \quad (53)$$

with the probability of F^2, and

$$|\phi_{2N}\rangle' = \frac{1}{\sqrt{2}}[|HH\cdots H\rangle_{2N-2}|VV\rangle(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle \times|0\rangle\cdots|\sqrt{2}\beta\rangle|0\rangle)_N + |VV\dots V\rangle_{2N-2}|HH\rangle \times(|-\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle\cdots|-\sqrt{2}\beta\rangle|0\rangle)_N], \quad (54)$$

with the probability of $(1-F)^2$.

In order to obtain the purified mixed state, they should measure the second photon in each party in the $+/-$ basis and decrease the amplitude of the coherent states in analogy with the previous purification. After performing a 45° rotation on each photon in the second N-photon system, Eq. (53) becomes

$$|\psi_{2N}\rangle = |HH\dots H\rangle_N\left(\frac{1}{\sqrt{2}}\right)^{\otimes N}(|+\rangle+|-\rangle)^{\otimes N} \times(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle\cdots|\sqrt{2}\beta\rangle|0\rangle)_N + |VV\dots V\rangle_N\left(\frac{1}{\sqrt{2}}\right)^{\otimes N}(|+\rangle-|-\rangle)^{\otimes N} \times(|-\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle|0\rangle\cdots|-\sqrt{2}\beta\rangle|0\rangle)_N, \quad (55)$$

and Eq. (54) becomes

$$|\psi_{2N}\rangle' = |VH\dots H\rangle_N\left(\frac{1}{\sqrt{2}}\right)^{\otimes N-1}(|+\rangle-|-\rangle) \times(|+\rangle+|-\rangle)^{\otimes N}(|\sqrt{2}\beta\rangle|0\rangle|\sqrt{2}\beta\rangle|0\rangle\cdots|\sqrt{2}\beta\rangle|0\rangle)_N + |HV\dots V\rangle_N\left(\frac{1}{\sqrt{2}}\right)^{\otimes N}(|+\rangle-|-\rangle)^{\otimes N} \times(|+\rangle+|-\rangle)(|-\sqrt{2}\beta\rangle|0\rangle|-\sqrt{2}\beta\rangle \times|0\rangle\cdots|-\sqrt{2}\beta\rangle|0\rangle)_N. \quad (56)$$

Finally, by decreasing the amplitude of the coherent states, they will get a new mixed state $|\Phi_N^+\rangle$ with the probability of $\frac{F^2}{F^2+(1-F)^2}$, if the number of $|-\rangle$ is even. Otherwise, they will get $|\Phi_N^-\rangle$ with the same fidelity, if the number of $|-\rangle$ is odd. We denote

$$|\Phi_N^-\rangle = \frac{1}{\sqrt{2}}(|HH\dots H\rangle_N|\beta\beta\dots\beta\rangle_N - |VV\dots V\rangle_N|-\beta-\beta\dots-\beta\rangle_N). \quad (57)$$

One can also convert the $|\Phi_N^-\rangle$ to the desired state $|\Phi_N^+\rangle$ by performing a phase-flip operation. In essence, the process above is used to correct the bit-flip error on the first photon. Those errors on the other photons and the coherent states can also be corrected in the same way and one will get similar results. The phase-flip error can also be corrected in this way, that is, to transform the phase-flip error to bit-flip error and purify it in the next round. In this way, you can completely purify the HES for multiphoton and multicoherent states.

IV. DISCUSSION AND CONCLUSION

We have explained our hybrid EPP for quantum repeaters. It is interesting to compare it with the PBS protocol [54]. In the PBS protocol, the PBS plays the role of a CNOT gate between a spatial-mode qubit and a polarization qubit. The spatial mode, which is flipped or not flipped, is a function of polarization. In fact, the PBS protocol tells us that to implement the purification protocol, the genuine CNOT gate is not necessary. We only need one function of the CNOT gate, that is, the parity check. In Ref. [54], the four-mode cases ask us to choose the case in which Alice and Bob are in the same even parity. The cross-combination items cannot satisfy this condition with one being in odd parity and the other being in even parity. In this EPP, the PBS can be seen in the same role as in the PBS protocol. It is clear that the BS cannot act as the CNOT gate, while it only makes a parity check. From Eqs. (4)–(7), after the BS, the even parity such as $|\beta\rangle|\beta\rangle$ and $|-\beta\rangle|-\beta\rangle$ will be in the upper mode $d1$, and the odd parity $|\beta\rangle|-\beta\rangle$ and $|-\beta\rangle|\beta\rangle$ will be in the lower mode $d2$. The parity check for coherent state is quite different from the single photon. After the BS, one of the output modes will get an amplitude amplification of the coherent state $|\sqrt{2}\beta\rangle$ or $|-\sqrt{2}\beta\rangle$, while another output mode will get $|0\rangle$. Therefore, after performing the purification protocol, not only does the fidelity increase but also the composition of the states are changed. Interestingly, the amplified coherent state is one of the advantages of our protocol. The probe beam in hybrid quantum repeater is used to create the entanglement in the neighbor node [31,32,35,39]. On one hand, one of the problems in a quantum repeater is the photon loss for coherent state. The fidelity of the mixed HES is proportional to the quality of quantum channel, that is, the worse the channel, the lower the fidelity. Therefore, the increased coherent state may improve the fidelity of the whole quantum system. On the other hand, after the transmission, the coherent state will interact with the second single qubit via the controlled displacement operation $D(\beta\sigma_z)$, which is used to create another entanglement. In an optical system, this displacement operation can be performed by cross-Kerr interaction [44,74]. The Hamiltonian can be rewritten as $H_{int} = h\chi a_s^\dagger a_s a_p^\dagger a_p$. The $a_s^\dagger$, $a_s(a_p^\dagger, a_p)$ are the creation and destruction operators, respectively. The χ is the strength of the nonlinearity. From the Hamiltonian, to generate the entanglement like Eq. (1), a larger probe beam is better. In this EPP, the PBS cannot distinguish the cases where both Alice and Bob are in the odd parity, because the two photons are in the same output mode. It makes the success probability the same as for the PBS protocol, which is half of Ref. [51]. In fact, the purification protocol for decoherent coherent-state superpositions has been proposed by Suzuki *et al.* in 2006 using partial homodyne detection [70]. Their protocol is used to deal with the single coherent state, and it cannot purify the hybrid entanglement state. Moreover, after purification, the amplitude of the coherent state will decrease and one should perform another amplification process to recover it. Certainly, we acknowledge that the practical noise is much harder to deal with theoretically. The true noise not only makes the states mixed but also makes the photon loss occur on both the single photon and the coherent state. Moreover, the noise will also make two or more photons have the bit-flip error in the multiphoton and multicoherent state HES. Therefore, it is worth performing the study in a future work.

In conclusion, we have presented an optical hybrid EPP for quantum repeaters. This protocol can purify not only the mixed state caused by bit-flip and phase-flip errors but also the coherent-state dissipation. Interestingly, after purification, the amplitude of the coherent state is increased, which is an advantage of this EPP. It is feasible for current technology. Moreover, we show that to achieve the task of entanglement purification, a parity-check gate is enough. One does not need to resort to the CNOT gate or similar logical operation. This protocol can also be extended to the system of multiphoton and multicoherent states. We hope that this protocol has useful applications in current quantum communication.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China under Grants No. 11104159 and No. 11175094, University Natural Science Research Project of Jiangsu Province under Grant No. 13KJB140010, the Open Research Fund Program of the State Key Laboratory of Low-Dimensional Quantum Physics, Tsinghua University, the Open Research Fund Program of National Laboratory of Solid State Microstructures under Grant No. M25022, Nanjing University, the Open Research Fund of the Key Laboratory of Broadband Wireless Communication and Sensor Network Technology, Nanjing University of Posts and Telecommunications, Ministry of Education (No. NYKL201303), the National Basic Research Program of China under Grants No. 2009CB929402 and No. 2011CB9216002, and the project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

[1] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Rev. Mod. Phys. **74**, 145 (2002).
[2] J. L. Duligall, M. S. Godfrey, K. A. Harrison, W. J. Munro, and J. G. Rarity, New J. Phys. **8**, 249 (2006).
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[5] A. Karlsson, M. Koashi, and N. Imoto, Phys. Rev. A **59**, 162 (1999).
[6] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**, 1829 (1999).
[7] L. Xiao, G. L. Long, F. G. Deng, and J. W. Pan, Phys. Rev. A **69**, 052307 (2004).
[8] F. G. Deng, X. H. Li, C. Y. Li, P. Zhou, and H. Y. Zhou, Phys. Rev. A **72**, 044301 (2005).
[9] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[10] C. H. Bennett, G. Brassard, and N. D. Mermin, Phys. Rev. Lett. **68**, 557 (1992).
[11] G. L. Long and X. S. Liu, Phys. Rev. A **65**, 032302 (2002).
[12] F. G. Deng, G. L. Long, and X. S. Liu, Phys. Rev. A **68**, 042317 (2003).
[13] X. H. Li, F. G. Deng, and H. Y. Zhou, Phys. Rev. A **74**, 054302 (2006).
[14] H.-J. Briegel, W. Dür, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **81**, 5932 (1998).
[15] W. Dür, H. J. Briegel, J. I. Cirac, and P. Zoller, Phys. Rev. A **59**, 169 (1999).
[16] N. Sangouard, C. Simon, H. de Riedmatten, and N. Gisin, Rev. Mod. Phys. **83**, 33 (2011).
[17] L. M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Nature (London) **414**, 413 (2001).
[18] B. Zhao, Z. B. Chen, Y. A. Chen, J. Schmiedmayer, and J. W. Pan, Phys. Rev. Lett. **98**, 240502 (2007).
[19] C. W. Chou, H. de Riedmatten, D. Felinto, S. V. Polyakov, S. J. van Enk, and H. J. Kimble, Nature (London) **438**, 828 (2005).
[20] C. W. Chou, J. Laurat, H. Deng, K. S. Choi, H. de Riedmatten, D. Felinto, and H. J. Kimble, Science **316**, 1316 (2007).
[21] N. Sangouard, C. Simon, J. Minář, H. Zbinden, H. de Riedmatten, and N. Gisin, Phys. Rev. A **76**, 050301(R) (2007).
[22] N. Sangouard, C. Simon, B. Zhao, Y. A. Chen, H. de Riedmatten, J. W. Pan, and N. Gisin, Phys. Rev. A **77**, 062301 (2008).
[23] C. Simon, H. de Riedmatten, M. Afzelius, N. Sangouard, H. Zbinden, and N. Gisin, Phys. Rev. Lett. **98**, 190503 (2007).
[24] Z. S. Yuan, Y. A. Chen, B. Zhao, S. Chen, J. Schmiedmayer, and J. W. Pan, Nature (London) **454**, 1098 (2008).
[25] T. J. Wang, S. Y. Song, and G. L. Long, Phys. Rev. A **85**, 062311 (2012).
[26] L. M. Duan and C. Monroe, Rev. Mod. Phys. **82**, 1209 (2010).
[27] M. Zukowski and A. Zeilinger, Phys. Lett. A **155**, 69 (1991).
[28] X. S. Ma, A. Qarry, J. Kofler, T. Jennewein, and A. Zeilinger, Phys. Rev. A **79**, 042101 (2009).
[29] L. Neves, G. Lima, J. Aguirre, F. A. Torres-Ruiz, C. Saavedra, and A. Delgado, New J. Phys. **11**, 073035 (2009).
[30] L. Neves, G. Lima, A. Delgado, and C. Saavedra, Phys. Rev. A **80**, 042322 (2009).
[31] P. van Loock, T. D. Ladd, K. Sanaka, F. Yamaguchi, K. Nemoto, W. J. Munro, and Y. Yamamoto, Phys. Rev. Lett. **96**, 240501 (2006).
[32] W. J. Munro, R. Van Meter, S. G. R. Louis, and K. Nemoto, Phys. Rev. Lett. **101**, 040502 (2008).
[33] T. D. Ladd, P. van Loock, K. Nemoto, W. J. Munro, and Y. Yamamoto, New J. Phys. **8**, 184 (2006).

[34] T. P. Spiller, K. Nemoto, S. L. Braunstein, W. J. Munro, P. van Loock, and G. J. Milburn, New J. Phys. **8**, 30 (2006).
[35] P. van Loock, N. Lütkenhaus, W. J. Munro, and K. Nemoto, Phys. Rev. A **78**, 062319 (2008).
[36] S. G. R. Louis, W. J. Munro, T. P. Spiller, and K. Nemoto, Phys. Rev. A **78**, 022326 (2008).
[37] P. van Loock, W. J. Munro, K. Nemoto, T. P. Spiller, T. D. Ladd, S. L. Braunstein, and G. J. Milburn, Phys. Rev. A **78**, 022303 (2008).
[38] W. J. Munro, K. A. Harrison, A. M. Stephens, S. J. Devitt, and K. Nemoto, Nat. Photon. **4**, 792 (2010).
[39] K. Azuma, N. Sota, R. Namiki, S. K. Özdemir, T. Yamamoto, M. Koashi, and N. Imoto, Phys. Rev. A **80**, 060303(R) (2009).
[40] K. Azuma, N. Sota, M. Koashi, and N. Imoto, Phys. Rev. A **81**, 022325 (2010).
[41] J. B. Brask, I. Rigas, E. S. Polzik, U. L. Andersen, and A. S. Sørensen, Phys. Rev. Lett. **105**, 160501 (2010).
[42] K. Azuma, H. Takeda, M. Koashi, and N. Imoto, Phys. Rev. A **85**, 062309 (2012).
[43] S. W. Lee and H. Jeong, Phys. Rev. A **87**, 022326 (2013).
[44] K. Nemoto and W. J. Munro, Phys. Rev. Lett. **93**, 250502 (2004).
[45] M. Fujiwara, M. Toyoshima, M. Sasaki, K. Yoshino, Y. Nambu, and A. Tomita, Appl. Phys. Lett. **95**, 261103 (2009).
[46] F. Bussières, J. A. Slater, J. Jin, N. Godbout, and W. Tittel, Phys. Rev. A **81**, 052106 (2010).
[47] J. T. Barreiro, T.-C. Wei, and P. G. Kwiat, Phys. Rev. Lett. **105**, 030407 (2010).
[48] E. Karimi, J. Leach, S. Slussarenko, B. Piccirillo, L. Marrucci, L. Chen, W. She, S. Franke-Arnold, M. J. Padgett, and E. Santamato, Phys. Rev. A **82**, 022115 (2010).
[49] W. A. T. Nogueira, M. Santibañez, S. Pádua, A. Delgado, C. Saavedra, L. Neves, and G. Lima, Phys. Rev. A **82**, 042104 (2010).
[50] K. Kreis and P. van Loock, Phys. Rev. A **85**, 032307 (2012).
[51] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[52] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Phys. Rev. Lett. **77**, 2818 (1996).
[53] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, Phys. Rev. A **57**, R4075 (1998).
[54] J. W. Pan, C. Simon, and A. Zellinger, Nature (London) **410**, 1067 (2001).
[55] J. W. Pan, S. Gasparonl, R. Ursin, G. Weihs, and A. Zellinger, Nature (London) **423**, 417 (2003).
[56] C. Simon and J. W. Pan, Phys. Rev. Lett. **89**, 257901 (2002).
[57] Y. B. Sheng, F. G. Deng, and H. Y. Zhou, Phys. Rev. A **77**, 042308 (2008).
[58] Y. B. Sheng and F. G. Deng, Phys. Rev. A **81**, 032307 (2010).
[59] Y. B. Sheng and F. G. Deng, Phys. Rev. A **82**, 044305 (2010).
[60] X. H. Li, Phys. Rev. A **82**, 044304 (2010).
[61] F. G. Deng, Phys. Rev. A **83**, 062316 (2011).
[62] F. G. Deng, Phys. Rev. A **84**, 052312 (2011).
[63] L. Xiao, C. Wang, W. Zhang, Y. D. Huang, J. D. Peng, and G. L. Long, Phys. Rev. A **77**, 042315 (2008).
[64] C. Wang, Y. Zhang, and G. S. Jin, Phys. Rev. A **84**, 032307 (2011).
[65] C. Wang, Y. Zhang, and R. Zhang, Opt. Expre. **19**, 25685 (2011).
[66] D. Gonta and P. van Loock, Phys. Rev. A **84**, 042303 (2011).
[67] D. Gonta and P. van Loock, Phys. Rev. A **86**, 052312 (2012).
[68] J. Clausen, L. Knöll, and D. G. Welsch, Phys. Rev. A **66**, 062303 (2002).
[69] H. Jeong and M. S. Kim, Quant. Inf. Comput. **2**, 208 (2002).
[70] S. Suzuki, M. Takeoka, M. Sasaki, U. L. Andersen, and F. Kannari, Phys. Rev. A **73**, 042304 (2006).
[71] Q. Lin, B. He, J. A. Bergou, and Y. H. Ren, Phys. Rev. A **80**, 042311 (2009).
[72] B. He, Y. H. Ren, and J. A. Bergou, Phys. Rev. A **79**, 052323 (2009).
[73] H. Jeong and M. S. Kim, Phys. Rev. A **65**, 042305 (2002).
[74] B. Yurke and D. Stoler, Phys. Rev. Lett. **57**, 13 (1986).
[75] S. J. van Enk and O. Hirota, Phys. Rev. A **64**, 022313 (2001).
[76] T. C. Ralph, A. Gilchrist, G. J. Milburn, W. J. Munro, and S. Glancy, Phys. Rev. A **68**, 042319 (2003).